**Integral and Vector Calculus**
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**Lecture – 54**
**Surface integral**

Hello students. So, in the previous class, we started with the integral aspects of Vector Calculus and we started with line integral actually. So, I left off the previous lecture at a very interesting example in my opinion actually. So, today we will continue with the similar example.

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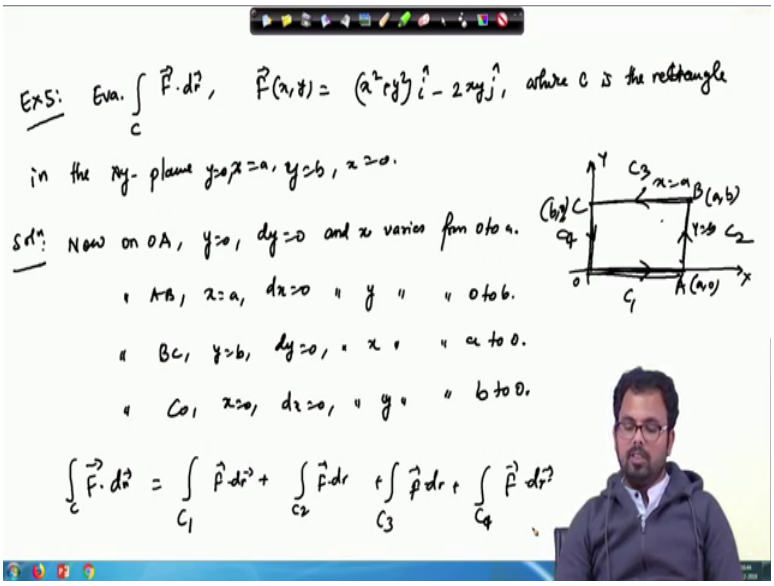

So, we were given a vector function and we had to find out the line integral of this vector function F along this rectangle. So, we have assumed that the orientation is in this way. So, O A, AB and then C O or yes C O. So, now let me write these points a 0 and then this is a b and then this is b 0 alright. So, now, on OA so on OA y is varying base a sorry x is varying, but y is 0 because the equation of the line of the x axis is y equals to 0. So, on OA x is on OA; our y is sorry our y is 0, our y is 0 and therefore, d y is 0 and x varies from and x varies from 0 to a right; so 0 to a.

So, 0 to a that is how x is varying. Now on AB our x is a and therefore, dx is 0 because x is constant. So, d x is 0 and y varies from 0 to b right ok. Now, on BC we have we have y equals to b and therefore, dy will be 0 and x varies from a to x varies from a to 0. And on

CO our CO x is 0 so, basically dx is 0 and y varies from b to 0 right. So, basically we have four smooth curves so, and along all these four curves we have these four criterias.

So, now instead of calculating the line integral of the vector function F along this curve C so, along this curve C, the line integral along the curve C 1 dot d r plus C 2 dot d r C3 F dot d r plus C4 F dot d r will be same because our curve C is actually the union of C C1 C 2 C3 C4. So, here we can write union of C1 C 2 C C3 C4 and from some of the properties from integral calculus or Riemann integral this union can be transferred into a summation. So, it can be changed into a summation. So, basically integral over the union of the domains is equals to the sum of the sub integrals over each of these domains alright. So, that is what we are doing here.

So, integral over C1 C 2 C3 C4 if you sum them then that will be the integral over the curve C. So, that is the formula which we have used here and we can call this line as our curve C1, we can call this line as our curve C 2, we can call this line as our curve C3 and we can call this line as our curve C4 and we have to now evaluate these integrals separately. So, what is the value of the integral along C1?

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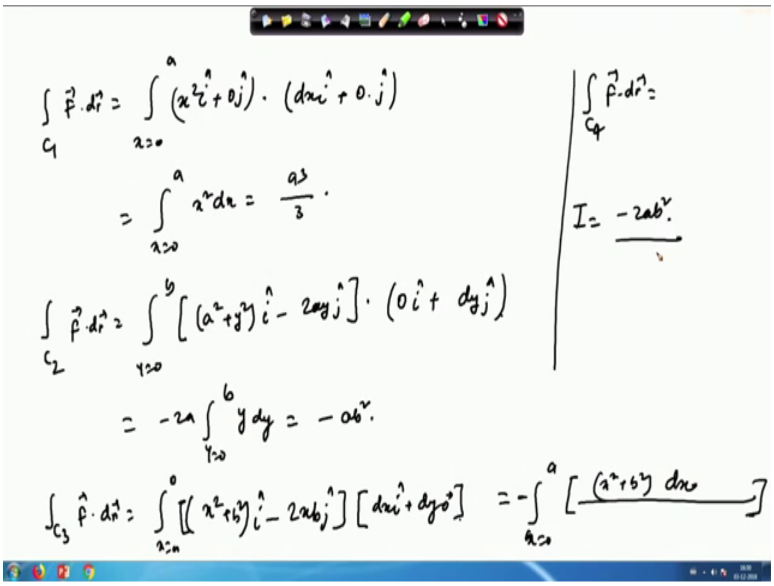

