Integral and Vector Calculus
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Lecture – 53
Line Integral

Hello students. So, in the previous class, we concluded the part where we saw how vector calculus can be applicable to mechanics or applied mathematics and there we derived several equations of motion for particles under certain laws of physics and we also derived formulas for momentum, angular momentum and things like that.

So, now we are at a stage that we can move to the Integral part of the Vector Calculus which is basically the line integral surface integral and volume integral and that they basically include Greens theorem, Gauss theorem and Stokes theorem. So, we will also learn about these three important theorems or of vector calculus.

So, today we are going to start with line integral. So, basically in vector calculus, any integral along a along a curve is called as a line integral in a nutshell actually.

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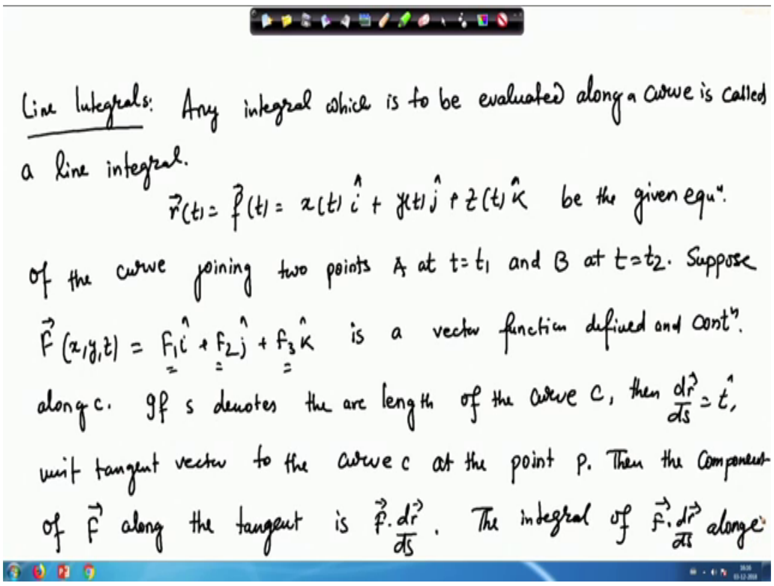

So, to start with, we will start with line integrals. We will start with line integrals. So, any integral which is to be evaluated which is to be evaluated along a curve is called a line integral; is called a line integral.

So, basically we know that equation of a curve. So, suppose the equation of the curve is given by r t is equals to F t which is equals to basically x t i. So, three components, x t i, y t j plus z t k be the given equation of the curve. And suppose we are integrating, so suppose we are integrating be the given equation of the curve joining two points. So, two points which are let us say A at t equals to t 1 and the second point is B at t equals to t 2. So, since these are obvious, I am not drawing any figure alright. And suppose, we have a capital F so we have a capital F which is a function of x, y and z and it is given as a capital F 1 x, y, z i plus capital F 2 F y x, y, z j plus capital F 3 x y z k. So, these are all functions of x, y and z alright.

Now, we are evaluating this function. So, suppose this is a function is a vector function, is a vector function defined and continuous and continuous along c. And if small as denotes the arc length, denotes the arc length of the curve c, then d r d s as we know is the unit tangent vector then d r d s t is the unit tangent vector, unit tangent vector to the curve c at the point p whose position vector is r, at a point p whose position vector is r. And , then the component of F along the tangent is F dot d r d s. So, this is the component and the integral basically and the integral of this component of d F along c along c from A to B is given by, so basically we integrate from A to B F dot d r d s.

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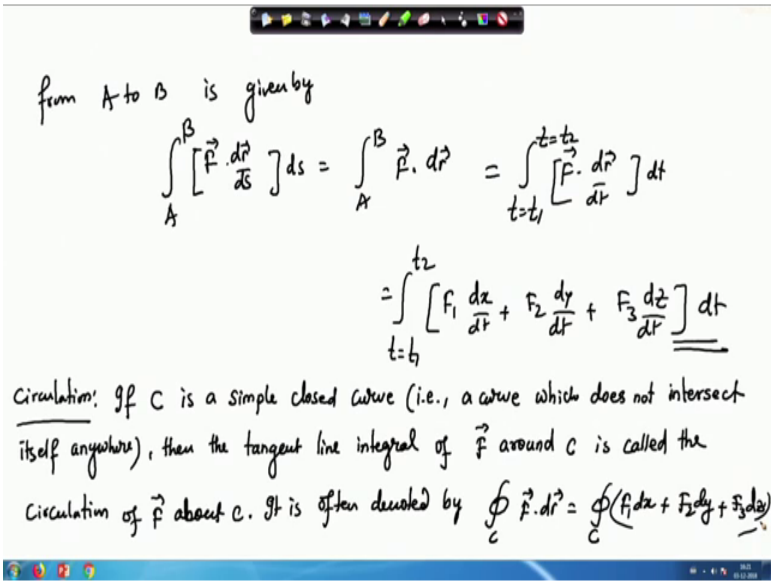

