**Integral and Vector Calculus**
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**Lecture – 48**
**Example on binormal, normal tangent, Serret-Frenet Formula**

Hello students. So, in the last class we started with derivation of Serret-Frenet formula and then we derived a result which actually connects the time derivative of a or the derivative with respect to the parameter of a given curve r is equals to f t its with curvature and torsion.

So, we saw that we really do not have to calculate dn ds db ds and all those things in order to calculate the curvature and torsion we just have to differentiate the given curve with respect to the parameter and we require derivative at least up to the third order to calculate these things and it just makes our life a little bit easier instead of going through all those the dn ds and db d s calculation.

So, today we will work out few examples. And we will see if we have some time then we can move on to our next aspect of vector calculus which is application to mechanics, but for the time being let us start with some examples alright. So, now, you might be asked in your exam to prove few relations.

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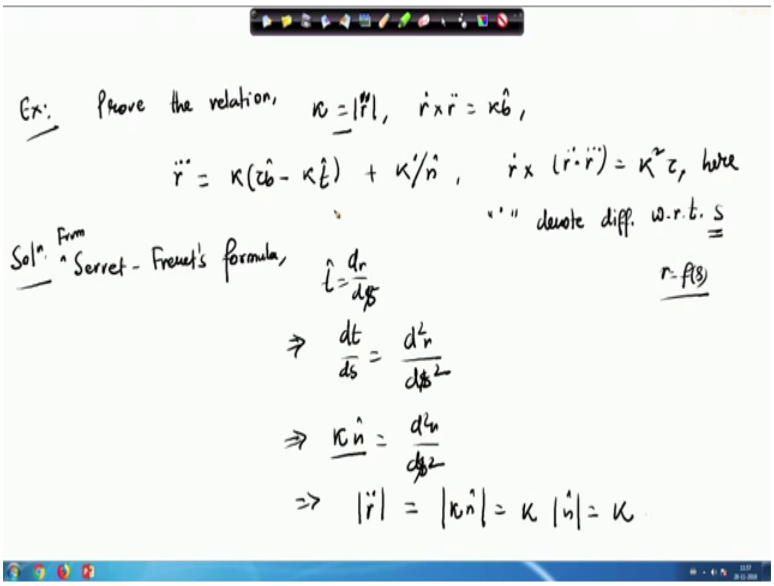

So, the first one is example. So, prove the relation kappa is equals to r double dot r dot cross product with r double dot equals to kappa times b cap r triple dot equals to kappa times tau b cap minus kappa times t cap plus kappa dash divided by n cap and r dot cross product with r double dot dot r triple dot equals to kappa square tau; so we will prove these relation.

So, for example, if you have a given curve r then we know that this formula is true. So, we know that in the if we saw that in the previous class that this formula is true; however, you can actually calculate d square r by dt square and mod of that d square by dt square will also give you the curvature alright. So, let us see how we can do that; so the first; so the first relation is to prove this one. So, from Serret-Frenet formula; so from Serret-Frenet formula Frenet's formula we have, first of all we have t cap equals to d r dt.

So, if we differentiate both sides then it will be it will be sorry dt ds. So, it will be dt ds if I differentiate both sides. So, dt ds equals to d square r by dt square now dt ds is from Serret-Frenet formula is kappa times n. So, it is kappa times n equals to d square r by dt square. So, now, if I take mod on both sides then in that case this will be r double dot which is basically d square r by dt square mod of kappa times n cap. So, kappa will come out and this will be simply mod of n cap, now n is the unit principal normal so its magnitude is 1. And therefore, this is our kappa.

So, the first relation is r double dot equals to kappa is proved all right our next relation is r dot cross product with r double dot. So, let us see what is our r double dot.

