Integral and Vector Calculus
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Lecture - 52
Example on derivation of equation of motion of particle

Hello students. I am so, in the previous class, we were sort of looking at the application of vector calculus in the mechanics and we try to derive equation of or formula for momentum, angular momentum, Newton's laws, Newton's second law of motion. And we saw that the force P can be written as m times acceleration where m is the mass of a particle and a p is the applied force or impressed force and a is the acceleration of the particle. So, today, we will try to derive some of the equations of motion where we see how vector calculus is useful to write the equation of motion of such particles are moving under some physical law and they will try to derive their path and things like that.

So, let us start with.

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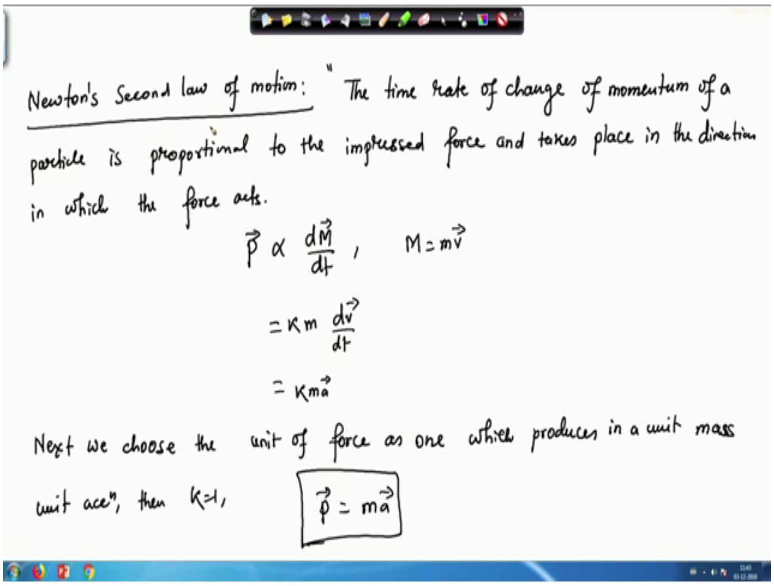

So, in the previous class, we had this formula p is equals to m a and this formula was obtained which is basically called as Newton's second law of motion. And sometimes, people also call it as equation of motion of a particle.

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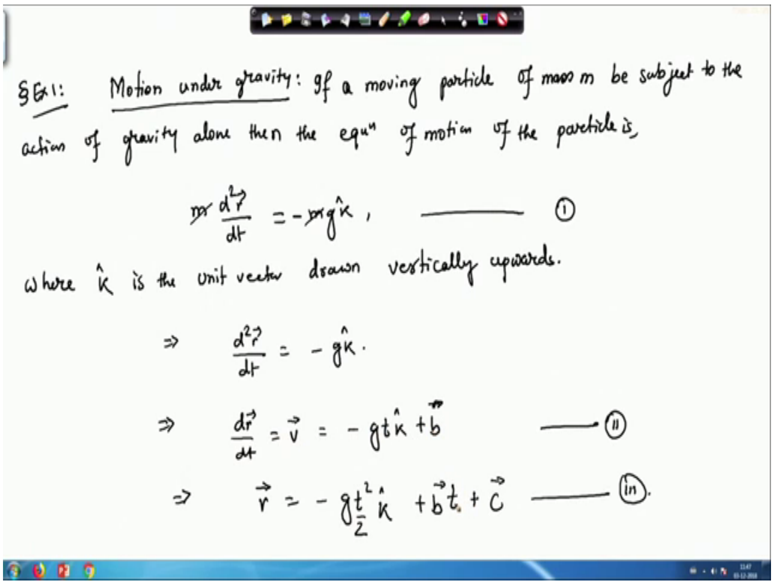

Now, let us start with our very first example, example 1. So, motion under gravity, so, motion under gravity means suppose you drop the particle or you threw a particle, then after a while, when you throw a particle, then of course, it starts more or less in a straight line manner, but over the time it follows a parabolic profile. So, be it mainly because the particle because of the gravity falls on the ground. So, basically the act, the gravity at that point is acting as a force and it is pulling the particle down towards it.

So, it does not matter what kind of force you have applied it may go far or it may fall right next to you, it will always follow a parabolic profile. And we will try to derive that how that parabolic profile and will first see how we can do that. Suppose, we have a moving particle. So, suppose if we have, we have a if we have a, if a moving particle of mass m be subjected to the action of gravity alone, then the equation of motion, equation of motion of the particle.