So, basically this here times d s, so we are integrating along the arc length and this can be written as integral from A to B F dot d r. And if we want to write in terms of the in terms of t, then we can be able to write this as integral t running from t 1 to t 2 F dot d r d t times d t. So, basically the point A is attained at the point at t is equal to t 1 and the point B is attained at t is equal to t 2.

So, we divide both sides by d t. And so, we divide the sorry, not both sides we divide this d r by d t and we multiply it by d t and then that is basically your integral F this F dot d r d t. We can also write it as a Cartesian, in the Cartesian form so, this one will be F 1 times d x d t plus F 2 times d y d t plus F 3 times d z d t times d t. So, this is the required integral of this function F dot d r d s along from the point A to B.

So, in this case, we are actually integrating along a curve. So, C is the given curve and which can be smoother, which can be piecewise smooth. So, when it is piecewise smooth, then we integrate along the one piece, then we integrate along the second piece. So, like in part we can do the integration. So, this is what we mean by; what we mean by a line integral along a curve and when we talk about, so there is a small definition here.

So, in the line integral we can also talk about circulation. So, what do we mean by circulation? So, if C is a simple curve, if C is a simple closed curve is a simple closed curve that is i e, that is a curve which does not intersect, which does not intersect itself anywhere. So, there is no looping there. So, it is not intersecting itself anywhere, then the tangent line; then the tangent line integral of F around C is called the circulation of F about C and it is often denoted by this symbol and we write it as F dot d r is equals to integral C F 1 d x F 2 d y F 3 d z. So, this is the required definition of this circulation of F around the curve C, alright.

So, now we will try to solve some examples on the line integral because line integral makes more sense when you actually try solving some examples than learning about the theory. So, let us start with our first example.

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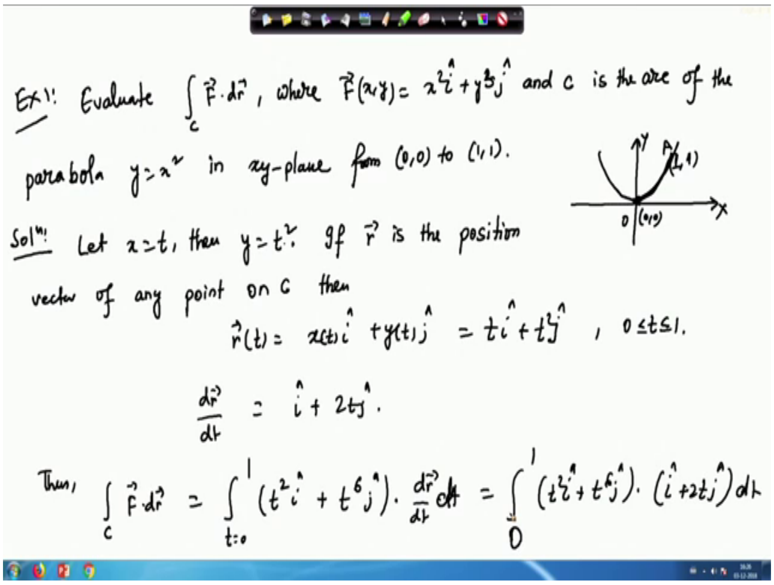

So, when we solve the examples, the concept will become even more clear. So, evaluate, the first example is evaluate integral over the curve C F dot d r where our F x, y is basically where F x, y. So, we are in 2 D. So, F x, y is equals to x square times i plus y is y cube times j where and sorry not where and C is the arc of the parabola, is the arc of the parabola y is equals to x square in in x, y plane from 0, 0 to 1, 1.

So, basically here we have a vector function F and we need to calculate the line integral of this function F along the curve C, where C is the arc of the parabola from 0, 0 to 1, 1. Now, here in this case and so, basically if we want to draw the figure, then this is our x, this is our y 0 and that is our parabola touching the origin and we have to calculate the line integral along 0, 0 and then 1, 1 so along this arc. So, let us say O, A, O basically. So, along this arc we have to evaluate the line integral for this function F.