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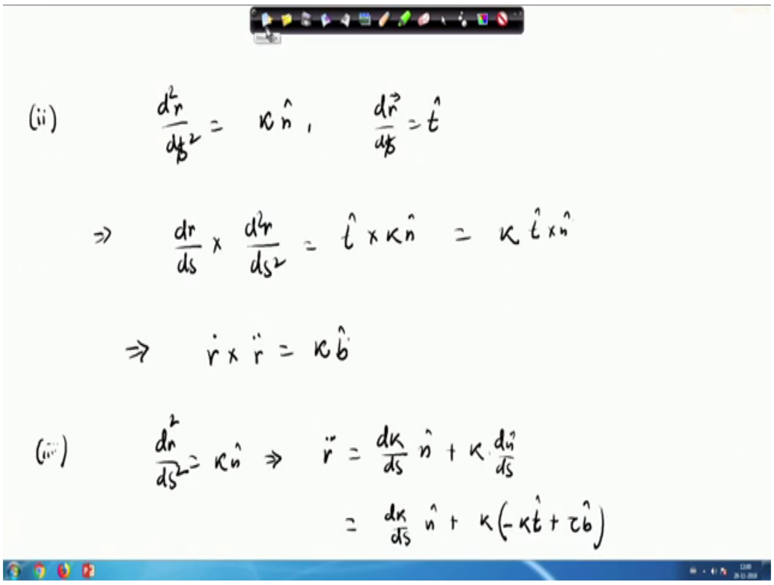


So, in the previous relation; so in the relation one we have shown that d square r by dt square equals to what is that? Kappa times n cap kappa times n cap and we know that dr dt is basically our t and wait wait wait this one we can take. So, here when we have dot I just forgot to mention, that here dots denote differentiation with respect to s right; so this one is d square ok.

So, here differentiation denotes the there is a dot denotes the differentiation with respect to s, not with respect to t. So, earlier the differentiation where with respect to the parameter t, but in this case the differentiation is with respect to the arc length s because we can also write r is equals to fs as a curve, where s is the arc length and then in that case these derivatives are basically derivatives with respect to s not with respect to t. So, of course, kappa is basically d square r by d s square.

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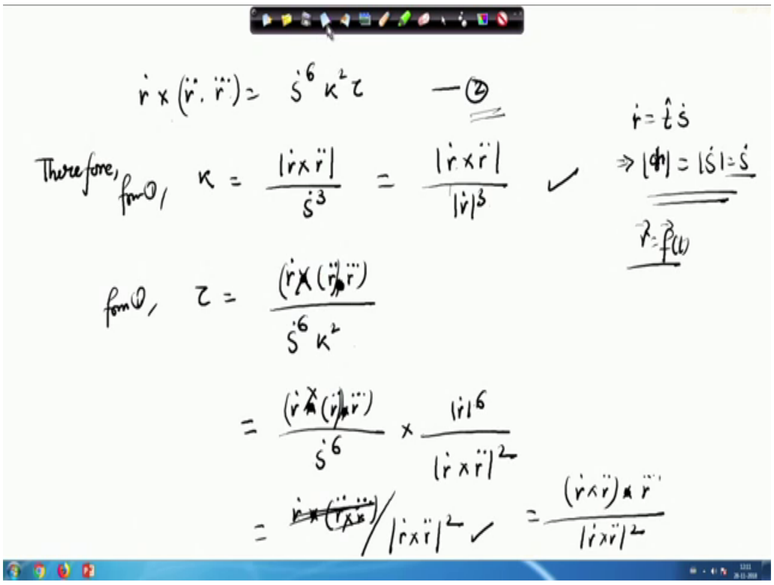

However, in the previous class we saw that kappa is r dot cross product with r double dot. So, here the dots are with is to t. So, when the equation of the curve is r is equals to f t alright, but in this case these dots are with respect to s. So, that was something I sort of skipped to tell you it is just that these derivatives are with respect to s only alright. Now, we have d r by d s square and dr ds equals to t cap alright.

So, when I take the cross product which is dr ds cross product with d square r by d s square then this is basically tau cap cross product with kappa times n cap. So, basically we have kappa times tau cap n cap alright and this one can be written as r dot which is of course, dr ds cross product with r double dot which is basically d square r by ds square kappa times. Now t cross n is basically our b cap. So, let me just verify the formula. So, t cross n is basically our b cap; so I can write b cap here.

So, you see r dot cross product with r double dot is basically kappa times b cap and these dots are taken with respect to s only. Now third relation is r triple dot. So, r triple dot is basically we know that we know that dr ds we know that d r d s equals to kappa times n; so I can differentiate twice.

So, this will be d k d s dk ds dr ds is basically t and d square r by ds square is kappa times n. So, d is d cube r by d s cube is d k d s times n cap plus kappa times dn d s; so now, the kappa times dn d s.

