**Integral and Vector Calculus**
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**Lecture – 50**
**Application to Mechanics, Velocity Speed, Acceleration**

Hello students. So, in the last class we were deriving the equations of oscillating planes, rectifying planes and normal plane and then I at the end I started to show you a small example where how you can calculate the length of a curve, but due to time constraint we could not continue. So, today I will continue with the same example and then we move on to our next example or the next topic we will see.

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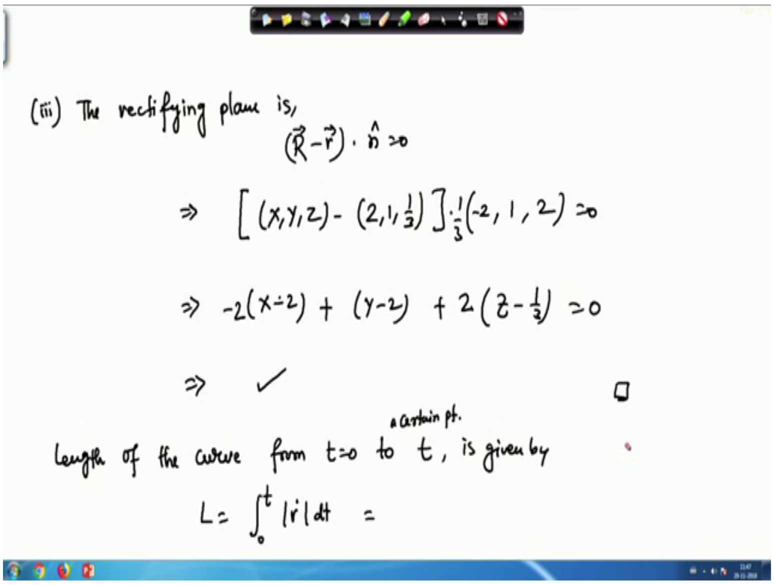


So, in the last class after deriving the equation of the rectifying plane I moved on to show you how you can calculate the length of a curve from a certain point for a from a point t goes to 0 to a certain point let us say t equals to some t 1. So, this is basically given by integral from 0 to t r dot dt. So, r dot is basically your, if you remember from integral calculus. So, if you remember from integral calculus the length of a curve from a point t goes to a 2 t goes to b is basically L equals to sorry is basically L equals to let me go to a new a new page.

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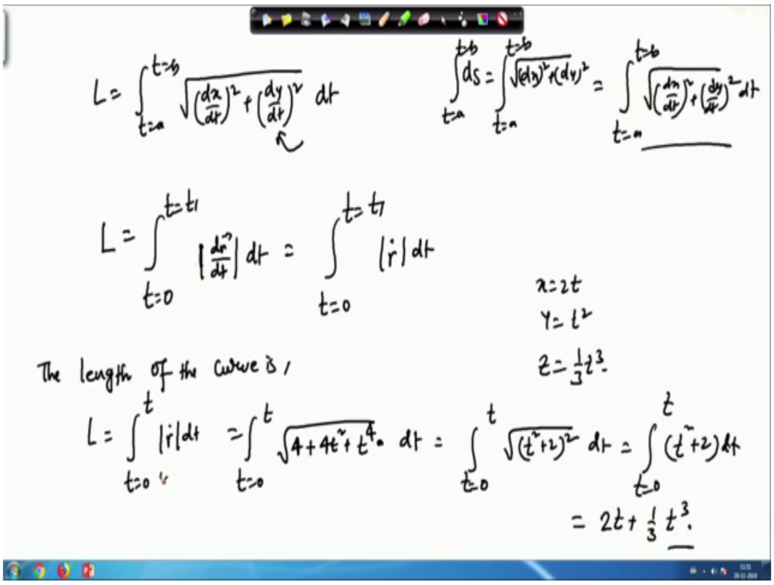


So, L equals to integral from t equals to a to t equals to b from integral calculus it was d x dt whole square plus d y dt whole square dt. And how this is coming here is mainly because we know that ds equals to d x square plus d y square.

