Integral and Vector Calculus

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Lecture – 49
Osculating Plane, Rectifying plane, Normal plane

Hello students. So, in the last class we were looking into the concepts of a very important formula in Vector Calculus which is Serret Frenets formula which actually connects the derivative of tangent normal and binormal with tangent normal binormal curvature and torsion. So, we derived that formula and we were also working our few examples today we will continue with that formula and we will look into three different planes where those three vectors lie actually.

Because we know that all we learnt in the previous class that the tangent vector, the normal and the binormal they are they form a right-handed sort of like screw system. So, each one of them are perpendicular to one another so; that means, the one each one of them will lie in one plane in their respective planes perpendicular to the other two planes in which the other two vectors lie and we will write down the equations for those planes and then we will try to work our few examples.

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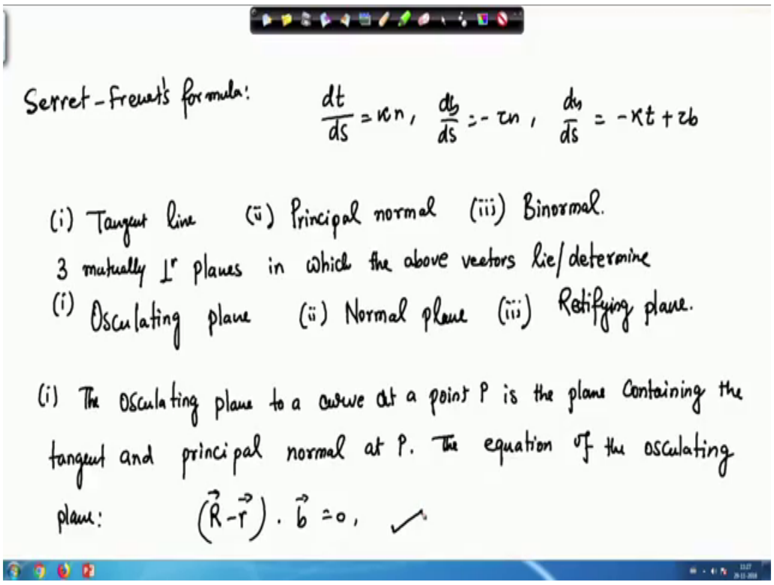

So, let me start we had the Serret Frenet formula as. So, we had the formula as, formula as dt ds, I am not writing any caps just to save some time kappa n and db ds is equals to minus of tau n and dn ds is equals to minus of kappa t plus tau b, where t b and n are the tangent vector unit tangent vector unit binormal and unit principal normal alright.

So, basically what we have is what we have is here is first of all the tangent line right. So, we have unit tangent vector or the tangent line, it is the next we have is the principal normal or unit principal normal and the third thing that we have is the binormal right. So, these are the three vectors that are connected with this equation.

Now, they, now that the form are sort of like a trihedral system or right handed screw system with; so, these three vectors must be lying in three different planes mutually perpendicular to one another. So, those three planes have also their names and the names are like first one is, so the three mutually perpendicular, mutually perpendicular planes in which the above vectors lie vectors lie or we can write determine the osculating planes osculating plane, second one is normal plane and third one is rectifying plane alright.

So, basically the normal plane will contain the normal vector perpendicular to tangent line and binormal, the osculating plane basically the osculating plane we will contain the oscillating plane we will contain your binormal basically and it will be perpendicular to t and n and the rectifying plane we will contain the tangent sorry here there is the c missing. So, rectifying plane we will contain the tangent vector and it will be perpendicular to both b and m alright.

So, so what we have is the, what we have is. So, if we want to draw a figure. So, the figure would be slightly. So, what let me write in words just to make the things clear? So, what do we mean by oscillating plane so oscillating plane. So, the first definition is the oscillating plane the osculating plane to a curve to a curve at a point P at a point P is the plane containing the tangent and the principle normal at P.

So, basically it will contain the normal, principle normal and the tangent at the point P and the equation of the oscillating plane. So, the equation and the equation of the osculating plane. So, basically it will contain both tangent and normal and so osculating plane is actually perpendicular to the by normal yes.

So, the equation of the osculating plane is; so, it is not actually containing, but it is containing two vectors t and n and its actually perpendicular to the binormal. So, oscillating plane is perpendicular to binormal; however, it will contain the tangent and the principle normal. So, now the equation of the oscillating plane is r minus small r dot b equals to 0.