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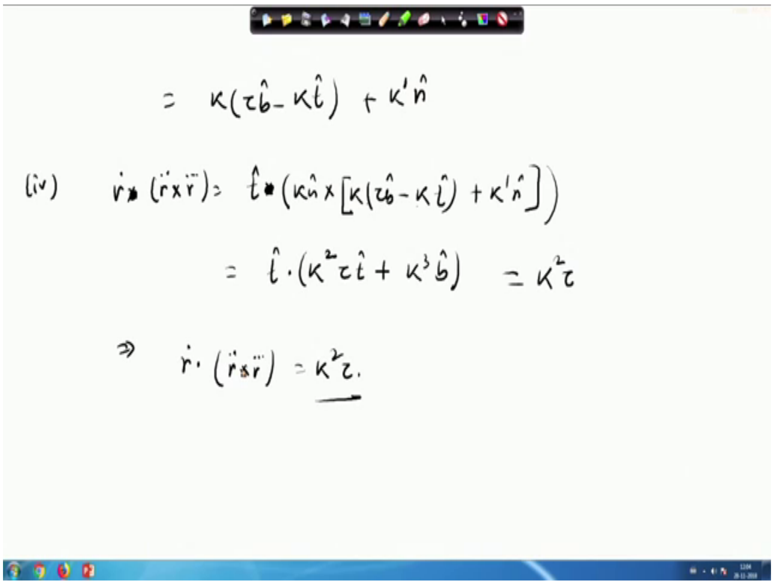

So, now this is basically d k d s, dk ds times n cap and then kappa times dn d s is from Serret-Frenet formula we have minus of kappa t cap plus tau times b cap. So, this is ultimately kappa times tau b minus k times kappa times t plus k dash n cap and if I multiply both numerator and denominator with n cap. So, or we can leave it like that.

So, in the formula we do not have to take a division. So, in the formula we can leave it like that. So, let us not divide it or anything; so this is just k dash dot n. So, k dash multiplied with n. So, this is this is what we are required to prove in the third relation, and the fourth relation is basically and the fourth relation is basically r dot cross product with r double dot r triple dot. So, here all of them are taken with respect to s. So, r dot is t cap, r double dot is kappa n and r triple dot is cross product of kappa tau b cap minus k t cap kappa t cap plus kappa dash n cap.

So, all you have to do is calculate this cross product. So, we do the cross product of this and this first and then cross product of this and this and all that. So, and keep in mind that b cross b and b equals to t cross n or n cross t whatever it is. So, just use those formulas and whenever you are using them use their proper sign whether you have to use b cross t or whether you have to use t cross b depending on that you will use the minus sign, and just substitute those signs here. It is just some vector algebra calculation nothing more and when you substitute all those things then it will simplify nicely into something like this.

So, here we have excuse me, here we have. So, we are calculating r cross, no this one has to be r dot and then here it should be r cross yes. So, r dot r double dot r 3 s. So, we will calculate because otherwise that will be a scalar that will be a vector so it will not be possible. So, we are calculating basically this thing here and here it should be a dot. So, here it should be a dot. So, ultimately when you calculate the whole thing if you calculate the whole thing then it will result into something like t cap kappa square tau t cap plus kappa cube times b cap and this t cap t cap will be 1.

So, this is ultimately kappa square tau and t dot b is basically 0 because they are mutually perpendicular to one another. So, the other term is 0 and therefore, this is what we needed to prove. So, we had to prove r dot dot with r double dot r triple dot equals to kappa square tau and this is what we needed to prove. So, remember in mind that here the dots are taken with respect to the arc length. So, instead of writing the dots we could have taken, we could have taken here a different notation let us say dash. So, dash dash dash and dash here dash here dash. So, the dash s basically denotes the differentiation with respect to s.

So, instead of taking the dot we can take dash just to distinguish it that we are not taking the derivative with respect to t instead we are taking derivative with respect to the arc length although its not something to confuse about since I told you several times and, but just for the clarity that in this example in order to calculate kappa and torsion and the curvature kappa and the torsion tau we have used derivative with respect to s all right.

So, this is one another example where you are asked to prove some relations we might may try to include examples motivated from this and now let us work out some examples; so some more examples actually.