So, if you integrate both sides with respect to; with respect to s actually. So, this will actually give you say let us say t equals to a to t equals to b you are integrating. So, here we substitute t equals to a to t equals to b and if you integrate then this will be s at b s at a. So, that is basically the length of the curve measured from t equals to a to t equals to b and on the right hand side we can be able to write t equals to a to t equals to b d x dt whole square plus d y dt whole square times dt. So, that is all so we are just at and dt.

So, now in case of vector calculus our L is let us say t equals to we can start from 0 and let us say we are at t equals to t 1 and r dot is basically d x dt whole square plus d by dt whole square. So, this thing can be written as; so, this thing can be written as instead of writing it again this thing can be written as mod of dr dt because r is a vector times dt or you can write it as a small notation simply r dot dt. So, these two formulas are same because your r dot is nothing, but the mod of r dot is nothing, but this thing.

So, in the example from the previous class where we had x equals to 2 t y equals to t square and z equals to 1 by 3 t cube if we wanted to if we want to calculate let us say length of the curve from a point t is equal to 0 to a point t then that case our L is basically t is equal to 0. So, the length of the curve, length of the curve as at any point t is r dot dt and our r dot is from previous class we can substitute the value of r dot.

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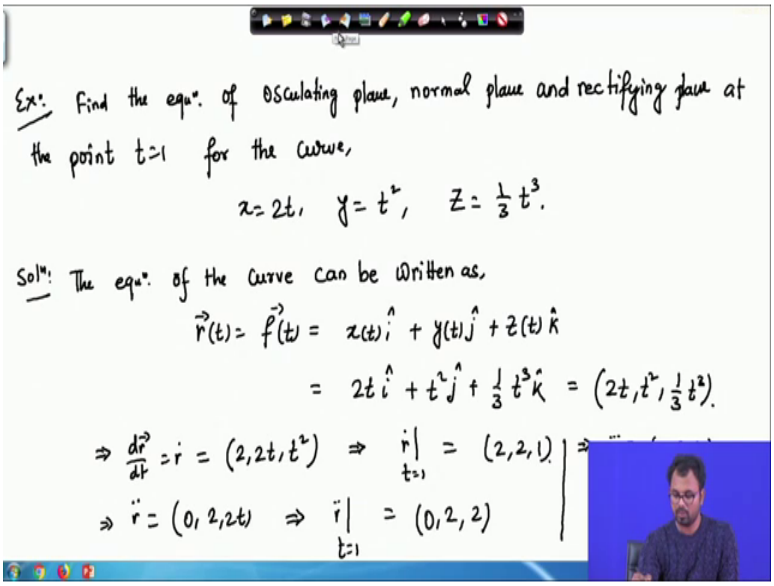


So, r dot is basically; r dot is basically 2, 2 t t t square. So, I can substitute the value 4, 4 t square and this one is t to the power 4 and dt. So, this can be written as t square plus 2 whole square right yes and this is basically if we integrate then this is basically.

So, first of all we have to take it out of the square root. So, this is t square plus 2 dt and if we integrate then this will be 2 t plus 1 by 3 t cube right. So, this is the required answer. So, just substitute instead of t if we substitute let us say t equals to 1 then this will be 2 plus 1 by 3. So, 7 by 3 would be the required length and of course, you can substitute the unit over there. So, to calculate the length of a curve all you have to do is calculate r dot and that r dot will give you the required length of the curve if we use this formula alright; if we use this formula.

So, next I will give you one example to practice because it follows the same logic and same method or method. So, its kind of tedious to do the same example again and again, but you can practice at your own these examples. So, I am going to write one example for you to practice.

