**Integral and Vector Calculus**
**Prof. Hari Shankar Mahato**
**Department of Mathematics**
**Indian Institute of Technology, Kharagpur**

**Lecture - 51**
**Angular Momentum, Newton's Law**

Hello students. So, upon the last class, we looked into the application of vector calculus in mechanics basically, so using the vector calculus, you can be able to express things like velocity acceleration in some vector notation. So, today, we will continue with those things and we will see there are several other concepts like momentum, angular momentum, areal velocity or some equations of motion of some particle under some kind of law basically can also be expressed using the concepts of Vector Calculus.

So, today, we will start with areal velocity and then we will move to momentum and then, we will try to write the equation of motion of some particles. So, will see how we can do that. So, to begin with, let me give you a formal definition of the Areal velocity.

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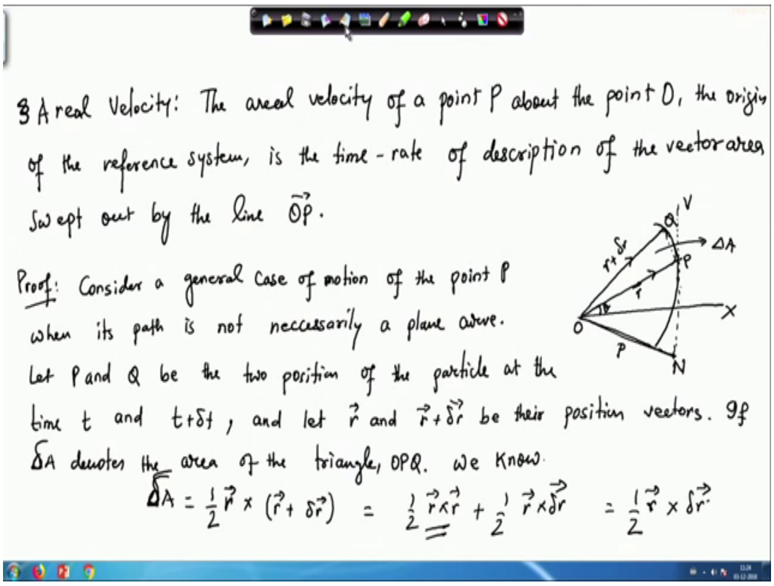


So, when I say areal velocity, it basically mean, so areal velocity. So, the areal velocity basically the areal, sorry I need to correct the definitions spelling basically, areal yes, a r e a l velocity.

So, the areal velocity of a point P about the point O, the origin of the reference system, of the reference system is the time rate of change, is the time rate of description or change basically, description of the vector area swept by the line O P. So, what do we mean by it? So, we basically mean that suppose we have a line, this is our reference line O X and this is our given curve. And I said this basically will be our point P. So, this is our vector r and this is our point let us say Q and this is r plus delta r and this is our angle theta and that is basically theta plus delta theta. And suppose here, we have a tangent. So, here we have a tangent. So, this is actually N and V.

So, what it means is that the areal velocity of a point P, so this is our point P about the point O, which is basically the origin for the reference system is the time rate of description of the vector area. So, this is basically the vector area in this direction. So, this is our r this is r plus delta r. So, this is the vector area that is being, this is vector area that is being swept out by the vector O P and that is actually the description, the time rate description of the vector area is actually called as the areal velocity.

So, how do we write in terms of the vector equation? So, we start with, let us say proof basically, so a small proof. So, we consider a general. So, consider a general case of motion. So, basically this particle P is moving along this curve; however, we assume that j consider a general case of motion of the point P, of the point P when its path is not necessarily a plane curve, alright.

And now, let P and Q be the two positions of the particle, of the particle at the time t and t plus delta t, at the time t and t plus delta t and let r and delta r plus delta r be their position vectors, be their position vectors alright. So, then if this area is delta a, so if this area, so if this whole area is delta n. So, if I call this area as delta m, so let me write if delta A denotes the area of the triangle. So, since delta r, since delta r is a very small arbitrary increment. So, then in that case this O P Q is actually forming a triangle. So, it is not actually a curve this P Q. It is basically considered as a line because this increment is considered to be very small in a in a very small increment of time. So, delta t delta r, they are considered in very small quantity. So, basically delta A is the area of the triangle O P Q, right.