So, C1 F dot d r. So, C1 is where y is let me go back. So, on C1 x is varying from 0 to a. So, on C1 x is varying from 0 to a and our F is given by this equation. Now, on C1 y is 0 so, d y is 0. So, when y is 0 this is 0 and when y is 0 so, this term is 0. So, we basically have x square i. So, we basically have x square i plus 0 times j and d r can be written as d x i plus d y j. Now d y j the d y is basically 0 because y is 0 so, d y is 0. So, we are ultimately left with i times d x. So, here we have d x times i plus 0 times j d alright. And now, if we take the dot product then this is basically x running from 0 to a x squared d x. So, this is a cube by 3; I believe yes. And similarly we can calculate the integral C 2 F dot d r. So, here y is varying from y is varying from 0 to b.

So, I can write y is varying from 0 to b and then x square here x is a. So, x square plus y square so, I can write this as a square plus y square times i minus 2 a y j and then dot product with d x i plus d y j. So, what will be d x? d x is 0 and this one will be so, 0 i plus this one is d y j. So, this one is d y j and when we take the dot product then this will be minus of 2 a while running from 0 to b Y d y. So, this is ultimately minus a b square. And then similarly if we integrates C3 F along C3 F dot d r that this will be basically again y equals to b and then d y is 0 and then we substitute x running from a to 0. We reverse them and then we reverse the minus sign. So, since we are integrating a is a positive number so, we cannot in so, it does not make much sense if you are integrating from a to 0. So, it always has to be 0 to a.

So, just do those formula changes. So, what we will get is basically this thing. So, we have x running from a to 0 F is our y is b. So, this one will be x square plus b square. So, x square plus b square times i minus 2 x b j 2 x b j and then this will be d x i plus d y j. So, d x is d x is that one is fine and d y is 0. So, we can write d x i plus d y 0. So, when we integrate, then this will ultimately give us or what we can do? We first change the formula. So, this will be a x running from 0 to a and then we write the entire dot product. So, when we take the dot product this will be basically x square plus b square times d x because i i will be gone.

So, now we integrate this x square plus b square from 0 to a and similarly we integrate. So, similarly we integrate the curve the function along the curve C4 and then you sum them and ultimately we will obtain the answer. So, ultimately we will obtain the answer i let us say is equals to minus 2 a b square and this is the required answer. So, here in this case instead of having a one instead of having one smooth curve, we basically had how to say a union of four smooth curves and those four smooth curves actually needed to be evaluated. So, the basically the line integral along those four smooth curves needs to be evaluated separately. We cannot do C in whole we have to do like C1 then integral on C 2, then integral on C3, then integral on C4. So, sometimes you might come across examples where there where the area or sorry the curve is actually composed of a parabola and a straight line.

So, you basically have to divide your integral into 2 sub integrals. So, you first have to integrate along that parabola and then integrate along that straight line alright. So, that is something I wanted to show you. So, of course, there are examples like that and due to time constraints we have to finish this course on time. So, we will move on to our next topic which is basically surface integral and greens theorem.