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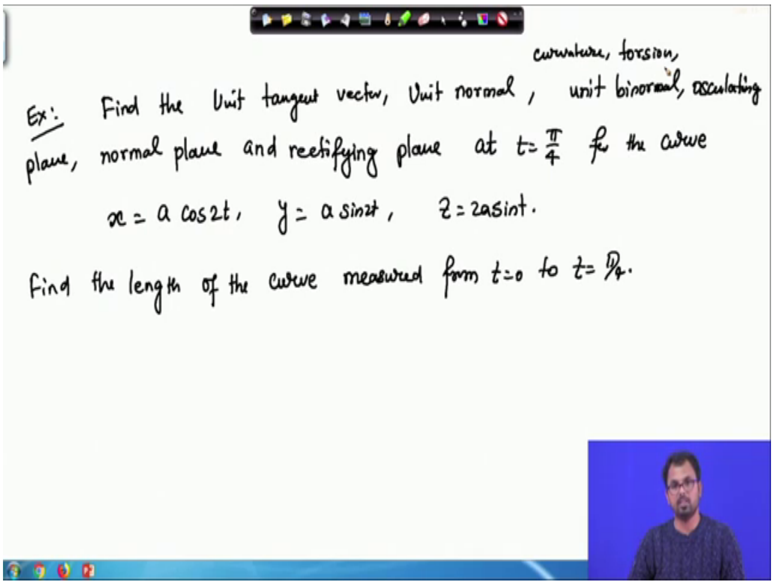


So, example, find the unit tangent vector unit tangent vector, unit principal normal or unit normal and unit by normal unit will not add let us say I am including everything. So, unit binormal, osculating plane, normal plain and rectifying plane at t equals to pi by 4 for the curve x equals to a cos 2 t and y equals to a sin 2 t and z equals to 2 a sin t alright.

And so and you can also try to calculate find the find or find also the length of the curve measured from t equals to 0 to t equals to pi by 4. So, this is a very nice example to practice all the things that we have studied before. So, first of all you can be able to write this x y z as a equation of a curve as r equals to f t, a vector equation and then from there you can calculate dr dt d square divided by dt square dr a triple dot and r triple dot after t equals to pi by 4. We probably do not need r triple dot, but it is a good to calculate up to r triple dot for any given equation because you might need at some point or may not need it and it is just that it is good to have it on your paper.

So, now once we have those equations we can be able to. So, I can put a necessity to calculate those things. So, I can also write curvature and torsion. So, now, you need those derivatives. So, calculate unit tangent vector unit normal unit by normal curvature torsion osculating plane normal plane rectifying plane at pi t equals to pi by 4. So, now, you need all those derivatives and once we once you have those derivatives you can be able to calculate curvature and torsion by using that r dot cross product with r triple dot and all that.

So, use that formula to calculate the curvature and the torsion and then from dr dt we can be able to calculate our unit tangent vector by dividing it with its magnitude r dot from there we can be able to calculate binormal, normal using that n equals to b cross t or whatever it is. And from there we will be able to calculate our osculating plane, rectifying plane and normal plane the way I just showed you and in the previous class and finally, we can be able to calculate the length of the curve from t is equals to 0 to t equals to pi by 4 by integrating 0 to pi by 4 mod of r dot dt. So, whatever r dot we will get we will take its magnitude and just integrated with respect to t and then substitute the value of t equals to 0 and t equals to pi by 4.

So, that is how you will be able to calculate all these terms here and it is a nice example to practice for you to practice so keep doing that. And now, we will move on to our next aspect of vector calculus which is basically application of vector calculus in mechanics. So, that was the one of the topics that I suggested in the syllabus. So, now, we will start with that.