So, now what we do? We form a parametric equation. So, that means, we involve a parameter t and we find the range for that parameter t which will satisfy this parabolic equation. So, let x equals to t and then y will be t square and if r is the position vector, position vector of any point on c, then our r t can be written as x t i plus y t j which is t i plus t square j. And t will be, so when x is 0, t is 0. When x is 1, t is 1. Similarly, when y is 0, t is 0. When y is 1, t is 1. So, basically t will be running from 0 to 1, alright.

Now, we will calculate d r d t from here, so our d r d t is i plus 2 t j, alright. So, we have got our d r d t. Thus integral over the curve C F dot d r is equal to integral t running from

0 to 1. Our F will be, so along this curve, along this curve we are we are integrating along this curve c. So, in that way x is basically our t square and y is basically our x is basically t and y is t square yeah. So, I mean, I squired after. So, so this is t square i plus t to the power 6 j times, I will make a d r d t all right.

So, I will make it d r d t. So, then in that case, this will be 1 to 0 t square i plus t to the power 6 j dot product with d r d t which is i plus 2 t j times d t. So, now, we will take the dot product and sorry this one will be 0 to 1.

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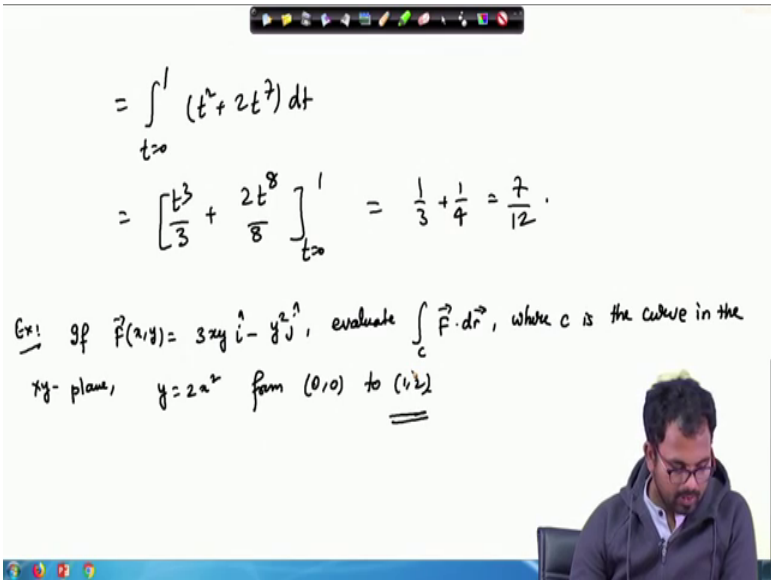

And now, we take the dot product and this will be integral from t running from 0 to 1 t plus 2 t to the power 7. So, t is square plus 2 t to the power 7 d t, right. And when we integrate, then this will be t to the power 8 divided by 8 whole thing evaluated from 0 to 1. So, this will be 1 by 3 plus 1 by 4. So that means 7 by 12 so, this is the required answer.

So, when we integrate the given vector function F along this parabola from the point 0 0 to 1 1, that will be our required answer. So, you see when, we are actually working out the example we have you get to see what I actually mean by this line integral. So, it is not always a line, I mean it is not it is not something like a straight line. So, when we say a line integral; that means, we are integrating along some curve which can be smooth, which can be piecewise smooth, which can a simple closed curve things like that. So, these are the certain properties our curve can have and based on that we can calculate the line integral. We will work out few more examples just to make the concept a bit more clear.

So, now let us consider an another example. So, if F x, y equals to 3 x y times i minus y square j, then evaluate line integral F dot d r where C is the curve in the looks y plane, y is equals to 2 x square from 0, 0 to 1, 1. So, this example is also pretty much similar to the previous example. So, here the given equation is again a parabola. So, we have to assume x equals 2 t and y is equals to 2, and y is equals to then will be 2 t square. And at both of these two points, the equation will satisfy. Therefore, t will run from 0 to 1 I and from there just integrating like before will give you the required answer. So, this example is exactly like the previous example.

