Integral and Vector Calculus
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Lecture – 47
Introduction and Derivation of Serret-Frenet Formula, Few Results

Hello students. So, in the last class we started with the concepts of a little bit words differential geometry aspect of Vector Calculus. So, we introduced the concepts of curvature and torsion and we also introduced how the unit tangent vector unit by unit principle normal and the unit binormal are connected with one another and we sort of derived some relations.

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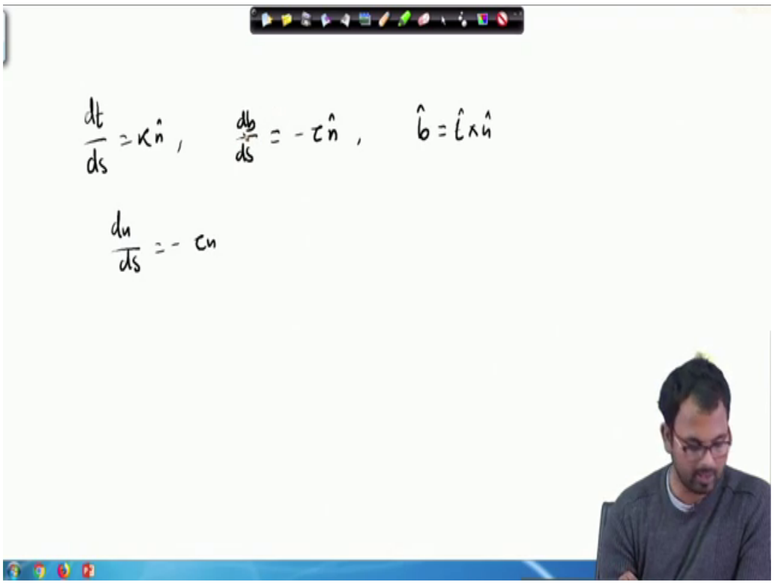

So, for example, we obtained relations like dt ds equals to; dt ds equals to Kappa n where Kappa is the curvature and then we obtained relations like db ds equals 2, where t is the unit tangent vector b is the unit binormal. So, db ds equals to be obtained minus of tau n where n is the unit principal normal and tau is the torsion, we also obtain b equals to t cross n I prefer to write a cap.

Because there are unit vectors, but you can also write a vector notation and make sure you specify that there are unit vectors; so, for example, b is a unit binormal. And similarly you can write t cap equals to b cross n or n cross b so, so whenever you are changing the order so, then in that case you have to put a minus sign. And of course, these 3 vectors they form a right handed screw system. So, they are mutually perpendicular to one another like the unit vectors i j and k along the coordinate axes.

So, we derived these 2 relations and we also derived something like dn ds. So, we derived something like dn ds equals to minus of tau n. So, this one was also obtained and I know we have obtained db ds sorry so we have obtained db ds. And now how do we obtain dn ds? So, we have obtained db ds excuse me and how do we obtain now dn ds.

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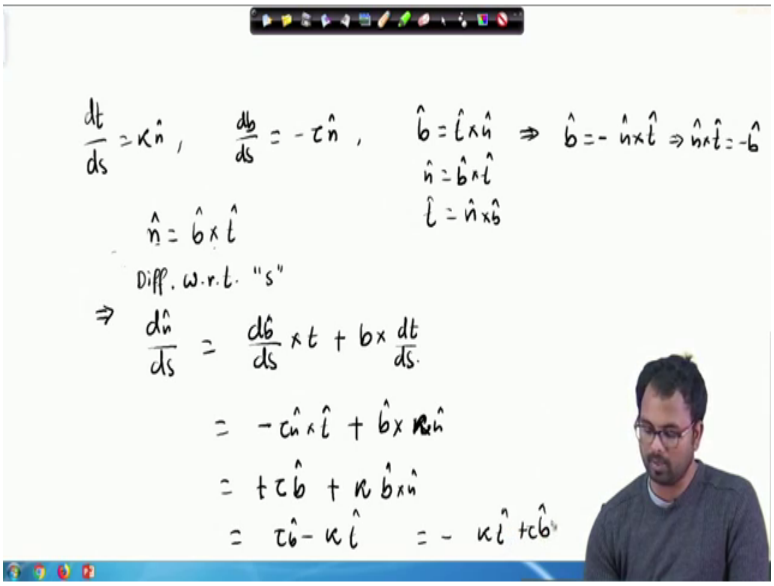

