Integral and Vector Calculus
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Lecture – 05
Examples

Hello students, so welcome to this class. So, up until last lecture, we looked into Riemann Integrable functions and their definition and several properties associated with a Riemann integrable function. For example, if your function is continuous, then it is Riemann integrable; if it is monotonic, whether it is monotonic increasing or decreasing, then in that case also it was Riemann integrable on the given interval.

We also looked into the definition of first of the fundamental theorem of integral calculus, we also looked into the definition or the statement of first mean value theorem and the second mean value theorem. The theory of Riemann integrable function is anyways too extensive, but I try to compressed it in a you know how to say shorter format.

Now, based on the theories which are the theorems which we learned in previous classes, today, we are going to work out few examples. And we will see that how we can be able to show that whether a given function is Riemann integrable or not, how we can calculate upper integral sum and lower integral sum of a given function.

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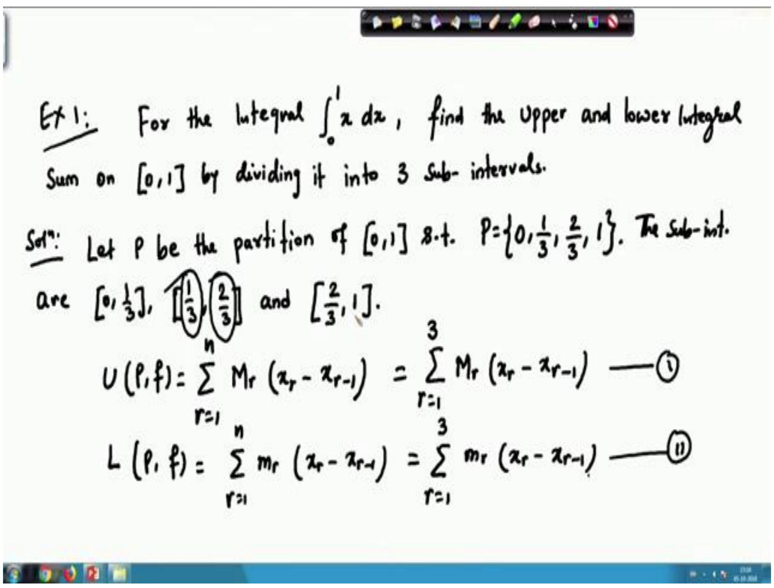

So, today's lecture will be all about the examples on Riemann integration. So, let me start all right. So, our 1st example is; for the integral $\int_0^1 xdx$, we will calculate so for the integral this, find the upper and lower integral sum, lower integral sum on [0,1] by dividing it into 3 sub intervals. So, that means, our partition will have four points and those four points, so we will constitute our how to say three subintervals.

Now, it is obvious that if you divide the interval [0 ,1] into 3 sub intervals, then your partition points will be 0, $\frac{1}{3}$, $\frac{2}{3}$ and $\frac{3}{3}$, which is 1. So, let P be the partition of [0,1], such that our partition P will be 0, $\frac{1}{3}$, $\frac{2}{3}$ and $\frac{3}{3}$, which is 1 such that and so this is basically our partition P. And the sub intervals will be $[0,\frac{1}{3}]$. So, the sub intervals are; $[0,\frac{1}{3}]$,$[\frac{1}{3},\frac{2}{3}]$ and $[\frac{2}{3}, 1]$.

Now, if we look into the definition of our upper integral sum and lower integral sum, which are basically U(P,f), and L(P,f). So, upper integral sum U(P,f) is given by a formula of this type. Since in our case n is the number of subintervals and in our case n is basically 3, because we have 3 sub intervals. So, $\sum_{r=1}^{3} M_r(x_r - x_{r-1})$ This is our upper integral sum.

And our lower integral sum L(P,f) will be $\sum_{r=1}^{n} m_r(x_r - x_{r-1})$, so in this case n is 3. So, we will substitute n=3 $m_r(x_r - x_{r-1})$. Note that M_r and m_r are the supremum and infimum or upper bound and lower bound of the function f, which is x here, in these individual sub intervals. So, let us name this equation as equation 1 and let us call this equation as equation 2.