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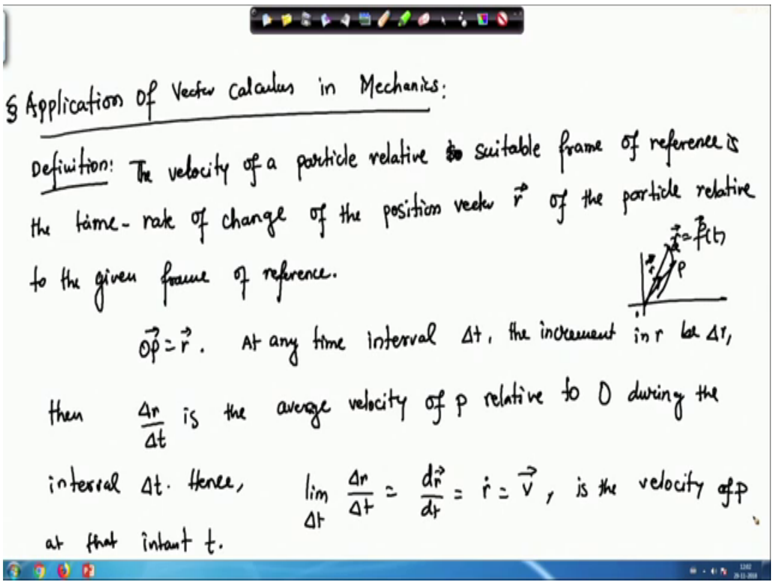


So, our next topic is application. So, why do we have to study the application of vector calculus in mechanics? The thing is whenever a particle moves it has a speed, but more precisely it has a velocity because whenever it is moving and not only that it has a speed it also has a direction that in which direction it is moving.

So, when once we talk about the direction that is when the vector quantity comes in. So, a vector quantity is something that has magnitude as well as the direction, but a scalar quantity has only the magnitude. So, when we talk about a motion of a particular or motion of a body we always talk about its direction and when we talk about its direction that is when the concept of vector vectors come comes in. Now, that direction, so, a particle moving has a speed and once we associate that speed with a direction then that is when we get a velocity. So, velocity is something which has a direction as well as a magnitude. So, you see how motion or a basically mechanics is connected to vectors.

So, we will start with those topics now. So, first of all what do we mean by velocity of a particle? So, definition and how do we express it in terms of vectors alright. So, the velocity the velocity of a particle relative to a suitable frame of reference is the time rate of change is the time, rate of change of the position vector r of the particle relative to the given frame of reference.

So, that means, the velocity of a particle when it is when a particle is moving is basically the time rate of change of the position vector so; that means, if we talk about position vector we are talking about the actually the direction in a way and how the direction of the particle is changing over time. So, that time rate of change of the position vector is actually called as the velocity alright.

So, if we consider let us say if we have a curve this r equals to f t and suppose this is my equation and that is origin that is the point P. So, this is our position vector r and if it is moving along this curve with respect to t or with respect to time then that is actually the velocity of that particle. So, we have the vector equation and that can be expressed as. So, with the difference with the difference to a frame if a particle P if a particle P has at any instance the position vector r as OP equals to r and if during the time interval delta t the increment is delta r.

So, let us say the increment here is r plus delta r for any time t any time delta t. So, this position vector is let us say q is r plus delta r. So, at any time; so, at any time interval delta t the increment in r b delta r then delta r by delta t is the average velocity; average velocity of P relative to O during the interval delta t. And hence limit delta t delta r by delta t is equals to dr dt is equals to r dot is called as the velocity, is the velocity of the particle P or of the point P at that instant it is also called as instantaneous velocity of P.

So, basically if we have a small increment for any time interval delta t we have a small increment delta r then basically delta r by delta t is the average velocity. So, how much it has increased is basically it is increased by delta r and delta r by delta t is actually the average velocity of P relative to this origin O during the interval delta t. And therefore, we can write in terms of limit and this dr dt or r dot is actually our velocity, at that point P at an instant t. So, basically if we have a given, if we have a curve let us say r is equals to f t.