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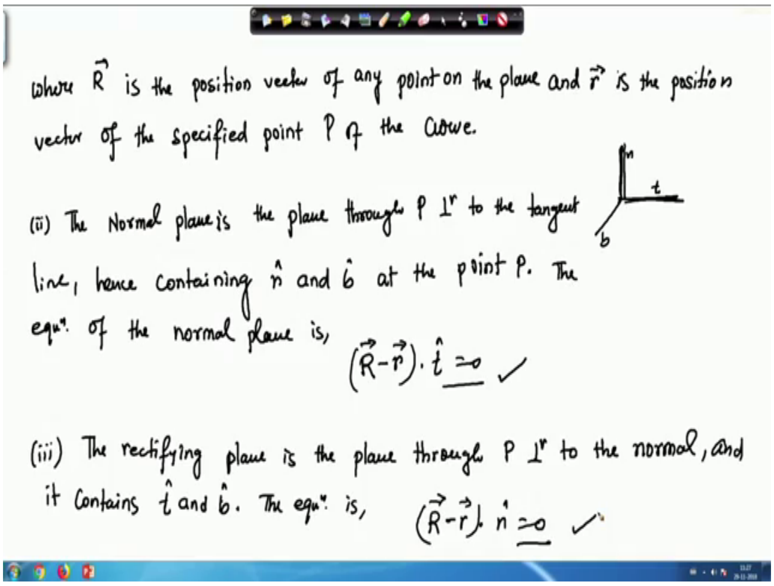

Where our capital R where capital r is the position vector of any point on the plane and small r small r is the position vector is the position vector of the specified point P of the curve. So, r is the position vector of any point on that oscillating plane and a small r is basically, the small r is basically the position vector of a point P where we are calculating that escalating plane alright.

So, this oscillating plane will be perpendicular to the binormal and it will contain t and n. So, if it is containing t and n ; that means, it is obviously perpendicular to b because t and n be are perpendicular to one another. So, if the vector is containing let us say if you have something like a three-dimensional geometry and if we have let us say this, this and this if this is my t this is my n and this is my b. So, if a vector is containing both t and n sorry t and n then in that case it is; obviously, perpendicular to b and if a vector is containing n and b then it will be perpendicular to t and if a vector is containing b and t then it will be perpendicular to n. So, now, we will define the normal plane.

So, the second definition is the normal plane the normal plane. So, the normal plane is the plane through P perpendicular to the tangent line so; that means, it will contain b and n alright. So, perpendicular to the tangent line hence containing hence containing normal and binormal at the point P. So, and the equation the equation of the normal plane is capital R. So, capital R is any arbitrary point on that normal plane minus small r. So, small r is the position vector of the point P where we are calculating the normal plane dot product with, since it is perpendicular to the tangent we will write t equals to 0. So, this is the required equation of the normal plane.

And third one is rectifying plane. So, the rectifying plane rectifying plane is the plane through P perpendicular to the normal and it contains. What are the lines it contains or what are the vectors it contains? So, it will contain t and n b.

So, it will contain t and b and the equation would be, the equation is and the equation is capital R minus small r dot product with n equals to 0. So, where capital R is any arbitrary point on the rectifying plane and a small r is the position vector of the point P and since it is perpendicular to the perpendicular to the normal we have the dot product equals to 0

So, going backwards this one is the equation of the osculating plane this one is the equation of the normal plane and this one is the equation of the rectifying plane. So, if you are given a equation of a curve in terms of r is equals to f t and if you are supposed to calculate any one of these planes, let us say at a point t is equals to a. Then in that case we have to calculate tangent binormal and normal and based on that of course, we can be able to give any one of these plane equations of any one of these planes.

So, we will work out one or two examples where we calculate these planes and of course, you can be able to express these equations in terms of the Cartesian coordinate. So, t can be written as so, this can be written as capital X capital Y capital Z minus r is that point x y z. And if you have a t equals to a given then in that case you can be able to calculate this small x y z and then you take dot product with t in terms of the Cartesian coordinate system and that will give you required equation in the Cartesian form. So, we will see how we can do that alright. So, let me look into my lecture note to get some examples alright.