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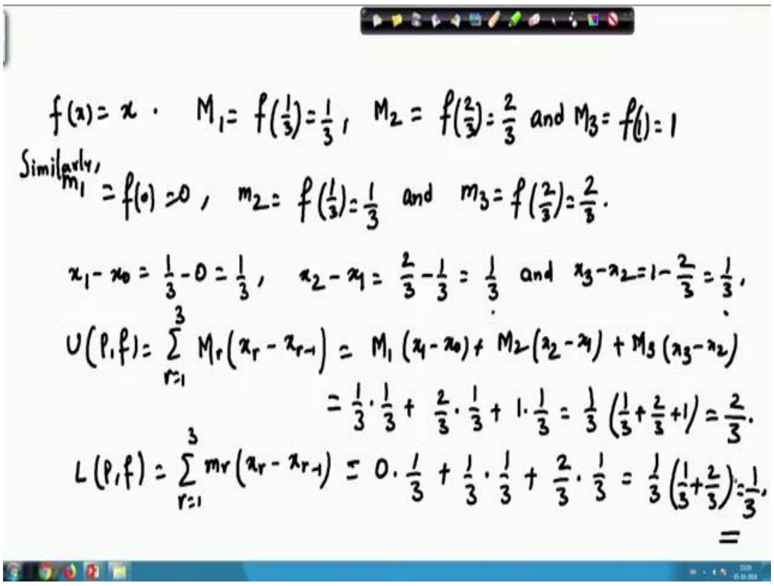

So, next we have to calculate the upper bound in each one of these sub intervals. So, we have $f(x) = x$ and from here; our capital M_1 is basically $f(\frac{1}{3})$, because $f(x) = x$ is a monotonically increasing function in the interval [0,1].

And therefore, the maximum value of this function $f(x)$ would be attend at the point $\frac{1}{3}$, because at all the other points at x=0, it the function will attain it is minimum value and so on. So, the maximum value of this function $f(x) = x$ will be attend at the end point $x = \frac{1}{3}$. So, what will be the maximum value or the upper bound of this function at $\frac{1}{3}$, it is $\frac{1}{3}$.

Next, we can calculate M_2. M_2 will be again attained at this end point $\frac{2}{3}$. So, it will be attained at $\frac{2}{3}$, because we have the function is a monotonically increasing and the points are increasing again, so in this case the maximum value or the upper bound will be attained at this end point. So, we can calculate the maximum value as $f(\frac{2}{3})$ and $f(\frac{2}{3})$ is again $\frac{2}{3}$. And M_3 is the maximum value of the function f, which is again attained that x=1. So, at $f(1)$, so this will be 1.

Similarly, we can calculate m_1. So, we can write similarly m_1 equals to the infimum of the value at x=0, because in this interval the infimum or the lower bound is attained at x=0. So, we can write f (0) ;f (0) is 0.

Next, we can calculate m_2 , which is the infimum or the lower bound. So, this is again attained at let us go back to this slide. So, the lower bound of this function f will be attained in this interval at the point $x = \frac{1}{3}$. So, we can calculate the lower bound at $x = \frac{1}{3}$, which is $\frac{1}{3}$. And that and m_3, so m_3 is the lower bound of this function f in the interval $[\frac{2}{3}, 1]$. So, this will be attained at $f = \frac{2}{3}$, which is basically $\frac{2}{3}$.

So, now that we have all these ingredients, which we have to calculate $x_1 - x_0$. So, x_1 is basically $\frac{1}{3} - 0$, which is $\frac{1}{3}$. Similarly, $x_2 - x_1$ will be $\frac{2}{3} - \frac{1}{3}$, so this is again $\frac{1}{3}$. And $x_3 - x_2$ will be $1 - \frac{2}{3}$, so this is basically $\frac{1}{3}$.

So, since we have divided the interval into three equal parts. These are basically the length of every sub interval and it is pretty much clear that if you have divided and interval into three equal parts, then in that case the length of each one of these sub intervals would be $\frac{1}{3}$. Because, your interval is [0,1], so that is what we are getting, $x_1 - x_0 = \frac{1}{3}$, $x_2 - x_1 = \frac{1}{3}$ and $x_3 - x_2 = \frac{1}{3}$.

So, now we are ready to calculate our upper integral sum. So, upper integral sum is $\sum_{r=1}^{3} M_r(x_r - x_{r-1})$. So, if we substitute the above values, then it will be M_1 . So, first of all we can expand this summation $M_1(x_1 - x_0) + M_2(x_2 - x_1) + M_3(x_3 - x_2)$ is it clear.