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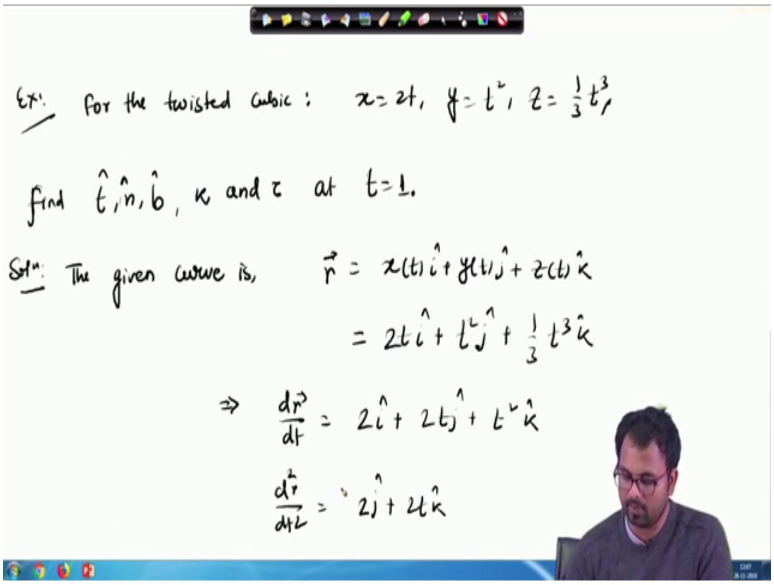


So, we will start with for the cube, for the twisted cube or cubic x equals to 2 t y equals to t square and z equals to 1 by 3 t cube, if you are given these then find t n b. So, they are all unit tangent vector unit principle normal unit binormal kappa and tau at t equals to 1. So, at t equals to 1 our given point can be basically 2 1 1 by 3. So, the given equation of the curve is, the given curve is r equals to x t i y t j plus z t k right. So, x t is 2 ti t square j plus 1 by 3 t cube k.

So, first of all we have to calculate t cap. So, in order to calculate t cap what we will do? We will first do the differentiation with respect to t. So, dr dt is 2 i plus 2 t j plus t square k and we will calculate d square r by dt square or r dot then this will be 0 i so we will not write that. 2 j plus 2 tk.

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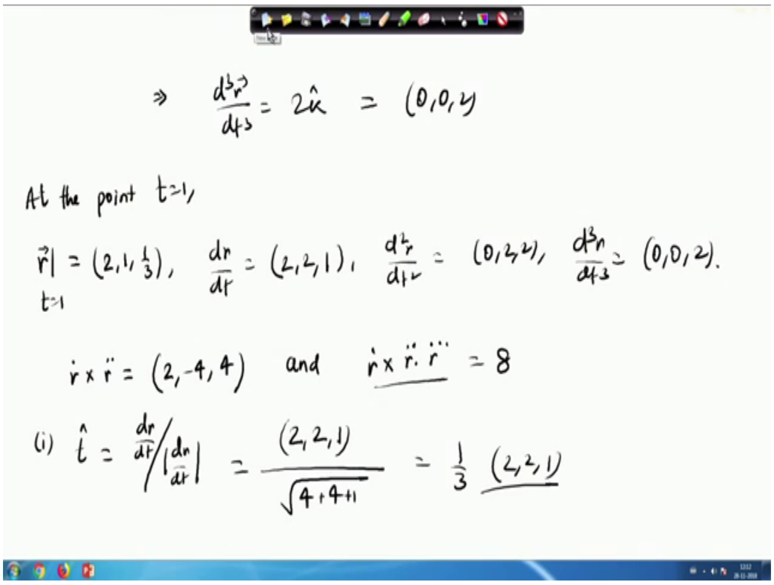


And then we will calculate d cube r by dt cube equals to 0 0 plus 0. So, we will avoid writing all those 0's and then we will just write 2 k right. So, this is ultimately 0 0 2, previously it was 0 2 2 and this is 2, 2 t, t square alright. Now at the point t equals to 1, at the point t equals to 1 r is r at t equals to 1 is the point p is basically 2 1 1 by 3 then the derivative is dr dt is 2 2 1 then d square r by dt square is 0 2 2 and d r cube by d 3 cube d cube r by d 3 cube is 0 0 2 alright.