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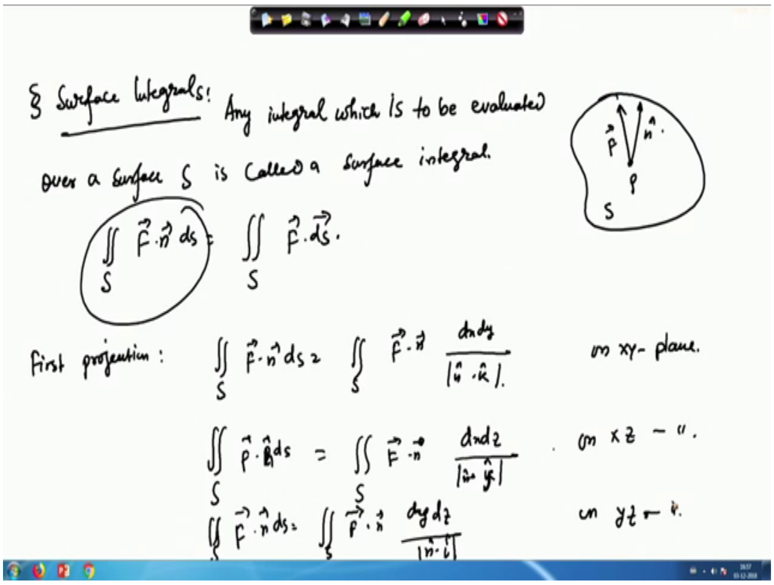


So, basically what do we mean by surface integral? So, let us assume that this is our surface s in three dimensional space. So, I call it as s and any integral and any integral any integral so, it is like line integral definition. So, any integral which is to be evaluated which is to be evaluated over a surface say S is called a surface integral. So, basically what do we mean by is there is a very nice theory going on at the back, but we generally write it as in this way.

So, we so when we are doing the surface integral, we usually denoted as surface integral over S F dot n d s which can also be written as surface integral over s F dot d s. So, here what we actually mean is that this F dot n is the component of F along the normal n and this d s. So, on the surface on the surface of this on the surface of this let us say this surface S, we basically have this is our vector and this is our point P. This is our vector F and this is our vector or the normal n. So, this F dot n is actually the normal component is the normal component of F at the point P and if we and the integral of F dot n, then over the surface S. So, this is the integral of the function F dot n over the surface S is given in this fashion. So, this is just like a notation. So, when you can you will be given as let us say a vector function and then from there you have to calculate the normal n; so, that you can evaluate this surface integral F dot n d s alright.

Now, when we evaluate the surface integral, we have to remember certain ways that how we can evaluate. So, this is of course, given this is of course, is given as an abstract definition that what do we mean by this component of F integral of component of F along the normal n over the surface S. But when we need to evaluate this we need some kind of formula and that will help us to evaluate. So, the formula is you have to take the projection of the surface either on x y plane or y z plane or z x plane. So, depending on your projection basically your evaluation will depend.

Of course, the answer would remain same its just that the complications are; I do not know the way you solve the problem will depend on which plane you are taking the projection alright. So, here we have F dot n. So, the first projection let us say we are projecting on XY plane. So, then this will be a surface integral over S F dot n d s which is surface integral over S F dot n d x d y divided by n dot K right; F dot n d x d y divided by n dot K. So, this is what we get when we take the projection on first projection or projection on x y plane. Similarly you can take the projection on XZ plane and we can take the projection on y z plane F dot n sorry XZ. So, if we are taking the projections that this has to be j and when we are taking projection and YZ plane so, this has to be i. So, that trick too on YZ plane. So, the trick to remember these formulas is that whenever you are taking projection on a certain plane so, that will be in the product d x d z and then the remaining axis. So, if you are taking the projection on XZ plane then basically y axis is perpendicular.

So, you take n dot j. So, j is the unit vector along the x is y. So, we take basically n dot j similarly if you are taking the projection on XY plane, then your Z plane is then your z axis is perpendicular to x XY plane and therefore, you take n dot K. And similarly if we are taking YZ plane then x axis is perpendicular to y z plane and therefore, we take the projection we take the product as n dot i. So, it is a very nice and convenient way to remember these formulas alright. So, now we will see how we can solve some examples.

So, this is of course, they are given in a bit abstract way we will see how we can work out an example. So, let us start alright.

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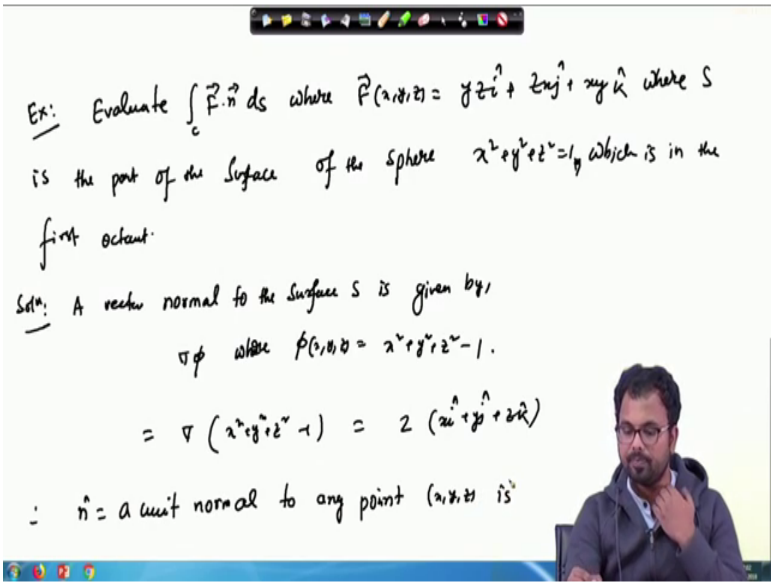