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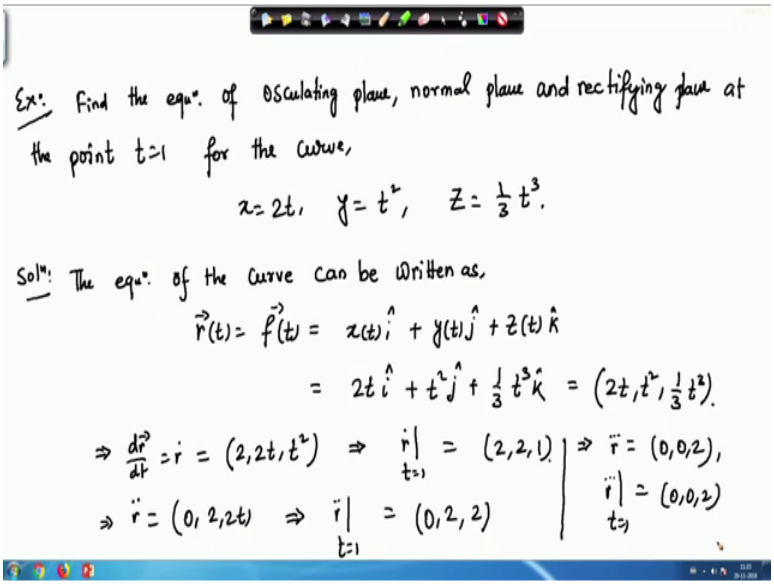

So, we have an example here. So, I will start with the same example that we considered yesterday. So, find the equation, find the equation of oscillating plane, normal plane and the third one was rectifying plane rectifying plane at the point t equals to 1 for the curve x equals to 2t y equals to t square and z equals to 1 by 3 t cube.

So, here we are given the equation in as x equals to y equals to and z equals to some function of t and we have to calculate oscillating plane, normal plane and the rectifying plane. And of course, for t equals to 1 the point p is given. So, at the point P which is which can be evaluated at t equals to 1 we have to calculate the equations of all of these planes.

So, let us see how we can do that. So, first of all we can write the equation of the curve the equation of the curve can be equation of the curve can be written as r t or r t equals to f t is equals to x t i y t j and z t k. So, this is our 2 t i t square j 1 by 3 t cube k or I can write it as a triplet alright. So, for the tangent we need d r dt in a way because if we have d r dt then that is basically the tangent vector and then you divide it with its magnitude and then we will get the unit tangent vector.

So, we first calculate d r dt. So, how do we calculate we differentiate both sides, I can also write r dot with respect to t. So, this will be 2, 2 t square and then t square and from here I can evaluate r dot at t equals 1. So, this is basically 2, 2 and 1 and of course, the point P would be at t equals to 1 r at t equals to 1 and that will be the point P next we can calculate r double dot.

So, now that we are calculating everything let us calculate r double dots. So, this is 0 2, 2 and 2 t. So, r double dot at t equals to 1 would be 0 2 2 and I can also calculate triple dot. So, I can calculate let me write it here in small. So, r triple dot is 0 0 2. So, r triple dot at t equals to 1 is 0 0 and 2. So, these are the three values for r dot r double dot and r triple dot at t equals to 1.

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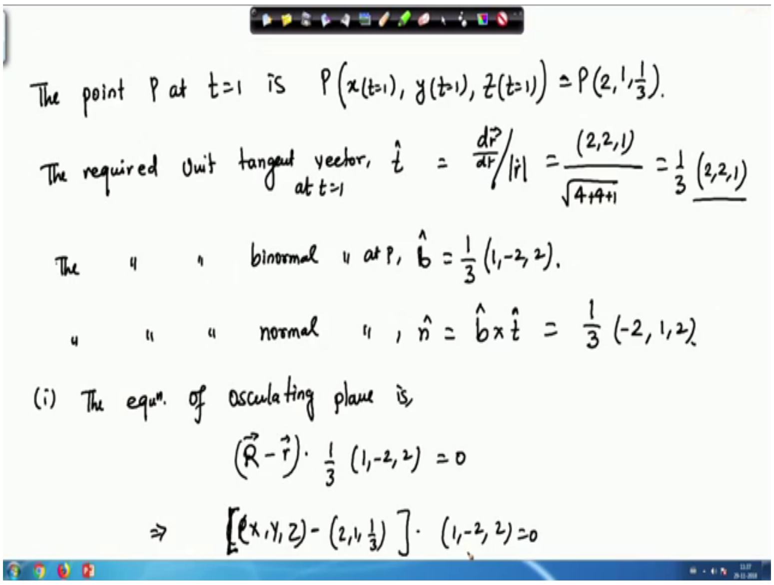

And the point P is the point P at t equals to 1 is p at x t equals to 1 y at t equals to 1 and z at t equals to 1. So, this is ultimately p at. So, when x is 1. So, when x is 1. So, this is 1, 1, 1 by 3. So, we have 2 1 1 by 3. So, this is my point P for t equals to 1.

So, now I will calculate. So, first I am being asked to calculate the we are being asked to calculate the equation of the osculating plane. So, we have all the ingredients, now we as we start calculating the equation of the oscillating plane. So, the equation of the osculating plane is if we go back, if we go back is r minus capital R dot b. So, b is actually our binormal unit binormal and rectifying plain is dot n and normal plane is dot t.