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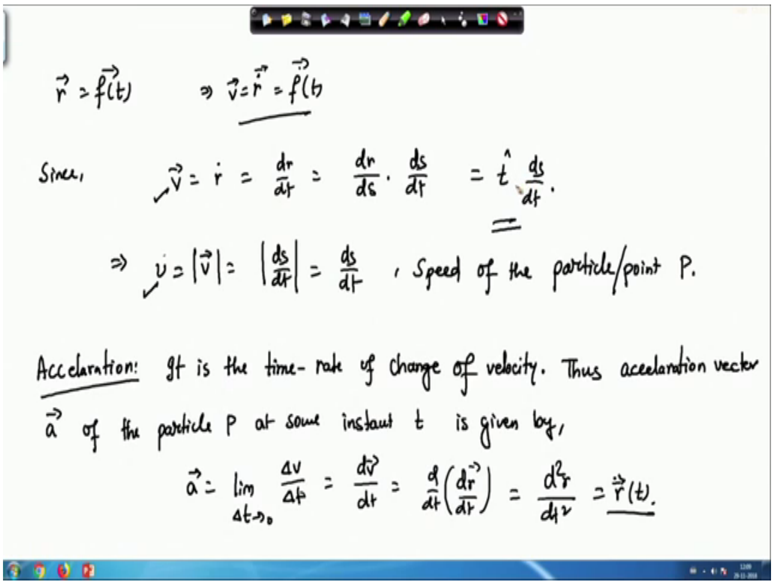


So, if we have a curve let us say r is equals to f t then of course, P is called as the trajectory. So, as the direction or the curve along which P is moving is called as the, it is called as the trajectory the locus of P is basically called as the trajectory. So, the curve or the path it follows so that is basically the trajectory of t and if we have a given curve r is equals to f t as the equation of the curve. Then basically we are doing is what we are doing is we are writing velocity dr dt is equals to f dot t f dot d sorry and that is how we calculate the velocity of the given point P on a curve r is equals to f t alright.

Now, since we can write; since we can write v is equals to r dot which is basically dr dt. So, I can write dr ds times ds dt. So, s is the arc length and rate of change of arc length is not relevant to the direction. So, in which direction the arc length is changing is not relevant. So, basically the rate of change of arc length is given by this ds dt and dr ds is basically our unit tangent vector. So, you see if I take mod of v then it is basically mod of ds dt. So, ds dt is rate of change of arc length.

So, it will always be positive. So, we have ds dt and this positive thing can be denoted by v and this v is actually the speed. So, as we were saying speed is a scalar quantity. So, mod of v or the magnitude of v will actually give us the speed of the particle P along that curve or the speed of the point P along that curve and that can also be given by ds d t.

So, differentiation of the arc length with respect to t will give you the speed that how fast that point P is moving along the arc length of the curve. So, speed of the particle or speed of the point particle or point both are same in this context at the point P alright. So, this is basically our velocity. So, this is our speed and now we can define our acceleration. So, the acceleration so after speed we have acceleration.

So, acceleration is it is defined as it is time rate of change of velocity. So, it is basically time rate of change of velocity and this acceleration vector this. So, first of all rate of change of r is called as velocity and now rate of change of velocity is called as acceleration. So, if it is increasing if the rate of change of velocity is increasing then in that case it will be acceleration, if it is decreasing then in that case it will be deceleration.

So, although we use acceleration here in this context, but those are the two how to say adjectives you use that is acceleration and deceleration. So, this acceleration vector let us say a of the particle P at some instant t is given by. So, basically we have a as limit delta t. So, time rate of change of velocity. So, limit delta t delta v by delta t.

So, at any time interval delta t with an increment of time interval delta t the increase or decrease in velocity is delta v. So, basically the delta v by delta t will give you the acceleration and making limit delta t goes to 0 this will go to dv dt and we know that our v is dr dt. So, this will be actually d square r by dt square or simply r double dot. So, r double dot is actually our acceleration and it is its also a vector quantity because r is a vector quantity alright.