So, in order to obtain the dn ds first of all we will write the we will write that right handed screw system n cap equals to b cap times t cap. So, this is also part of that right handed screw system relation. And now we differentiate both sides with respect to the arc length all right. So, let us see what happens differentiating with respect to s all right. So, when we differentiate this will be dn cap by d s equals to d b ds, I am not writing cap for the moment, but it is understood all right just to save some time now b cross dt d s. So, now db ds is minus of tau n all right cross product with t cap of course and b times dt ds is k times Kappa n.

So, this is basically minus of tau now n cross Kappa t. So, if we do n cross Kappa t n cross t, then in that case if we do n cross t, then in that case it will be minus of n cross t right. So, if you change the order of that cross product then in that case you have to put a minus sign. So, here this is nothing but minus so this is nothing, but n cross t is minus of b cap. So, I can substitute n cross t as minus of b cap so minus minus this will turn into a plus and Kappa times. So, this is not K this is basically Kappa make sure you see that Greek letter in some literature.

So, then so basically any mathematical books on notations, there you can be able to find this Kappa its a Greek letter and yes so this is Kappa and then we have b cross n. So, we have so, we know that b equals to t cross n; n equals to b cross t and t equals t equals to n cross b all right. So, let me just confirm if this is true so, t equals to n cross b that is correct. So, now that we have this here I can substitute b cross n as minus of t.

So, this will be tau b cap minus Kappa b cross n so t cap. So, this is basically we can write as minus of Kappa t cap plus b tau times b cap. So, therefore, dn ds is basically minus of Kappa times t cap plus tau times b cap.

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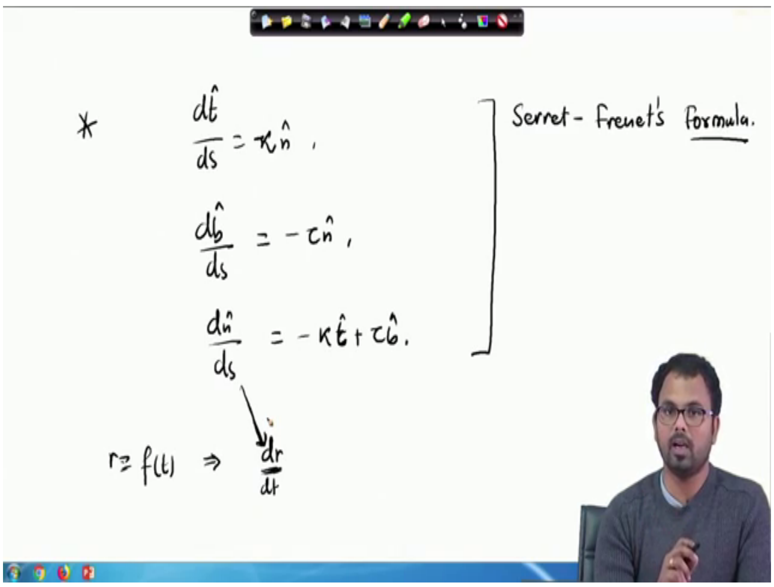

And if I write these 3 relations all together so, if I write these 3 relations that is dt ds of course, t cap equals to minus equals to Kappa n cap and then db ds equals to minus of tau n cap and dn ds equals to minus of Kappa t cap plus tau times b cap. So, these are the 3 relations that connects the derivative of t cap b cap and n cap. So, basically derivative of unit tangent vector derivative of binormal and derivative of unit principal normal with n cap t cap and b cap.