So, first of all we can be able to calculate our tangent. So, the required unit tangent vector is the required unit tangent vector t cap is d r dt divided by r dot mod. So, d r d r dt or r dot is basically at t equals to 1 is 2 2 1. So, at t equals to 1 it is at t they required in a tangent vector at t equals to 1 or at p we can write at P. So, this is 2 2 1 and then divided by 4 plus 4 plus 1.

So, this is ultimately 1 by 3 2 2 1. So, that is our required unit tangent vector. So, from here the required binormal. So, the required unit binormal vector at p is. So, do not write it is just because two is equals to vector perpendicular to t.

So, I can write 1 by 2 and of course, I have to consider a unit vector. So, this will be 1 minus 2 and 2 and the required unit normal, a unit principal normal at P is n equals to b cross t. So, we can calculate this cross product and this will ultimately yield 1 by 3 minus 2 1 2. So, we have been able to calculate the tangent vector, unit tangent vector unit by normal and unit principle normal. So, now, we have all the ingredients to write the equation of the oscillating plane. So, let us do that.

So, the first the equation of osculating plane is. So, we have capital R minus a small r do not forget the small r is actually a point P dot oscillating plane. For the osculating plane we need to have here we need to have here our where is that equation. So, we need to have here our b or binormal actually. So, the binormal what is that. So, the binormal is 1 by 3 1 minus 2 2 equals to 0. So, from here I can write 1 by 3 will go to that side and it will be 0 and this is capital R. So, I can write capital R as X Y and Z.

So, I choose the any point on that plane on the osculating plane as capital X capital Y and capital Z, minus a small r is basically the position vector. So, I can write 2 1 and 1 by 3 and then this one is dot product with 1 minus 2 2 equals to 0. So, this one will be x minus 2 y minus 1 z minus.

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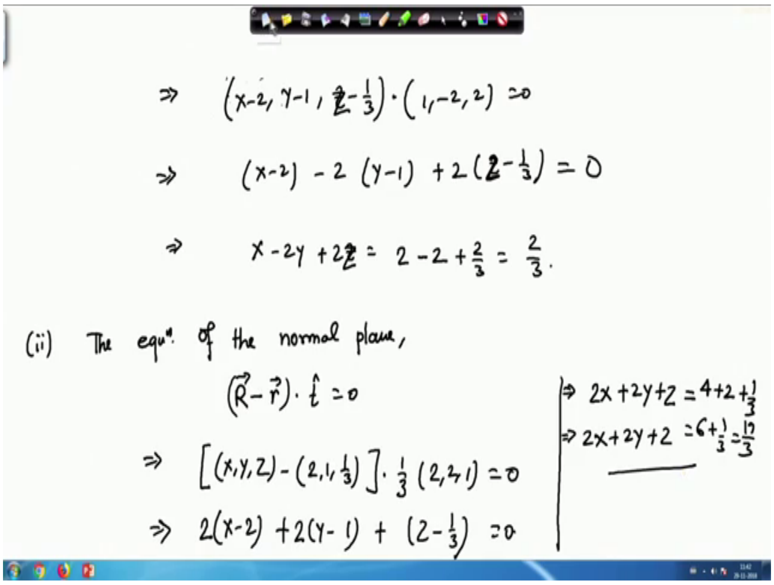

So, this one will be x minus 2 let me write it in a it clear way. So, this will be what do we have x minus 2. So, x minus 2 y minus 1 and z minus 1 by 3, the dot product with; so, 1 y 3 dot product 1 minus 2 and 2. So, 1 minus 2 and 2 equals to 0. So, this will be x minus 2 minus 2 times y minus 1 plus 2 times z minus 1 by 3 equals to 0. So, this is the required equation of the of the osculating plane. If you want then you can keep all these x y z on one side and z on the constant on the other side.

So, we can simplify this equation a little bit it will be x minus 2 y plus 2 z equals 2 I might make a mistake while calculating. So, this will be plus 2 minus 2 and then plus 2 by 3. So, this will be ultimately 2 by 3, I do not know from here to here if it is correct or not, but that will be the you can leave it here or you can leave it here. So, this is the required equation of your oscillating plane in terms of x y z you can also use a small x now a small y small z and. So, this is capital Z capital Z capital Z. So, if you want you can use this small x y z now and its totally safe to do that.

So, that is how we give the equation of the osculating plane, now let us go move on move on to our next task which is calculating the equation of; what was that? equation of the normal plane. So, for the normal plane so the equation of the normal plane equation of the normal plane. Capital R is again a point on the normal plane minus a small r is the position vector of the point and for the normal plane we need, for the normal plane we need dot t. So, dot t so what is our t t is 1 by 3 minus 2 2 1.