So, we have delta v sorry a is equals to the r double dot. So and this actually defines our acceleration. So, if we are asked to calculate the acceleration of a moving particle along a curve r is equals to f t all you have to do just differentiate it twice with respect to the parameter whether it is t or theta whatever it is and that will actually give you the acceleration. And if you are given a point let us say t equals to 1 then it will give you the acceleration at that point t equals to 1. Now, in this equation v is equals to in this equation v is equals to let us say t cap times small v.

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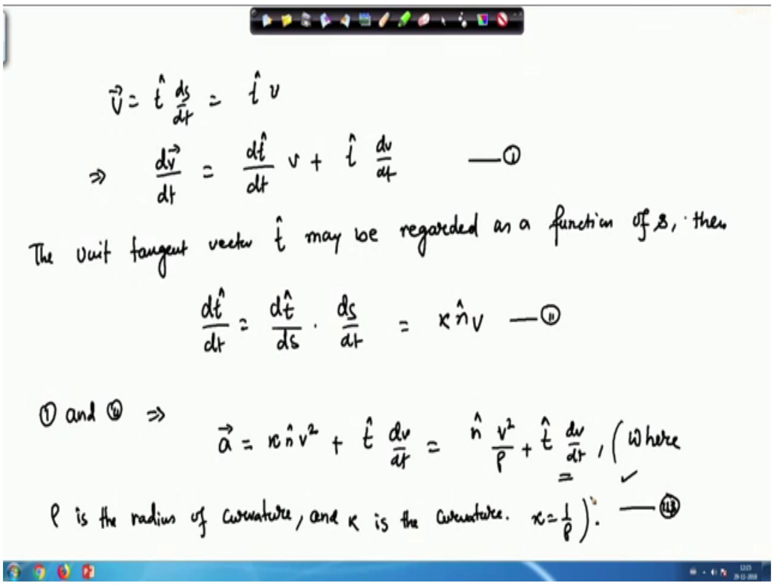

So, we have v is equals to t cap times ds dt. So, I can be able to write it as t cap small v. So, we in this case this v is scalar this v is vector. So, if I differentiate both sides then this will be dv dt is equals to dt dt times v and then t cap times dv dt, so, this v is scalar.

So, now the unit tangent vector. So, and the unit tangent vector the unit tangent vector t may be regarded as a function, regarded as a function of the arc length s; of the arc length s then our dt dt is basically dt ds times ds dt right. So, and dt ds is basically our Kappa n from that Serret Frenets formula and ds dt is our velocity v. So, this is Kappa n and if I substitute all these values here.

So, this will reduce to so equation let us say 1 from 1 and 2, we will have 1 and 2 we will have dv dt which is basically our acceleration is equals to dt dt; dt dt is actually Kappa n times v square Kappa n v square this v scalar these are vectors plus t times dv dt. So, dv dt and dv dt is dv dt and I can be able to write Kappa n as 1 by rho.

So, this can be written as n cap and v square by rho plus t times dv dt where rho is the radius of curvature radius of curvature and kappa is just a curvature and kappa is the curvature. So, I have used kappa is equals to 1 by rho. So, this is what I have used. So, kappa is equals to 1 by rho is taken here. So, this is basically v square by rho so; v is our sorry a speed by radius of curvature plus t cap times dv dt and this is our unit normal and.

So, what does it say is that acceleration vector so what does it say is that acceleration vector. So, you can see acceleration vector can be written as the combination of n cap and t cap. So, n cap and t cap means, if we go to our previous definition.