So, from here the unit tangent vector; so first of all and we can also calculate these factors. So, r 3 dot so r dot cross product with r double dot equals to we will take the cross product of this and this which is fairly easy to calculate I leave this task up to the students. So, 2 minus 4 4 I have the results calculated in my lecture notes. So, I am just writing it from there and r dot cross product with r double dot dot r triple dot. So, we have r dot cross product with r double dot dot product with r triple dot; so this is ultimately 8 all right.

So, now we have all these three ingredients. So, what are we going to do? So, now, and there is small error so dot dot double dot. So, r dot think we had r dot r double dot r dot r double dot ok. So, this r this is the dot product and then this is the cross product. So, this is the dot product and then this is the cross product.

So, dot product and this is the cross product. So, because we cannot so r dot r double dot cross r triple dot yes. So, because we cannot we cannot simply r dot cross product with r double dot. So, r dot cross product with r double dot dot product with r triple dot. So, r dot cross product with r double dot dot product with r triple dot.

So, ultimately this formula is r dot r double dot cross product with dot product with r triple dot alright. So, we have r dot cross product with r double dot and dot product with r triple dot yeah. So, there was a slight modification in the calculation of tau. So, this we have to keep in mind; so there is a slight vector notation issue here.

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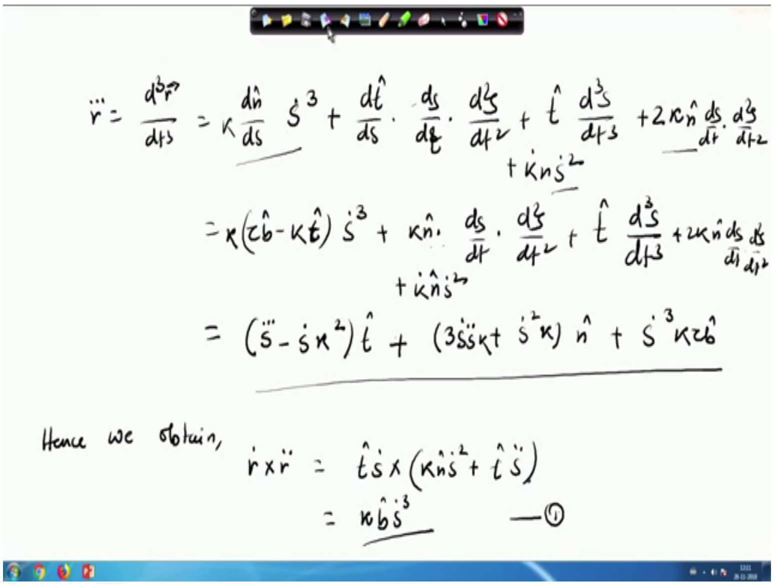

So, I copied it wrong; so, it is basically r dot cross product with r double dot. So, I had to copy it r dot cross product with r double dot dot product with r triple dot so the rest of the things are same.

So, here I made a small error while copying the results from the previous slide. So, that can happen to alright. So, r dot cross product with r double dot dot product with r triple dot. So, this is basically our 8. So, now, our unit tangent vector; so the first thing that we will calculate it calculate is unit tangent vector which is t cap and t cap is nothing, but dr dt divided by mod of d r dt. So, dr dt is 2 2 1 divided by mod of dr d t is 4 plus 4 plus 1. So, this is ultimately 1 by 3 2 2 1.

So, that is the unit tangent vector now that we have the unit tangent vector for the for unit binormal for the binormal we have for the binormal we use the formula we use the formula kappa. So, we use the formula for the binormal we use s dot. So, we for the binormal we use d b ds. So, you know that binormal is given by this relation here t times n cap.

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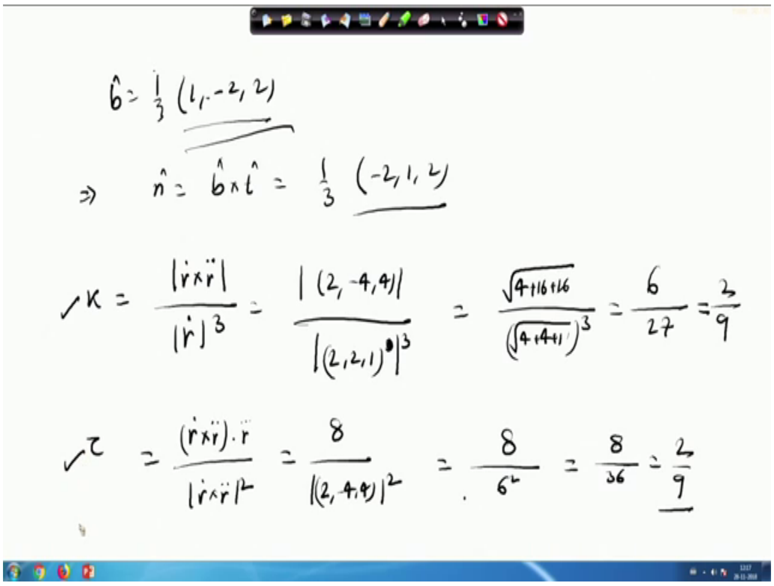