So, t is dot t is equals to 0. So, here we have X Y Z minus a small r s what is our small r 2 1 3, 2 1 1 by 3. So, 2 1 1 by 3 dot product with t. So, what is our t 1 by 3 2 2 one. So, 1 by 3 2 2 1, now this 3 will get absorbed here and then this will be x minus 2 times 2 plus y minus 1 times 2 plus z minus 1 by 3 z minus 1 by 3 times 1 equals to 0 and I can try to collect the terms at one place and so, this will reduce to I am not sure if I will end up doing the calculation correctly. So, 2 x 2 y plus z equals to 4 to 1 by 3. So, 6 plus 1 by 3 so, this will be 19 by 3. So, you can check from here to here that is not an is it is a difficult task to do. So, this is the required equation of our normal plane alright.

And now you can use the small x y z as well and the third task that is given to us is the rectifying plane, the plane that is perpendicular to t i guess.

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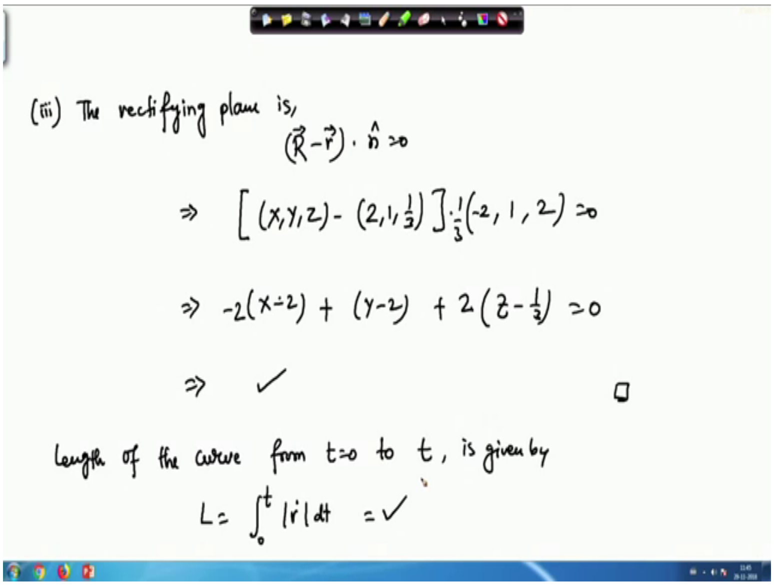

So, the rectifying plane now it is perpendicular to n the rectifying plane is perpendicular to n rectifying plane is capital R minus small r dot product with n right. So, capital R is again x y and z minus 2 1 1 by 3 and dot product with n. So, n is so 2 1 3 and n is 1 by 3 minus 2 1 2. So, 1 by 3 minus 2 1 and 2.

So, 3 will get reabsorbed and here we will have x minus 2 times minus 2 and then we have y minus 2 times 1. So, that we do not have to write it and then 2 times z minus 1 by 3 equals to 0. So, if we multiply both sides by minus and then we do some simplification you can be able to obtain the equation of the rectifying plane. So, rectifying plane if we I just have to check once, rectifying plane is dot n yes.

So, you see for this given curve which is also called as to state to a state cube we were able to obtain the equation of the osculating plane which is basically osculating plane is basically a plane containing tangent and principal normal and it is perpendicular to the binormal. And then we calculated the normal plane, for that given curve which is basically a plane perpendicular to the tangent and then we calculated the rectifying plane which is a plane containing t and b and perpendicular to n.

So, we just have to calculate these three vectors. So, if you know the tangent vector we can easily calculate binormal or normal because they are mutually perpendicular to one another and then the equation of the oscillating plane will be capital R minus small r dot t. And then just substitute the value of t and that will give us the required osculating plane; similarly the normal plane can be given by can be given by this equation. So, we just substituted the value of P and the tangent and then that is the equation and similarly the equation of the rectifying plane can be given by this way.

You can also calculate, you can also calculate the length of the curve measured from t up to any point. So, the length of the curve. So, this example was up to here we can also measure the length of the curve from t equals to 0 to any point. So, the length of the curve for the same problem the length of the curve from t equals to 0 to certain point let us say t to a point to any certain point t is given by capital L equals to integral from 0 to t r dot dt.

So, we will we will try to work out an example based on this in our next class. So, in today's class we were able to determine the oscillating plane normal plane and the rectifying plane of a given curve. And we will stop here for today and then in the next class we will continue with calculating the length of a given curve from a certain point to a certain point and I look forward to you in your next class.

Thank you.