So, to begin with so, evaluate F dot n d s where our function F x y z is y z i plus z x j plus x y k where S is the part of the of the surface of this sphere x square plus y square plus z square equals to 1 which is in the first octant in the first octant. So, here in this example what we have is we have a given vector function F and here we have to calculate the surface integral F dot n d s and our surface along which we have to integrate basically or on which we have to integrate is given by x square plus y square plus z square equals to 1. However, there is a small cast that we have to just consider a surface that is in the first octant.

So, if you consider let us say a circle in 2D, then it actually has 4 how to say quadrants. So, and in case of a sphere since it is a three dimensional geometry, you will have actually eight octants and we have to limit to the just the first octant alright. So, let us start. So, first of all if we want to calculate F dot n, we have to find out n. What is our n and unless you calculate this n we cannot proceed any further. But we remember from the gradient divergence and curl that in gradient of a function or the gradient of a of a given. Let us say you have a you have the equation of a surface then the gradient of that that F equals to that surface is basically a normal to the surface. So, the gradient of F is actually a normal to the curve F is equal to some constant or F equals to 0.

So, basically a vector normal to the surface S is given by S equals to gradient of phi where phi x y z equals to x square plus y square plus z square minus 1 alright. So, this is the required equation of the basically phi is cos 2 x square plus y square plus z square equals to minus 1. So, now we do not have to write S here. So, a vector normal to the surface S is given by grad phi and this grad phi can be calculated now. So, grad of x square plus y square plus z square minus 1. So, this is basically 2 times x i y z z k alright. So, we take the gradient and that is what we obtain. Now this is normal to the surface S, but n is the unit outward do normal; however, gradient of phi is just the outward do normal.

So, we have to now calculate n cap. So, to calculate n cap a unit normal therefore, a unit normal a unit normal to any point to a unit point to any point actually. So, a unit normal to any point x y z is basically n equals 2 grad phi divided by gradient of phi.

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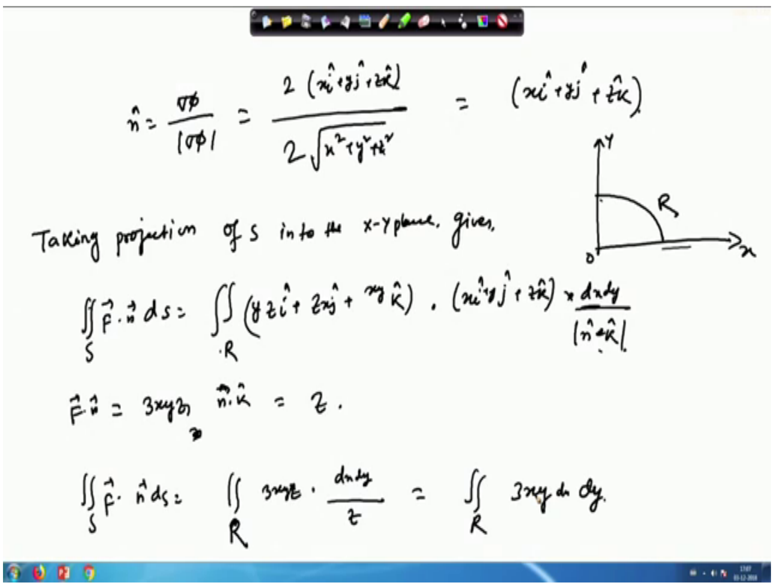


So, grad phi is 2 times x i y j plus z k and when we take square so, this is basically 2 times. So, 2 times x is square plus y square plus z square. Now, x is square plus y square plus z square is 1. So, this is basically x i plus y j plus z k. So, that is our normal n and now we will take projection on XY plane. So, if we take the projection on XY plane. So, this is our plane y and that is the first octant r because if you take the projection of first octant in the sphere; if you take the projection on XY plane, then it will be a circle in the first octant. So, taking projection and of course, a vector perpendicular to a vector perpendicular to this XY plane is the k axis right alright.