So, dt ds is Kappa times n cap db ds is minus tau times n cap and dn ds is minus of Kappa times t cap plus tau times b cap and these 3 relations all together are called as

Serret Frenet’s formula. So, these are the required Serret Frenet formula. So, if you have a let us say a right handed screw system of a unit tangent vector, unit principle normal and unit binormal, then they are connected with that b cap equals to t cross n and then t cap equals to n cross b or b cross n something like that.

So, that is the relation amongst the unit tangent vector unit principle normal and unit binormal and the relation between their derivatives will be given in this fashion and this relation is basically called as Serret Frenet formula. It is also very useful because if you know the unit tangent vector, then from there you can be able to calculate your curvature. And if you have a unit principle unit binormal given then from there you can be able to calculate, your torsion and if you have how to say your torsion and your binormal. And all the other things are given on the right hand side of this equation, then you can be able to calculate the derivative of a unit principal normal.

So, this relation is very useful and its widely used in vector calculus and also in some parts of mechanics as well. So, this is what we wanted to derive and its also part of our syllabus. So, we are glad that we were able to derive this relation. So, this is one of the important relations of vector calculus keep in mind so, we will put a star here all right.

Now that we have this relation of course, we would like to solve few examples. So, let us start with; let us start with our examples. So, how do we; how do we calculate these things Kappa and tau?

So, there must be some kind of formula that would relate these derivatives with the curve itself. So, the equation of the curve if we remember, its r is equals to f t and from there we were calculating dr dt which is actually the tangent vector. So, we were calculating dr dt then from there we can calculate dr ds and all that.

So, this dt ds the obvious question is this dt ds db ds and dn ds must connect with some derivatives of r and nothing else. So, if we; so, that we do not have to calculate those dr ds and dt ds and things like that. We do not want to calculate those things we want to calculate the derivative of r with respect to t only. And it could be first order derivative, it could be second order derivative, it could be third order derivative whatever it is, we just want to express these parameters like torsion and n curvature in terms of the derivative of r we do not want to go to these dt ds db ds. So, how we can do that? Alright.

So, we will of course, use these formulas to derive that relation where only r and its derivatives are connected with Kappa and tau all right.

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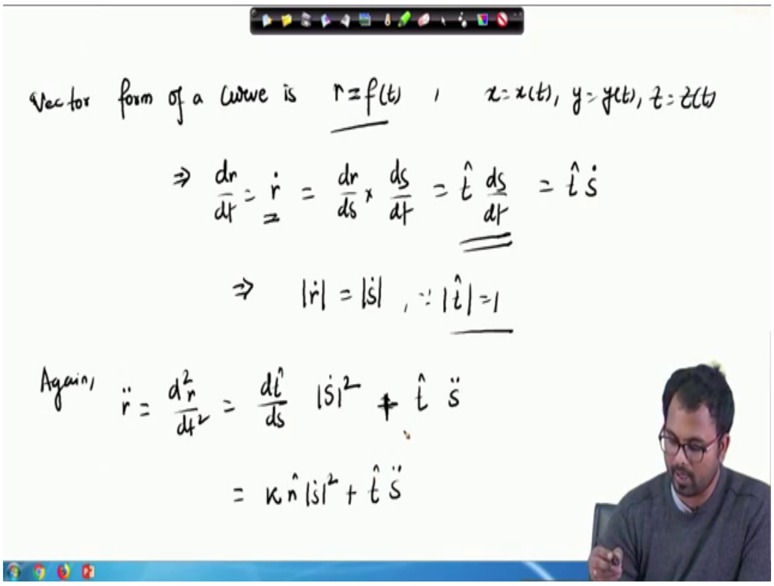

So, let us go. So, the vector form, so the vector form of a curve is r equals to f t right. And then we can be able to write here x equals to x t of course and y equals to y t and z equals to z t. So, this is the Cartesian form, this is the vector form all right. Now, if I differentiate this thing here, so then in that case this is dr dt.