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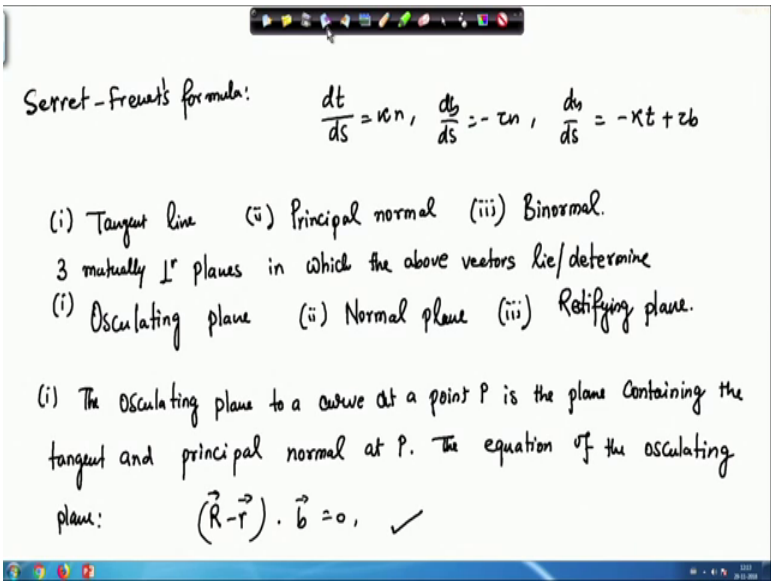


So, basically the acceleration vector is given as the linear combination of the normal and the tangent vector; that means, the acceleration vector is perpendicular to b, because if we take here. So, if we take; if we take dot product with b then in that case n dot b will be 0 and t dot b will be 0 and on the left hand side we will have a dot b equals to 0.

So, what does that mean? That mean that the; that means, that this vector a or the acceleration vector lies in the osculating plane and that is by taking the dot product with be equals to 0. So, this relation I can name it as let us say 3 equation a relation number 3. So, sorry relation number 3.

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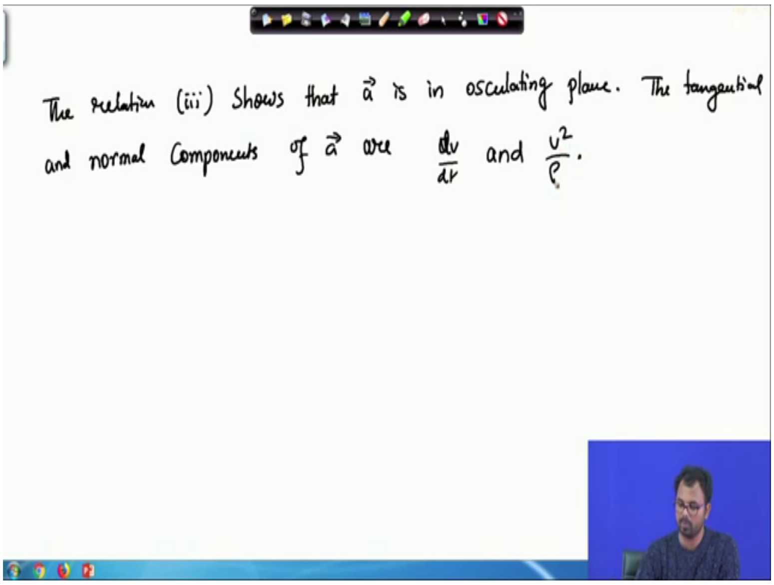


So, the relation 3 the relation three shows that vector a is in osculating plane because it is given as the combination of with it is given as the combination of n and t. So, if we take the dot product then a dot b will be 0 lies in the or is in the osculating plane.

And the tangential and the tangential and normal components of acceleration components of acceleration are v tangential component is dv dt where v is a scalar and normal component is v square by rho where v is the speed and rho is the radius of curvature.

So, from this relation we can see clearly that the acceleration vector actually lies in the lies in the plane of t and n and it is perpendicular to the vector by normal b and so; that means, the vector a lies in the osculating plane. And the component for the tangential and the normal components for a are given as dv dt where v is the speed and v square by v square divided by rho with rho is the radius of curvature.

So, this was this is an another a nice formula to remember that the acceleration vector lies in the osculating plane. So, we will stop here today and in the next class we will derive several other how to say relations keeping vector calculus in mind and mechanics in mind and we will see how vector calculus is widely used in mechanics and we will also try to work out few examples if time permits. So, we will stop here for today and I will look forward to you in your next class.

Thank you.