So, now M_1 is $\frac{1}{3}$ times $(x_1 - x_0)$ is also $\frac{1}{3}$, M_2 is $\frac{2}{3}$, $(x_2 - x_1)$is again $\frac{1}{3}$, M_3 is 1 and this 1 is again $\frac{1}{3}$. So, we take $\frac{1}{3}$ common, this one is $\frac{1}{3} + \frac{2}{3} + 1$, and this will be ultimately $\frac{2}{3}$.

And similarly, we can calculate L(P, f) which is our lower integral sum. So, this can be given as $\sum_{r=1}^{n} m_r(x_r - x_{r-1})$, we expand in the similar fashion $m_1(x_1 - x_0) + m_2(x_2 - x_1) + m_3(x_3 - x_2)$. So, our $m_1$is 0 and $(x_1 - x_0)$ is $\frac{1}{3}$+ m_2 is $\frac{1}{3}$ and $(x_2 - x_1)$is again $\frac{1}{3}$, then m_3 is $\frac{2}{3}$ times $\frac{1}{3}$. So, we can take $\frac{1}{3}$ common, this is $\frac{1}{3} + \frac{2}{3}$, so ultimately we will get $\frac{1}{3}$. So, this is our upper integral sum $L(P,f) = \frac{1}{3}$, and $U(P,f) = \frac{2}{3}$ are our respective lower integral sum and upper integral sum, which we wanted to calculate.

So, this is how we can calculate the upper integral sum and lower integral sum, it is we have seen several times that all we have to do is to find a partition of the given closed interval and you need to calculate the upper bound and lower bound of the function in those sub intervals and then put it in this formula. So, this is basically the formula. And that will give you the U (P, f) and L (P, f) of a given function. So, I hope you were able to understand this example and we will probably have such kind of examples in your exercise or in your assignments and hopefully that will make the concepts even more clear all right.

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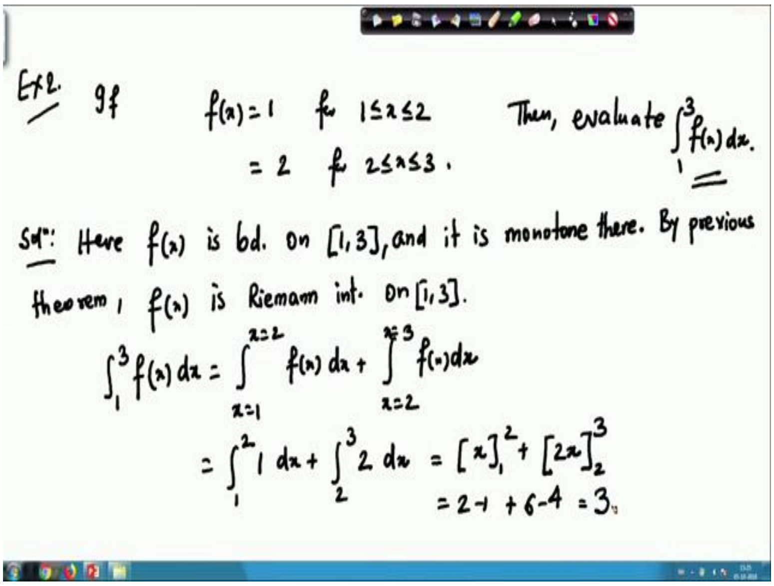

So, now that we have looked into L(P, f) and U(P, f) example. let us consider an example, where we actually say whether our given function is Riemann integrable or not. So, example 2; example 2 is if $f(x) = \begin{cases} 1 \; for \; 1 \leq x \leq 2 \\ 2 \; for \; 2 \leq x \leq 3 \end{cases}$, then evaluate $\int_1^3 f(x)dx$.

So, first of all the given function is f (x)=1 between [1,2] and f(x)=2 between [2,3]. So, obviously it is a discontinuous function at x =2. Now, having just one point of discontinuity that would not create any kind of problem, I mean you can still talk about the Riemann integrability. Remember all we have to have is whether the function is first of all the function is bounded or not. And second of all I mean if it is monotonic, then in that case also we can say that the function is Riemann integrable.