So, we know that derivative can also be written as r dot and this one is basically dr ds. So, we can be able to write dr ds times ds dt and dr ds is our unit tangent vector. So, let us write this as unit tangent vector. So, dr dt or r dot is basically t cap times ds dt.

So, this is what we know. So, from here it follows that mod, this one can also be written as t cap times s dot all right, equals to mod of s dot all right. So, derivative of s with respect to t and the mod of t is 1. So, since t is a unit tangent vector we can write mod of t as 1. So, this is this we can always write. Now, again we have d square r by dt square so; that means, r double dot equals to d square r by dt square and that will be dt ds d square r by dt square.

So, this will be ddt of t cap s. So, then we can write dt ds times s dot square, because it will be ds dt ds dt whole square. So, this will be square plus t cap times s double dot, because then we have d square s by dt square. Now, dt ds if we go to the Serret Frenet formula dt ds is Kappa times n cap. So, let us substitute that here. So, this is Kappa times n cap and this is not cross product actually. So, do not confuse it with this cross product, its just a usual multiplication. So, I do not know; let us leave it like that. So, s square plus t cap times s double dot all right and lastly we will calculate r dot 3.

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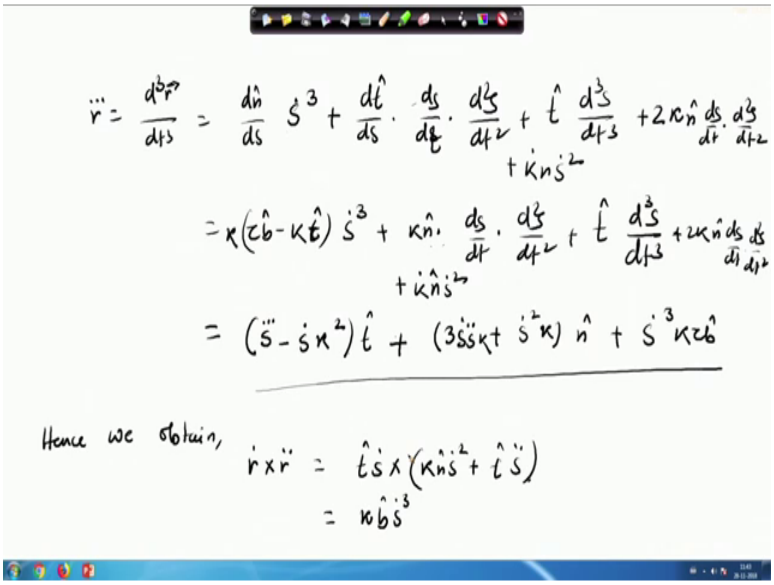

So, r triple dot which is basically d square d d 3 r by dt cube and this will be basically if I go here then, this will be dn ds times s dot to the power 3. So, this will be dn ds times s dot. So, this will not be mod in this case. So, we can write s dot to the power 3. So, it will be ds dt ds dt ds dt so; that means, ds dt to the power 3 plus, then we have dt ds times dt ds times ds dt times d square s by dt square plus t cap and d cube s by dt cube. So, now, dn d s if I go back here; so, dn ds is tau b minus Kappa t.

So, let us put here; its tau b minus Kappa t; minus Kappa t times ds dt whole to the power cube and dt ds is Kappa n ds dt times d square s by dt square and t cap d 3 s by dt cube. So, let me just match; if I have done it correctly, so, yes. So, now we rearrange the terms.

So, if we rearrange the terms; then we will obtain s triple dot minus s dot Kappa square s dot Kappa square times t cap. So, if I match the formula. So, here there is a Kappa already. So, Kappa so, here we have Kappa already. So, minus of Kappa and then alright; this is fine plus, now we have 3, so, we have 3, so, here we had 2.