So, we know that the area of a triangle is given by 1 by 2 times, 1 by 2 r which is basically this vector times our cross product with r plus delta r right and this is basically

1 by 2 r cross r plus 1 by 2 r cross delta r. So, since r cross r is 0 vector, so this is 0 vector plus this quantity ultimately we have is 1 by 2 r cross product with delta r alright. And now if I define this, so the rate of change of, so of the definition says the time rate of description of. So, the time rate of this change of this area of this description of the vector area will be basically we have to take d a d t, delta a delta t.

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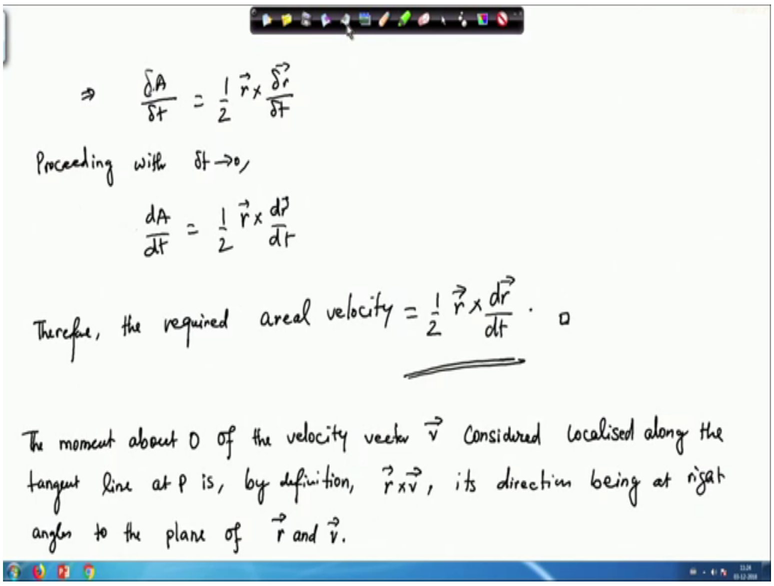


Because that is what we are denoting the rate of change of the areas by the vector O P. And just for the notation point of view, I think it is a good idea to take it as small delta, just for the sake of notation.

So, small delta s, small delta, so we are basically in delta how to say notation, so delta, delta, this delta, so I will also take it as small delta, alright. So, it is this basically notation but it makes the whole thing look how to say nicer in a way. So, delta A by delta t is equals to, since delta t is scalar we can put this delta t here, so delta r by delta t. Now, if I take delta delta t goes to 0. So, proceeding with delta t going to 0, we will have limit this thing will actually reduce to, we do not have to write the limit, we can write simply as d A d t is equals to 1 by 2 r cross product with d r d t. So, that means, the required therefore, the required areal velocity; therefore, the required the required areal velocity which is basically the time rate this d t is signifying the time rate of description of the area swept by the vector O P right.

So, this is basically the time rate of description. So, the areal velocity is basically 1 by 2 r cross product with d r d t and this is the required areal velocity for this for this point P with whose position vector is given by vector r alright. And similarly, we can calculate, so here in my lecture note I have also calculated the moment basically. So, how do we calculate the moment ah? So, this proof is complete up to here. Let me also give you the moment. So, the moment you may have studied these things in your physics course. So, the moment about O of the velocity vector say v.

So, this is not the areal velocity, this is the usual velocity vector v considered localized along the tangent line, along the tangent line at P is by definition. So, the moment is basically a moment O, the moment about O, so the moment of the particle about O of the velocity vector basically. So, the moment of the velocity vector considered as a, considered localized along the tangent vector and it is given by. So, by definition it is given by r cross v right. And its direction, being at right angles or perpendicular basically right angles to the plane of r and v, alright. So, it is actually at the right angle to the plane of r and v. And if we want and if we consider this as the, so this is our N. So, if we consider this, so this as the perpendicular whose length is small p.