So, first of all here so, here our f (x) is bounded on [1,3] and it is monotone; and it is monotone there right, which is obvious. So, between [1,2] it is 1 and then between [2,3] it is 2. So, obviously, it is a monotone function with one point of discontinuity at the point x=2.

And therefore, the theorem or the properties which we studied earlier that if the function is monotonic, then in that case it is Riemann integrable. So, by that theorem you may have to look into the number. So, then by previous theorem, I would expect you to write that theorem number here, f(x) is Riemann integrable right on [1,3] just looked into that theorem, which say that a monotonic function is Riemann integrable.

So, with the help of that theorem, we can say that this function is at least Riemann integrable that means, this integral here it makes sense that if someone asks us to evaluate such kind of integral, first of all we have to make sure that whether the function is integral or not, whether the integral exists at all exists or not.

So, since by this theorem the integral exists. We can now calculate this integral. So, how do we calculate? So, to calculate we can write the integral $\int_1^3 f(x)dx$, now remember 2 is the point of discontinuity. So, we divide the integral into 2 sub integrals at the point x=2. So, $\int_1^2 f(x)dx + \int_2^3 f(x)dx$.

Now, between [1,2] f(x) is 1 and between [2,3] f(x) is 2. So, if we calculate, then this is basically $\int_1^2 dx + 2\int_2^3 dx$.. So, this will be (2-1) + (6-2), so ultimately it is 5 sorry this one is 4 so, it will be 4 , so 6-4 and therefore, this will be 3, so the answer is 3 yes.

So, you see first of all we have to check whether the given function is bounded and monotonic or not or whether it is continuous or not, so that we can at least talk about that the function is Riemann integrable or the integral at all exists. And once we have that then we can calculate the integral the way we please. So, this is how to say of one such example, where we use those theorems. Now, next we will move to let us go to new page.

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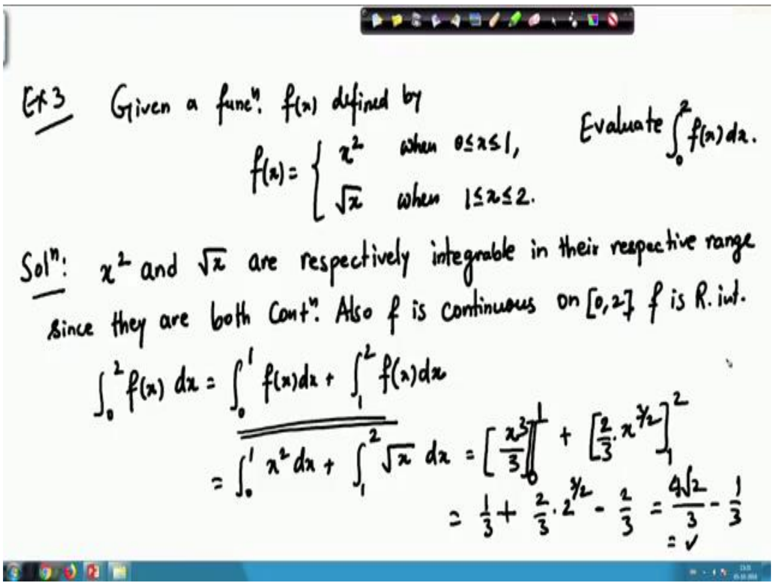

Now, here an example; so, another example is given a function f(x) defined by

$f(x) = \begin{cases} x^2 \; for \; 0 \leq x \leq 1 \\ \sqrt{x} \; for \; 1 \leq x \leq 2 \end{cases}$. So, then we have to find out what is so evaluate $\int_0^2 f(x)dx$?

So, here our given function f is x^2 between [0,1] and $\sqrt{x}$ between [1,2]. So, obviously this function is continuous from [0,2]. And therefore, we can use that continuity theorem that every continuous function is Riemann integrable. So, we can write x^2 and $\sqrt{x}$ are respectively integral or Riemann integrable in their respective range, since they are both continuous right, it is very obvious to see that so that means, the function f is continuous.

And if the function f is continuous, then this function is Riemann integrable. So, also f is continuous on [0,2]. So, at least the continuity part is settled. And if the function is continuous on that interval [0,2], then obviously it is integrable or Riemann integrable.

So, let us calculate the Riemann integral. So, since we have two different functions from [0,2], we can divide the range in 2 parts f(x)dx. So, we have divided the range or the limit how to say the interval [0,2] into two parts. So, after division, it looks like this. So, one whole integral is divided into two sub integrals.