So, t t is basically the unit tangent vector and n cap is the unit unit normal. So, unit normal can be can be given by or simply we just have to find a unit normal which is perpendicular or a unit vector that is perpendicular to t.

So, a vector that is perpendicular t the unit vector that is perpendicular to t is 1 minus 2 and 2. So, I can write simply b as 1 by 3 1 minus 2 and 2 because then n and b at t and b will be perpendicular they are unit vectors and therefore, it is a unit binormal and from here I can calculate my n cap.

So, n cap is b cross t and b cross t can be given by 1 by 3 minus 2 1 and 2. So, they are all perpendicular to one another. So, we first calculated t, from there we can write our b and based on the b we can calculate our n by just simply calculating b cross t. So, we know our t is this one, and we know our b is this one.

So, just calculate their cross product one by three will come up front and then we will have this as the as the required unit principle normal. So, we have these 2, now we can calculate our kappa. So, kappa is basically from this relation kappa is from this relation let me go back to my lecture note and there I have kappa as r dot cross product with r double dot divided by r dot whole to the power 3. So, what is my r dot r dot cross r double dot? So, 2 minus 4, 4 so they are mod.

So, 2 minus 4, 4 and then r dot cube. So, what is r dot? So, our r dot is at the point t equals to 1 r dot is where is that 2 2 1 alright. So, let me go back here 2 2 1, whole to the power 3. So, this will be 4 16 16 and this will be 4 4 1 whole to the power 3.

So, ultimately from here we will get 32 plus 4. So, 36; that means, 6 and here we will get 3 and 3 to the power 3 is 27. So, this will result into I believe 2 by 9 and tau can be calculated as using this formula from our previous class from our previous class.

So, this formula r dot cross product with r double dot dot product with r triple dot. So, we have formula as r dot cross product with r double dot dot product with r triple dot and divided by r dot cross product with r double dot alright. So, if I take the dot product with r triple dot what is our r triple dot? So, r triple dot at t equals to 1 is just 0 0 2.

So, if I take the dot product with this here I will only obtain 8 and in the denominator I will have mod of 2 minus 4 4 whole square. So, this will be 8, and this will be square root of 4 16 16. So, 6 6 whole square; so this will be ultimately 8 by 36. So, I can cancel by 4 or 2 by 9.

So, therefore, we have calculated the kappa and tau. So, all you have to do is just use some results what we have studied in Serret-Frenet formula use the appropriate relations to calculate dr dt. And do not forget to use the perpendicularity condition or the relation between b tau and n in half an hour I am sometimes how to say making a small errors in the calculation, but I am pretty sure you are getting what I am trying to say.

So, you just have to calculate those dr dt use the relation between b t and n and from there if you know one of the vectors you can calculate the other one by simply using the perpendicularity condition. And from there you can calculate the third one using that relation that b cap is a t cap cross product with n cap. And similarly n cap is cross product of b cap and t cap and t cap is a cross product of b cap and n cap.

So, if you know any one of them the calculating other 2 would not be difficult and to calculate curvature and torsion you just have to differentiate the curve with respect to t if the given equation is in terms of the parameter t. However, if it is given in terms of arc length then we have also worked out an example where curvature and the derivative of r with respect to arc length is connected. So, it depends it depends what kind of curve equation is given to you based on that you use that if you use the respective formula and then you can calculate the curvature and torsion.

So, that is what we did in today's class and in the next class we will probably do one or 2 more examples on Serret-Frenet Serret-Frenet formula and then we will move on to our next topic. So, I will stop here for today and I look forward to you in your next class.