So, 2 will come at the front and then s dot and then, so, here we had; so, 2 will come at, at the front. So, I am using basically the product rule. So, this will be dn ds and then we will have 2 times Kappa. So, there is one more term here. So, the another term is 2 times Kappa and then n cap times ds dt Kappa n cap times ds dt times d square s by dt square, the second term is dt ds ds. So, the other things are fine. So, there was one more term missing.

So, I can write this term here 2 Kappa n cap and dt ds dt times d square s by dt square and here then this will be 3 actually yes. So, here we have 3 ss double dot s dot is double dot plus s dot squared times Kappa and there is another Kappa here, n cap plus s dot cube Kappa tau and b cap.

It is a little bit complicated formula, but not really that much complicated. So, ultimately basically we will obtain dt ds. So, let me just match again. So, I am matching with my lecture notes. So, we have t times s 3 dot and then kn times s dot d square s by dt square and then, I have Kappa times minus of K dot tau s dot q and then Kappa n, Kappa dot; Kappa dot n. So, then I have Kappa dot n, Kappa dot n s that s dot. So, I have this thing, so, How many terms would I get? Basically 5 terms ok.

So, I can differentiate Kappa as well, so, here I can differentiate Kappa as well. So, ultimately I will obtain Kappa dot n s dot ok. So, I will obtain. So, there will be another term, Kappa dot n s dot square. So, it's just how to say Kappa is also a function of n. So, we can differentiate Kappa as well s dot square. So, ultimately we will obtain this expression all right.

So, s dot Kappa and n cap now its fine. So, basically at the end you will obtain this relation it is a little bit complicated. And now, that we have this relation here, we can write hence; we obtain. So, if we take the cross product of r dot with r double dot, then we just know what is our r dot; r dot is t cap s dot and I am taking cross product with r double dot, so, basically Kappa times n.

So, we have Kappa times, so, r dot is; r dot is t cap s dot t cap s dot cross product with Kappa times n cap s double dot; Kappa times n cap s double dot plus t cap s double dot; t cap s double dot. So, if we take the cross product, then this will be basically s dot s dot then this will be whole to the power s dot s s dot to the power 3 and then t cross n will be our; t cross n will be our b. So, ultimately I can be able to write this as Kappa times b times s dot cube plus t cross t will be 0. So, ultimately r dot r double dot is Kappa times b cap dot s dot to the power 3.

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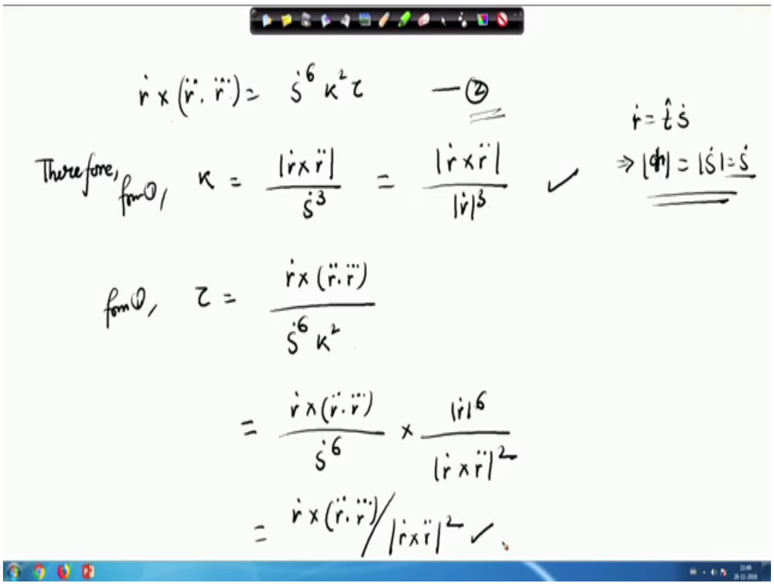

And if I take; if I take r dot cross product with r double dot dot product with r triple dot. So, similarly we can calculate and this will yield s dot whole to the power 6 Kappa square tau cube, Kappa square tau simply. So, we have r dot cross r double dot, dot product with r triple dot is basically s to the power 6 Kappa square tau. And therefore, so, this is let us say my relation number 2 and this is my relation number 1.