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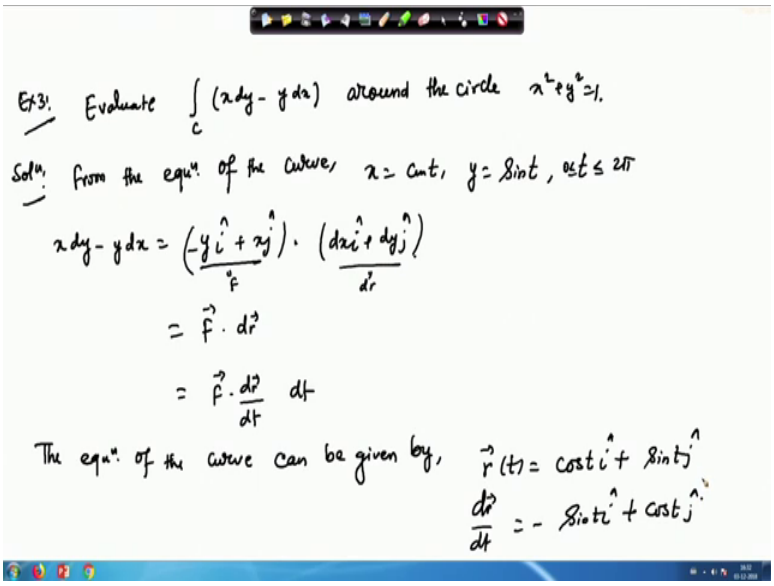

So, I leave this example up to the students for them to practice. We will now move on to our next example which is this one, example 3.

So, the example 3 is evaluate C x d y minus y d x around the circle x square plus y square equals to 1. So, here the given equation of the curve the given equation of the curve C is basically a circle, x squared plus y squared equals to 1. So, this circle is actually a simple closed curve because it is not forming any kind of loop. So, it is not intersecting itself. It is actually a simple closed curve. So, this is nothing but the circulation of the function F. It is also very easy to get the function F from here, let us see. So, the given equation of the circle is x square plus y square equals to 1. So, from here, from the equation of the curve of the curve, we have x equals to so, basically the radius of the circle.

So, radius is 1 obviously. So, we have cos t and y is equals to sin t where t is running between 0 to 2 pi because we have a closed circle. So, our t will be running from 0 to 2 pi and this function here x d y minus y d x can be written as, can be written as y minus y i plus x j dot product with d x i plus d y j and this is nothing but F dot d r where this is actually F and this is actually d r, right. And from here, I can be able to write F dot d r d t times d t. So, this d r d t can be calculated very easily. So, d r d t is nothing but d x d t i plus d y d t j which is minus sin t plus cos t. So, from here what we will obtain is, so the equation of the curve can be written as or can be given by r t equals to x t i plus y t j. So, x t is basically cos t i plus sin t j, right. And now, we will take d r d t. So, d r d t is minus sin t i plus cos t j, alright.

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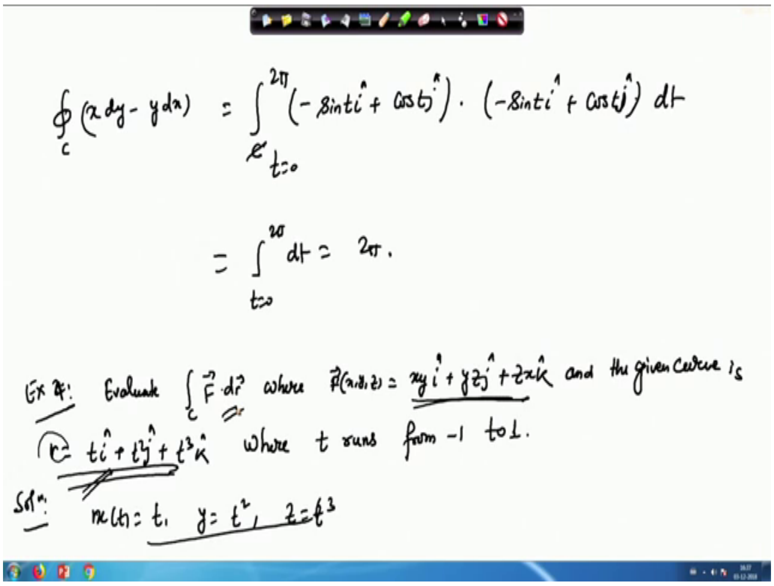

So, now I will do the integration. So, now, we have integration over the curve C x d y minus y d x which can also be written as integral over the curve C f dot d r d t. So, what is our F? F is minus y i and y is sin t and x is cos t. So, we substitute these values. So, y is sin t and x is cos t dot product with d r d t. So, d r d t is minus of sin t i plus cos t j. So, this is minus of sin t i plus cos t j times d t and here t is running from 0 to 2 pi. So, we will take the dot product and that will result into sin squared t plus cos square t. So, ultimately the whole thing will be 1.

And therefore, we will have integration of d t which is something like this right. And when we integrate, then we substitute the value and then this is basically 2 pi. So, the circulation of the vector function F which is given by this way which is given by this way is actually equals to 2 pi along that circle C.