So, equation of motion as you remember p is equals to m times a. So, a is the acceleration and p is the force and the force is given by m times d square r by d t square right. So, the force is given by m times the d square r by d t square and the right hand side is minus m g k where k is where k is the unit vector is the unit vector drawn vertically upwards, alright.

So, now, if we cancel m from both sides, then this is basically our d square r by d t square is equals to minus g times the vector k. And if I integrate both sides, then in that case this will be d r d t which is basically our velocity v d r d t means v and this side will be minus of g t k plus some constant vector b. And if I integrate again or if I integrate again, then in that case this will be r times r.

So, r is equals to, so, if I integrate again. So, r is equals to this will be minus g t square by 2 times k plus b t plus some vector c. So, let me call it as equation number. So, this is our equation number 1, this is our equation number 2, this is our equation number 3. So, I integrated both sides of these equations and when you integrate, I am assuming you are familiar with that ordinary differential equations. So, I integrate it with respect to t. So, will get a constant vector b and then I again integrate it and then I got a constant vector c. Now, we have to find the values for b and c.

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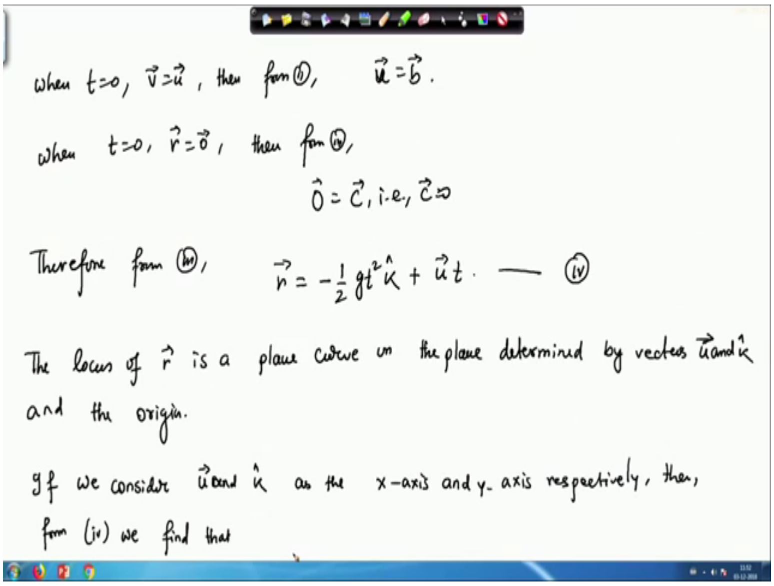

So, if, so when t equals to 0, so when t equals to 0, so when t equals to 0 at time t 0 at that time the velocity v and the initial velocity would be same. So, the initial velocity at t equals to 0 is same as the velocity v. So, I can write v equals to u. So, u is my initial velocity. So, then from 2 we will have, what do we have? So, v is equals to u, this term will be 0 and b. So, this will be v u equals to vector b. So, if I use this in this equation, so I can substitute u here. Now, when t equals to 0, now again when t equals to 0, then the vector r will be 0 because when t equals to 0, then the particle is not moving. So, at that time basically the position vector r would also be 0. So, when t equals to 0, r is equals to 0 and then from 3, we will have 0 and here we will have 0, 0. So, c is also a 0 vector.

So, c is equals to 0 vector, not implication, but we can write that is c equals to 0. Therefore, from 2, therefore, sorry from 3 actually therefore, from 3 we will have r is equals to minus half g t square k plus; what do we have? u t, u is the initial velocity and c is 0. So, this is the required equation of the motion of the file. So, this is the required how to say locus of the point b. So, I can write the locus of r is a plane curve on the plane determined by the vectors back at the, determined by the vector you vectors u and k and the origin.

So, the origin and the vector u and k will determine the locus of r. And if we consider and so, this is the required equation of equation of the particle p actually moving along the curve and the basically this is the equation of the curve along which the particle is moving. And if we consider, so one particular case, so if we consider u and k as the x axis and y axis respectively, then from this one is equation number 4, then from 4 we find that. So, if we equate, so here I can substitute here I substitute basically. So, if it is x axis, then this one will be u comma 0 and if it is y axis, then this will be just 0 and 0 comma k and then I can equate the coefficients.