So, from both of these 2 relations, what we have is; what we have is? Kappa is basically; Kappa is basically r double dot cross product with r, r dot; cross product with r double dot divided by s dot cube right. And s dot is basically; s dot is basically r dot right. So, we have r dot is equals to t cap s dot. So, s dot is basically the arc length whose derivative is with respect to t.

So, if I take dr d r dot mod and this is basically s dot mod which is just we write s dot alright, because it does not have many components. So, it just have one component, so, this will remain just like s dot. So, here I can write r dot whole to the power cube. So, ds dt or s dot simply is nothing but mod of r dot all right. So, we have this thing here and now tau can be calculated as; tau can be calculated as r dot cross product with r dot dot r triple dot divided by so, from relation number 2.

So, this is from relation number 1. Now, this is from relation number 2 I have s to the power 6 Kappa square tau. Now Kappa square is basically; Kappa square is basically r dot r double dot times with our triple dot s dot to the power 6 and so, tau is there is there should not be any tau here. So, Kappa square; Kappa square is r dot to the power 6 r dot r double dot whole square all right.

So, s dot is basically mod of r dot that is what we have here. So, basically these two will get cancelled and then we will obtain r dot r cross with r double dot, dot product with r triple dot divided with mod of r dot cross product with r double dot whole square.

So, basically Kappa is r dot cross product with r double dot mod divided by r dot whole to the power 3 and tau is r dot cross product with r double dot, dot product r triple dot divided by r dot cross product with r double dot whole square. So, this is how we obtain the curvature and the torsion of the given curve and you can see here, that I mean now we do not have to calculate any db ds dn d s or dt ds or anything like that. We just have to differentiate the curve given curve r is equal to f t with respect to t at least thrice, because for torsion you need to calculate r triple dot; that means, derivative of r up to third order with respect to t.

So, we just have to calculate the time derivative or the derivative with respect to the parameter t for the curve r. And we just calculate these dot products cross product and divided by this mod of dot product sorry cross product whole square and that will give us the torsion. And similarly, for the curvature we can calculate the r dot cross r double dot mod divided by r dot whole cube and that will give us the curvature.

So, of course, it was a slightly how to say tricky here, its not tricky actually, you just have to spend some time calculating this derivative. So, just remember Kappa is also; Kappa is also a function of s. So, when we were differentiating the first term, we should have for example, here we had a Kappa times n s dot. So, Kappa times n. So, dn ds and then there should have been a Kappa here; there should have been a Kappa here and then when we are differentiating Kappa then n dot s dot will be unchanged. So, we have k dot; that means, dk ds n s s dot whole square and then we are differentiating s dot. So, this will come as 2 Kappa n cap ds dt times d square s by dt square.

So, that is what we have here 2 Kappa and ds dt d square s by dt square. So, we had to pay a close attention, so, that is what we will get all right. And then we are differentiating t dot s t cap s double dot. So, this will be dt ds times ds dt. So, dt ds times ds dt d square s by dt square and for the second term it will be t cap s triple dot.

So, t cap s triple dot is basically d 3 s by dt is dt cube and this is the required derivative. And now we will use the formula for dn ds from Serret Frenet formula. We substituted the value for dt ds dn ds and then we are just calculating the terms and we just take the cross product with r dot and r double dot that will give us this relation and we take the cross product of r dot with r double dot r triple dot that will give us this relation and this comes from I and this comes from II.

So, from I and from II, these are the required relations. So, we see that when we want to calculate the curvature and torsion of a given curve we really do not have to go through all those dn ds and db ds. We just differentiate the curve with respect to the parameter t up to 3 times and then we calculate this dot product and cross product and that will give us the required curvature and torsion of a given curve. So, we will practice a few examples based on these formulas in our next class.

And I thank you for your attention and I look forward to you in your next class.