So, sometimes you are not given the function F and some and there you have to actually obtain the vector function F. So, it is also a little bit small trick. It is not actually complicated it is very easy to see what can be our function F d. So, just have a look at the given integral and from there, you can be able to evaluate this will be my vector function F which I need to take dot product with d r d t and from the given equation of the curve where you have to calculate the integral, you can be able to find out the equation of the curve in vector form. And then, you calculate d r d t and then the rest of the things are same as in the example one. So, like we did in this example. So, we calculated the circulation of the function F which was not given. So, we calculated the vector function F along the given curve C alright.

So, we can have examples similar to this, many examples similar to this actually. So, I am just looking into my lecture note which one to consider. So, now what we will do, let us consider and this example. So, here in this example, example 4 I believe. So, evaluate F dot d r where F x, y, z is basically x y i plus y z j plus z x k and the given curve is r is equals to t i plus t square j plus t cube k where t runs from minus 1 to plus 1. Now, this is not a very complicated example. So, from the given equation, from the given equation of the curve which is this one, we can be able to obtain our x t which is t, our y t which is t square and our z t which is t cube.

And then, we substitute x t y t and z, then we substitute x t y t and this z t there basically in this equation and we do d r d t and then we calculate d r d t from here and then we substitute the value of d r d t and we integrate from minus 1 to plus 1. So, here in this example, it is just that the equation I mean how to say in the Cartesian form x t, y t plus z t, the equation of the curve is not given, but the vector equation of the curve is given. So, from there extracting x t, y t plus z t is relatively simple.

So, here, we see that in this example, you can actually be able to write x t, y t plus z t although they are not given. So, you see these examples, I mean one thing is not given, then you can be able to obtain the other thing. Similarly, the other thing is given, then you can be able to obtain the first thing. So, it is all about doing practice and getting used to the examples of this type they are not complicated they are just playing with some parameters. That is, pretty much it and from here on onwards, you can be able to solve this example very easily; just substitute these values just substitute these values here, do the d r d t from here and then integrate from minus 1 to plus 1 and then everything is like piece of cake alright. So, this is also very simple, yet how to say a nice example.

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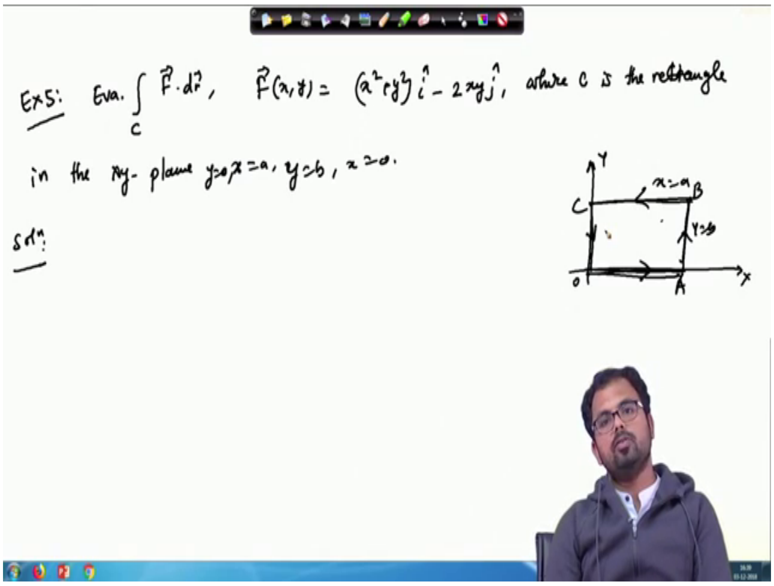

Next, I am going to consider an example which is I personally like. So, here we have to evaluate like before F dot d r where our given vector function F x, y is x square plus y square times i minus 2 x y j where C is the curve, where C is the rectangle in the x, y plane; x equals to a, y is equal to 0, x equals to a, x, y is equals to b and x equals to 0.

So, the solution, so here as you can see we have a rectangle so, obviously, it is not a smooth curve because then you have four corners here. So, if I draw this curve. So, that is our x axis, this is origin, this is y axis. So, this is my x equals to 0 line, this is x equals to a and that is y is equal to 0 and that is y equals to b. So, I can write it as a, b, c. So, this is our curve right not the middle portion the curve this one all right. So, now, it depends on us which direction are we following? Are we calculating in this direction or are we calculating in this direction? So, that is one thing.

Now, the second thing is that whatever we get along this direction, we just have to take the reverse direction actually when we are on the upper side, alright. So, this is a very interesting sample and we will actually try to solve this in our next class because it will require some time to see the interesting part of this example. So, I will stop here for today and in the next class we will continue with this example. And I thank you for your attention and I look forward to your next class.