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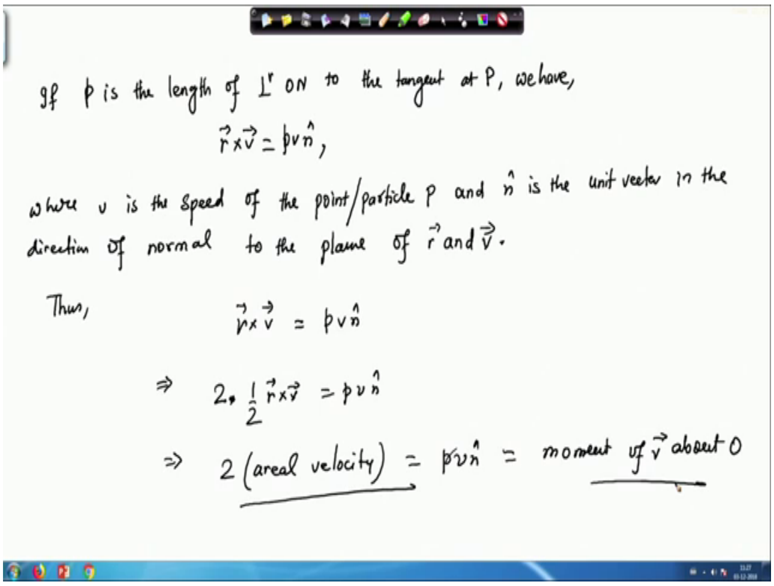


So, if p is the length of the perpendicular O N, so I can write if small p is the length of the perpendicular, length of the perpendicular O N to the tangent at P. Then, in that case, since our cross b is perpendicular to the plane of r and v, I can be able to write r cross v is parallel to that normal. And that is basically p times, small p times small v. Small v is basically the magnitude of this velocity times n cap because if r cross v is perpendicular to the plane of r and v, then it must be parallel to the normal you know way and these are basically the scalars that signifies that it is basically a normal, alright.

So, here we can, so, here basically where v small v is the speed of the particle or of the point P, the point or particle whatever you prefer from the particle P and n is the unit vector in the direction of normal to the plane of r and v vector v. So that means, basically so; that means, we can write thus, I can write r cross v r cross v is equals to is equals to p v n.

So, that means, from here I can write two times, one two times 1 by 2 r cross v p v n. So, this v is scalar and this is nothing but two times areal velocity which is equals to P times v times n. So, this v is speed actually. And this shows that, this shows that this is nothing but moment of v about O. So, this is actually our moment of v about O given by this formula and that is actually twice the areal velocity.

So, this is one of the important formulas of in this motion of motion of a particle on a curve. And here we can see that with the help of vector calculus we can be able to show that the areal velocity is half the moment of the velocity of the particle at the point velocity of the particle P about the point O, about the origin O. So, this is another important result that we can express using the vector calculus, it would also help if you have some knowledge of physics from your previous plus two level just to make these things a little bit more clear alright.

Next is momentum and linear momentum and then we will also write a Newton’s second law of motion using vector calculus so or vector notation basically.