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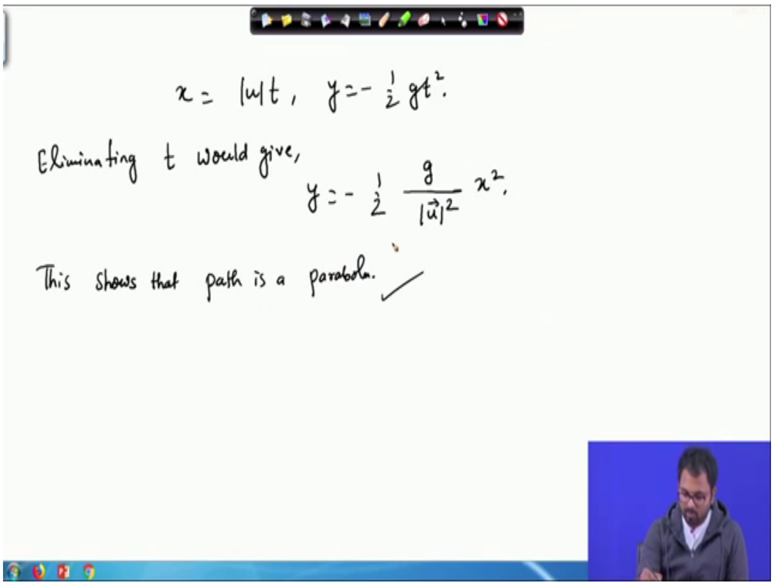

So, basically we find that x equals to mod of u times t and y equals to minus of half g t square. So, eliminating t, so if I substitute for t here, so this will be x by use of eliminating t will be. So, eliminating t would give y equals to minus of half g by mod of u square x square. So, this shows that the path is a parabola; so, this shows that path is a parabola. So, that means, when you throw this the stone or a particle like this, then so, if you throw like this, I do not know if you can be able to see in that video not. So, if you throw like this, then basically your u is x axis and your k is let us say y axis. So, then in that case, the particle is actually following a parabolic path and that is what we were trying to establish.

So, you see using just some simple ca concepts of vector calculus. We did not even complicate things, we were able to derive the equation although, I sorry the curve along with the particle p or in that case and a stone p let us say move along that curve, alright. So, this is one of the interesting applications of vector calculus basically in mechanics. And let me see in my notes, if I have anything other, anything else interesting to show.

So, next we can derive the harmonic motion. So, let me give you a small example, what do you mean by harmonic motion.

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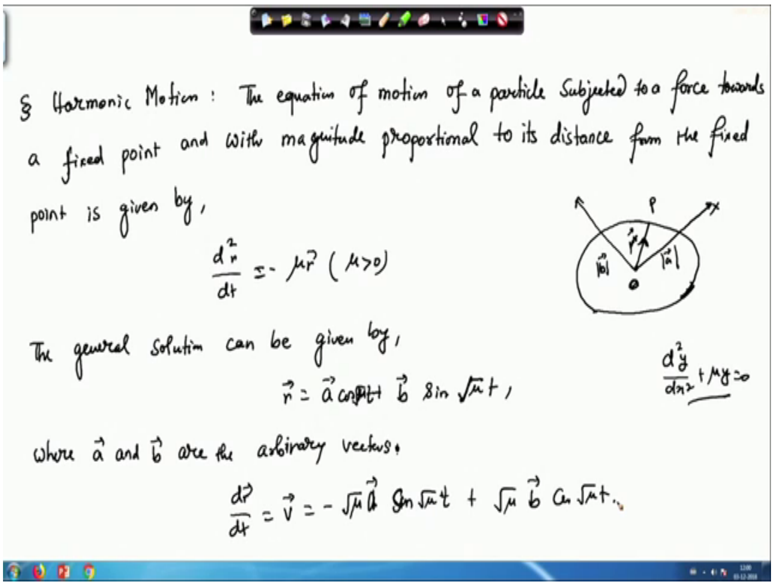

So, equation of motion for the harmonic motion, so harmonic motion, so basically the equation of motion of a particle subjected to a force subjected to a force towards a fixed point towards, a fixed point and with magnitude and with magnitude proportional to its distance, proportional to its distance from the fixed point is given by.

So, basically what we have is, we have a let us say a closed curve. In this case, we are choosing an ellipse alright. This is our origin O and let us say this is the point p and this is x axis and this is our y axis and the magnitude is b and here in this case the magnitude a is a and this is basically our, this is basically our mu times our the position vector, alright. So, what it says is the equation of motion of a particle subjected to a force towards the fixed point.

So, this is our fixed point O and with magnitude towards a fixed point and with magnitude for point proportional to it is distance from that fixed point is given by d square r by d t square minus mu r. So, basically since it is, so this vector is the r. So, since it is subjected towards that fixed point O; that means, it is subjected towards that fixed point; that means, that the vector is in the opposite direction. So, basically d square r by d t square is equals to it should have been O P. But since it is subjected in the in the fixed direction O, then it should be ma basically P O and that is basically minus of O P.

