Integral and Vector Calculus

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Lecture – 46
Unit normal, Unit binormal, Equation of Normal Plane

Hello students. So, in the last class we learned about the concepts of a unit tangent vector and the equation of a normal plane at a point p to a given curve r is equals to f t.

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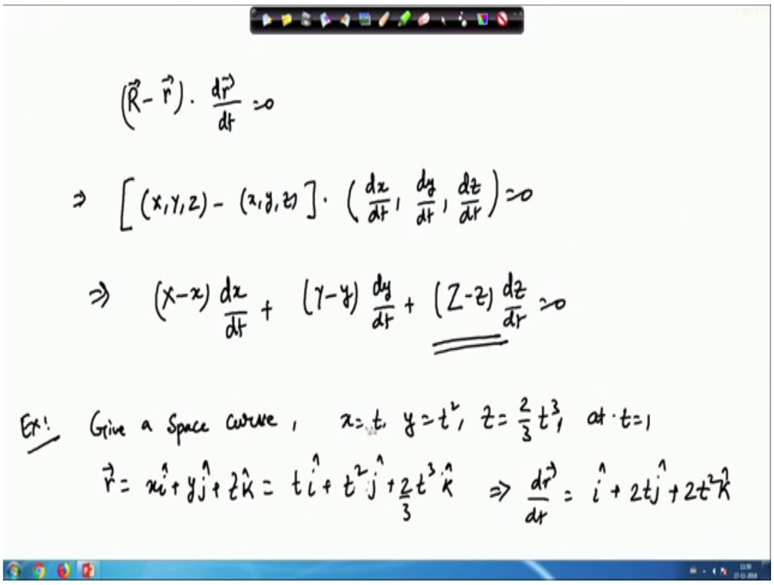

And our equation of the normal plane was something like capital R minus small r dot product with dr dt. This is not, a 0 vector because the dot product is always, the dot product always yields a scalar quantity.

Now, I can also write this equation in terms of the in terms of the Cartesian coordinate system. So, how we are going to do that? We can write capital R as that capital X, Y, Z and small r as that point p which is basically small x, small y, small z, dot product with dr dt can be written as dx dt, dy dt, and dz dt, equals to 0. So, I can take it as x capital X minus small x dot product with dx dt capital Y minus small y times dy dt and capital Z minus small z times dz dt. So, this is basically; this is just some simple dot product formula. So, this is the required equation of a normal plane at a certain at a given or at a point p to the curve r is equals to f t, alright.

So, let us work out an example just to make things clear. So, given a space curve we take the same example x equals to t, y equals to t square and z equals to 2 by 3, t cube. So, here we have to find out the equation of the normal plane. So, of course, we can write r is equals to x i, y j and z k where all these variables are function of t. So, I can write t i plus 2 t j plus this one will be 2 t square k and from here I sorry 2 by 3, so we are not differentiating it, so let I thought we are differentiating. So, so far we are not differentiating we are just writing the equation of the curve. So, let us write the equation of the curve first.

So, this one is basically t square and 2 by 3 t cube. So, from here we can write dr dt equals to i plus 2 t j plus 2 t square k, alright. So, this is the required equation of the vector of this is the required tangent vector dr dt.

Now, from here we can calculate, so from here we can calculate our how to say the equation of the normal plane in the vector form. So, capital R is our so capital R is basically capital X, capital Y, capital Z.

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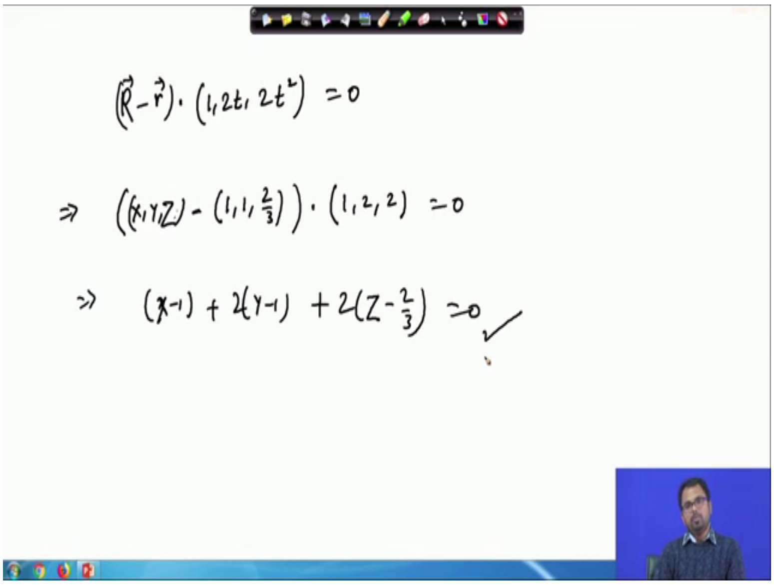