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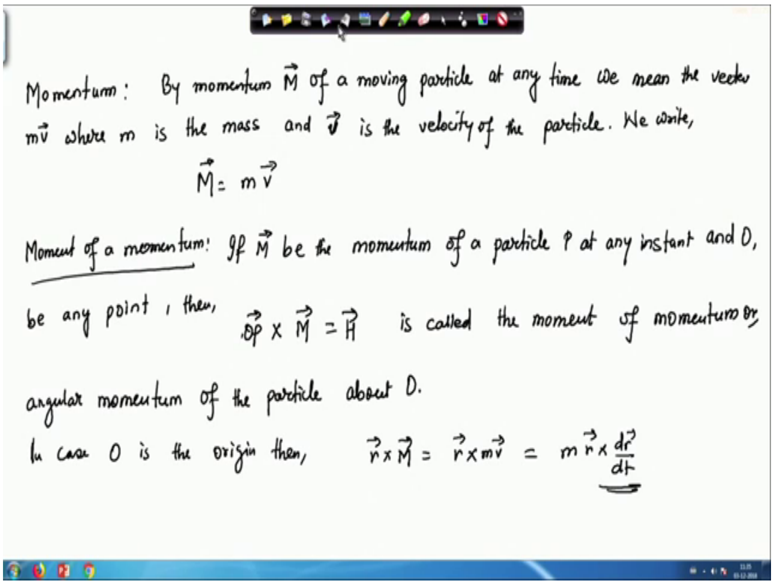


So, what do we mean by momentum? So, the definition you must remember by momentum M; by momentum capital M which is a vector of a moving particle of a moving particle at any time, at any time, we mean that we mean the vector m times v where m is the mass and v is the velocity of the particle right. And in notation or we write simply just write, we write capital M is equals to small m times capital v. So, small and basically is the mass of the particle and we mean the vector v where m is the mass of the particle and v is the velocity of the particle.

So, for example, if you have let us say an iron ball and if you just touch it on your hand and it will not hurt you that much. So, suppose the weight of the iron ball is like 500 grams. So, if you just touch it with your hand, then it will not hurt your hand, but if you throw it that ball on your hand and there is a strong possibility that it will hurt your hand. The thing is that since it got some velocity, the momentum is higher. So, when it is hitting, that is when you are feeling the glow actually.

So, even though the mass is very huge if the velocity is not higher than in that case, the momentum will always be small. So, that is why when something hit is us or when someone gets hit by something then in that case that at that point momentum is high basically because the velocity was high. So, momentum is directly proportional to the velocity or it depends on the velocity basically. So, higher the velocity is, higher the momentum would be and m is the mass of course. So, of course, it also depends on mass.

So, if your mass is as big and if the velocity is higher, then the blow will be higher. And if the mass is small and velocity is higher, then at that time also the low will be higher, but it will not be as high as when you had a bigger mass. So, mass also plays an important role, but in case of momentum, I would say that the velocity plays an important roles because velocity can increase and decrease and based on that for a given mass, the momentum can also increase or decrease alright.

Now, next we define the moment of a momentum, the moment of a momentum. You probably know all these things, it is it might be a recapitulation from your physics class, but I am just writing the advantages of vector calculus here. So, if M be the momentum, if M be the momentum of a particle P momentum of a particle, let us say P at any instant and O be any point, then O P, so then the vector O P cross M. So, this is not necessarily origin.

So, that this can be any arbitrary point, so I could take q which or let us let us keep it as O. So, O P cross M is equals to vector H because this is a vector this is the vector. So, the cross product will definitely be a vector is called the moment of momentum or sometimes they are also called as angular momentum of the particle about, alright. So, this is called the angular momentum of the particle about O.

Now, and so here basically so here O P is the position vector of the point P about O. So, with respect to O, O P is the position vector and in case and in case O is the origin in case, O is the origin, then this can be written as vector r times capital M which is basically cap vector r cross product with m times v. So, this can be written as m times r cross d r d t. So, v can be written as d r d t and this is basically our angular momentum. So, M the momentum time cross product with d r d t or m times r cross d r d t is basically your angular momentum. So, if I go back to this formula where I have the areal velocity, so basically areal velocity and angular velocity. So, I can take two times areal velocity. So, this is basically two times m times areal velocity. So, that is basically our angular momentum alright.

And, so, the thing and another thing is angular momentum will be different for every point that we choose this O. If we choose O as a different vector point, then in that case this O P vector will change and then of course, our angular momentum will change. So, these are the small here and there details which you have to be careful about. So, here if we choose instead of one a different point, then in that case this angular momentum will change. So, yeah this is our moment of momentum. The next topic is Newton's second law; Newton's second law of motion.