So, when you change the direction of a vector, then in that case you have to put a minus sign. So, that is why you have taken minus mu r or we have taken minus mu r where mu is the where mu is the constant basically. So, the general solution of this equation can be written as, so if you solve this equation, then basically the general solution can be given by. So, if you solve this equation, then in that case this is basically you can assume r is equals to a cos t, r is equals to a cos t and vector a and here cos since we have mu. So, this will be square root of mu t and this one will be sine square root of mu t. So, if you substitute this solutions here, then it will satisfy this equation. So, this is basically the general solution where a and b are the arbitrary vectors where a and b are the arbitrary vectors.

So, this equation is similar to what we remember from our ordinary differential equation. So, d square y by disc d x square plus mu y equals to 0. So, if you solve this equation, then we usually obtain a y equals to a cos root mu x plus b sine root mu x. So, it is the same thing. You can write r as the as x plus x i plus y j and then you equate the coefficients of i and j and then you solve the individual equations and then you sum them to get the vector r. And then, you take a 1 plus a 2 as a new vector and a 2 b 1 plus b 2 as the second vector and that is basically how you obtain this general solution. It is not complicated but it is a little bit lengthy. So, I am pretty sure you can be able to do that.

Now, this is our arbitrary and so this is our, these are our arbitrary vectors. And from this equation, basically if we differentiate then we will obtain d r d t which is basically our velocity v and this can be given as minus of square root of mu times vector a cos square root of me sorry sine square root of mu t plus square root of mu vector b cos square root of mu t, alright. So, these are the formulas.

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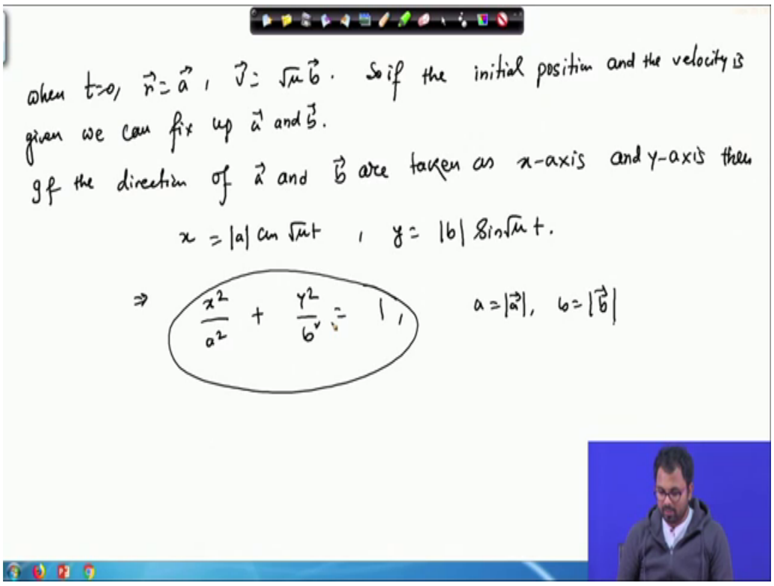

And if I choose at t equals to 0, so when t is 0, then in that case when sorry. So, when t is 0, when t is 0 then basically our r is this vector a. So, in that case r is the vector a and v is if I substitute t equals to 0, then sign t will be 0 and then this one will be cos t will be 1. So, v is equal to square root of mu b. So, square root of mu b and so, if the initial position and the velocity is given. So, that means, if the initial position and the velocity is given, we can fix up a and b.

So, if we have the velocity of the particle and the initial position, let us say that vector r, then in that case we can fix up these constant a and b. And if the directions of and if the directions, if the directions of a and b are taken as x axis taken as x axis and y axis then, so if they are taken as x axis and y axis, then we can equate the first of all the coefficients from here and then it is x equals to mod of a cos square root of mu t and y is equals to mod of b sine square root of mu t, alright. And this will be basically if I divide, then this will be basically x s square by a square and a y square by b square equals to 1 where a is equals to mod of a and b is equals to mod of b. So, if you choose x axis and if you choose vector a and vector b, those are basically our reference vector initially.

So, if you choose those reference vector as x axis and y axis, then basically the curve along which the particle p is moving for which it is for which the it is subjected to a force directed toward the origin, the path which it follows is basically and ellipse. So, this is the path or this is the curve along which the particle is moving. This is somehow also related to our planetary motion that it follows an elliptic path and the force which is basically in this case the gravity or acting between sun and the planets. It is actually directed towards that fixed point and therefore, the path along which this planet is moving is basically an ellipse or the curve is basically an ellipse.