So, taking projection taking projection of S into the XY plane; so, take a projection of S into the XY plane gives so, this is basically our unit normal and now taking projection of S into XY plane it will be basically. So, our F is y z i plus z x j plus x y k y z i plus z x j plus x y K dot product with d x y d d x y d i divided by n dot k. So, first of all here I will write our normal which is x y y j plus z k they did and this one is basically. So, we have n and we have F and now d x is basically d x d y divided by n dot k right. So, here in this case what we will obtain is n dot K. So, this is nothing, but so R is basically the R is basically our after projection, the new the new idea basically or the new place where we are doing the integration.

And here in this case if our n is n is K basically so, n is the unit normal perpendicular to it. So, n is the unit normal and the K is the given in k is the k is the vector perpendicular to XY plane. So, if you take n dot k then this will reduce to this will reduce to just a z. So, from here what we have is F dot n is if you break the dot product, then this will be 3 x y z and k will be n dot K will be basically n dot K will be basically z right. And therefore, if I substitute everything here then surface integral F dot n d s is equals to integral over R 3 x y z and then times this is basically times it is not a cross product d x d y and n dot K is again z. So, this whole thing will reduce to integral over R3 x y d x d y right.

And now, what we will do is we substitute since we are in the 3 in 2 dimensional geometry and it is basically sort of like a surface in this integral in the in the circle. This whatever you want to call, we basically substitute x equals to let us say small r cos theta and y equals to small r sin theta.

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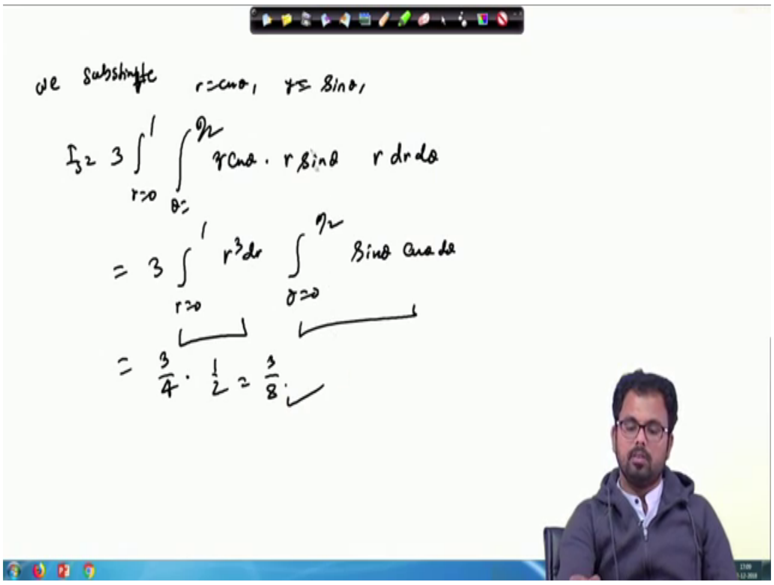

So, basically we substitute we substitute x equals to r cos theta y plus 2 r sin theta and therefore, the required line integral sorry that requires surface integral. Let us say I am calling it as I s equals to sorry equals to I have r running from 0 to one because it is a unit radius and theta since we are in the first octant than theta our first quadrant, then theta will run from 0 to pi by 2. So, that I am writing here and then i have 3 3 is here x. So, x is. So, this is x x is r cos theta. So, 3 r cos theta and times this one is y y is r sin theta right.

So, y is r sin theta and everything is transferred to spherical sorry polar coordinate system. So, we have r d r d theta. So, if you multiply this whole thing and take the term for r on one side so, this is r running from 0 to 1; this will be r cube d r and this one will be theta running from 0 to pi by 2 sin theta cos theta d theta. So, we integrate this thing here and then we integrate this thing here and this will give us 3 by 4 times half. So, this is ultimately 3 by 8. So, we are just a half with 2 here and then we do sin 2 theta and all that. So, this is the required answer.

So, in this case we basically have to find out the projection of the surface on a certain plane and just having to know this the projection you can be able to solve the rest of the example. We will solve some more examples on surface integral in our next class just to make the concept clear and then we will move on to volume integral. So, I will stop here for today and we will continue with our examples on surface integral in our next class.

Thank you.