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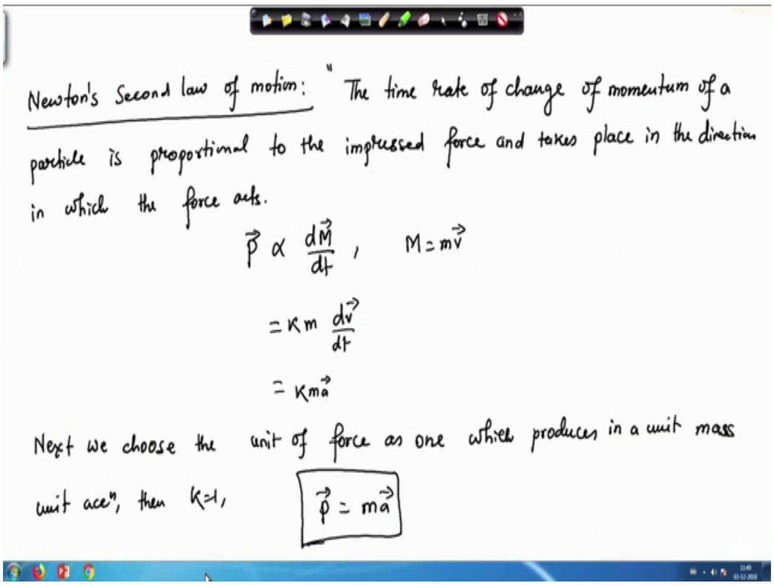


So, you probably heard about there are three laws of motion and when you say three laws of motion it obviously, means that we are talking about Newton's three laws of motion because everything around us they obey Newton's laws of motion. So, the first one we all we know that it is called law of inertia. So, if a body is in motion, then it will remain in the motion. If it is in rest, then it will remain in the rest unless it is being how to say compelled by some external force to change it is state. And the second law is that force is perpendicular to, proportional to acceleration basically and and the third law is that to every action there is an equal and opposite reaction.

So, these are the three basic laws and which has a huge, I mean which has huge, huge, huge application everywhere around you. So, anything that is moving it has in one way or another some application of our direct application of Newton's law or laws of motion. So, basically all three of them are important, but today we will just derive or learn the second law of motion. So, the second law of motion says that the time rate. So, the statement is the time rate of change of momentum of a particle is proportional to the impressed force and takes place in the direction in which the force acts, alright. So, if P is the impressed force or if P is the applied force, then in that case let us call P because if you are applying a force, then it has a direction. Because if you apply a force and expects the chains to happen in this direction, that will not happen. So, that, when you apply the force in a certain direction, the change would also happen in that direction. So, the force is a vector quantity. So, it is always having a direction.

So, that is what P is the force which is chosen as a vector and it is directly proportional to the momentum. Momentum is a vector quantity because it involves acceleration sorry velocity and m is basically the velocity capital M v t. So, I can I can change this proportionality to a constant K times m d v d t. So, rate of change of velocity and I can write this as K m where K is the constant is a scalar constant given constant of proportionality m is the mass and a is the acceleration.

So, next if we choose if we choose the unit of force, unit of force has one which produces in a unit mass which produces in a unit mass, unit acceleration, unit acceleration, then our K will be constant. Then, K will be 1. So, of course, it is constant then K will be 1 and hence we will have P is equals to m times a. So, this is basically our Newton's second law of motion. So, that means, force is equals to mass into acceleration and this means that acceleration produced in a motion of a particle of a constant mass has the same direction as the force producing it.

So, it will happen, the change will happen in the direction of the force and this equation is called as the equation of motion of a particle. So, this today, so this is an another application of vector calculus or we saw that how with the help of vector calculus we can be able to write Newton's second law of motion. So, these are some of the applications of vector calculus in the applied mathematics or in mechanics. In the next class, we will continue with that and if time permits, then will move on to our next topic.

So, I thank you for attention today and I will see you in the next class.