So, an a and b are the length of the semi major axis and semi minor axis. So, this is an important example which I chose to show you all. And do we have some other interesting examples? Of course, I mean, so here I have a lot of examples actually, but if we try to cover all of them, then they probably run out of time to cover the other parts of this labours.

So, let me give you an another example and I will leave the derivation of the equation for optima of the curve up to the students. So, it is you have to derive the formula for that, alright.

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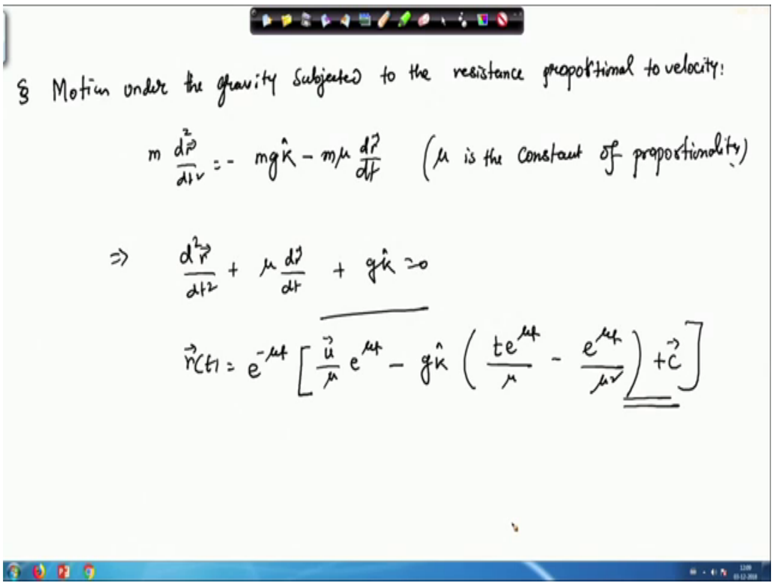

So, the third problem is motion under the gravity subjected to the resistance proportional to velocity, proportional to velocity. So, first of all motion under gravity we know how to write d square r by d t square is equals to minus of m g k. Now, the thing is that we have a resistance. So, that means when resistance against velocity. So that means, the particle cannot move freely. It got some kind of resistance. So, that a distance is actually a force which we have to subtract. So, it is a quantity or it is a term that we have to subtract from this from the right hand side because now, you have got resistance. And since it is proportional to velocity, we can write it as m times mu times d r d t.

So, that is the velocity m is the mass and mu is some constant of proportionality. So, mu is the constant of proportionality for proportionality alright. So, if you cancel m from both sides, then this is basically d square r by d t square plus mu d r d t plus g k. So, this is your required vector equation or ordinary vector differential equations I would say. Although there is no such thing, you can call it simply a differential equation or let us say ordinary differential equation in terms of vectors and from here you have to solve this vector r.

So, if we saw if we how to say write these equations as x, y, z and then if we try to solve it, then you basically obtain r is equals to we will obtain r is equals to ultimately. So, here I have small answers. So, here we will obtain r is equals to, but I would ask you to derive this e to the power minus mu 2, u is the initial velocity divided by mu e to the power mu t minus g k times t times e to the power mu t divided by mu minus.

So, this is basically t e to the power mu t divided by mu minus e to the power mu t divided by mu square and plus c. So, c is a vector which we have to determine. So, this is the required equation of the part where we have to determine this constant c. And this is another example where we have used the vector calculus to derive the equation of the curve along which the particle is moving.

So, bit like this, you can have several examples from vector calculus where from mechanics where you will have the application of vector calculus. So, for example, here I have the inverse law of attraction, then you can also write equations such as a planetary motions, speed of a particle on any orbit, things like that. So, it is not just limited to these two or three examples. There are like I mean thousands of examples where you can apply the concepts of vector calculus all with the help of vector calculus, you can be able to write their equation of motion and just try to solve to know the curve along which the particle is moving, use some initial conditions things like that.

So, will probably stop this chapter here because I think I have covered enough examples and give you gave you how to say enough idea where we use the vector calculus in applied mechanics. So, we will move on to our next topic which is a vector integration like line integral, surface integral and volume integral because we also have only 6 or 8 lectures left. So, I will try to include some examples in your assignment sheet. And you can solve them and you are also, I advise you to look into some vector calculus books where they are addressing some problems from mechanics and practice them and I am pretty sure you will be able to have a same learn some new things from there and it would not be any problem for you.

So, thank you for attention for today and I look forward to you in your next class.