Now, we can write it as $\int_0^1 x^2 dx + \int_1^2 \sqrt{x} dx$. So, this will be $\frac{x^3}{3}$ from 0 to 1 + this will be so $\sqrt{x}$. So, it will be $\frac{2}{3}x^{\frac{3}{2}}$ yes from 1 to 2. So, then we can calculate these and after calculating, we will be able to obtain this as $\frac{1}{3} + \frac{2}{3}2^{\frac{3}{2}} - \frac{2}{3}$, so this will be I believe $\frac{4\sqrt{2}}{3} - \frac{1}{3}$. So, we can calculate whatever this is so I leave this up to you.

So, you can see here the given function f was actually given by how to say two different functions. So, f is of course it is continuous, but it is given with the help of two different functions. And all we have to check, first of all whether they are continuous or not? If they are not continuous or if there is even one point of discontinuity, then we check I mean whether they are monotonic or not. And if they are monotonic and bounded functions, then they are Riemann integrable.

However, in this case we are very lucky, because both the functions are continuous in their respective range and therefore the function f is continuous. And if the function f is continuous, then it is Riemann integrable. So, here I can write a small point f is Riemann integrable so f is Riemann integrable and now we can calculate the integral from [0,2]. So, and this will be whatever you get after the calculation here that will be the answer. So, $\frac{4\sqrt{2}}{3} - \frac{1}{3}$ would be your answer. So, this is one such example.

Now, what I am talking about is with this discontinuity part that let us say, you have one point of discontinuity or if you have two points of discontinuity, then what will happen to the remain integrability of the function. And in order to do that we have a small theorem, which talks about these finite points of discontinuity. And even though if you have finite points of discontinuity, you can still talk about the Riemann integrability.

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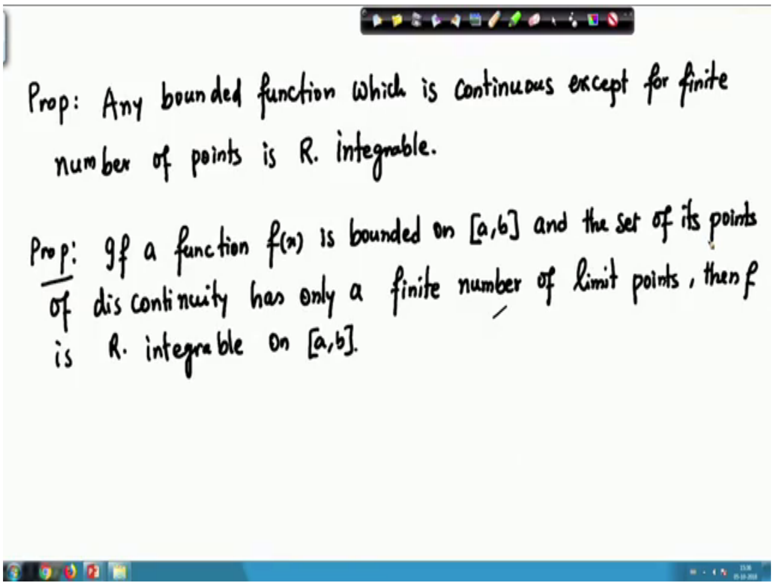

So, let us state a very small result all our properties. So, the result goes like this, I would list it as property all right so, property so it says that any bounded function, which is continuous except for finite number of points, so that means, it is not continuous at these points is Riemann integrable.

So, if you have a bounded function, which is considered continuous except for finite number of points that means, there are finite number of points, where the function is not continuous even in that case your given function would be Riemann integrable. There is a little bit generalized form of this theorem. So, the generalized form goes like this.

If a function f(x) is bounded on the closed interval [a,b] and the set of it is points of discontinuity; has only a finite number of limit points, then also f is Riemann integrable on [a,b]. So, that means, if you have a given bounded function on a closed interval [a,b] there and the set of points, where the function is discontinuous has only a finite number of limit points and then in that case the function is also Riemann integrable. So, it is in a way a modified version of the previous theorem.

And we will see via some example in our next lecture, what do we mean by these two theorems and where how can we apply them. So, today we will stop here and in the next lecture, we will continue with the examples on Riemann integration.