So, first of all we will write this thing here. So, dot product with dr dt, so dot product with dr dt, dr dt is 1, 2 t, 2 t square. So, this is 1, 2 t and 2 t square equals to 0, alright. And from here we can choose let us say any point where we want to calculate the normal vector.

So, suppose for t equals to 1, the point will be 1, 1 and 2 by 3, ok. So, let us calculate that normal plane at the point t equals to 1. So, at the point t equals to 1 the point p will be 1 1 and 2 by 3, so I can write this as capital R X comma Y comma Z minus the point will be 1, 1 and 2 by 3 and our derivative this how to say the dr dt will be 1, 2 and 2, alright equals to 0. So, from here I can write X minus 1 capital X minus 1 plus capital Y minus 1 times 2 plus capital Z minus 2 by 3 times also 2 equals to 0. So, these are all capitals. So, these are all capital Z, alright.

So, this is basically the equation of the tangent plane sorry equation of the normal plane at a point t equals to 1. So, that at t equals to 1 the point is 1, 1 2 by 3 and the derivative this dr dt is basically 1, 2, 2. So, just take the dot product and that will give you the required equation of the normal plane. So, it is really not complicated to calculate, we can leave it here or we can also express it in terms of the Cartesian coordinate system, so that is up to us, alright. So, this is how we calculate the normal plane at a certain point p to a curve.

Next like a unit tangent vector we will move to unit normal and binormal. So, what do we mean by these things? So, the thing is we are slowly moving towards Serret Frenet formula, alright.

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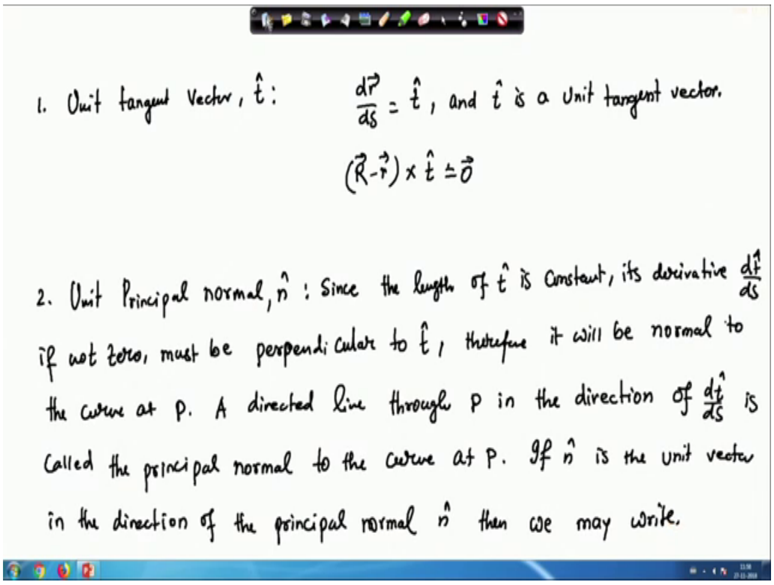

So, first of all we know unit tangent vector, right. So, we know unit tangent vector which is given by t cap. And we have shown that or we know that dr ds if we have a curve given in terms of the arc length then dr ds is basically t cap, and t cap is a unit vector or unit tangent vector, alright.

And we also know the equation of the tangent plane as capital R minus small r cross product with dr dt. So, dr dt can be replaced with, can be replaced with t cap because we divide dr dt by its magnitude and then we multiply it on the right-hand side, so that will be 0, so ultimately we will have a t cap here as well. So, this is not very complicated to see, alright.

And next we define the unit normal. So, unit principal normal; so, we have normal plain or normal now we are talking about unit normal n cap, alright. So, from here there is a small statement, so the statement is suppose this length of t, the length of t is constant, so since how to say the length of the tangent in this unit tangent vector is constant then in that case dt ds the its derivative will be 0, and if it is not 0 then in that case dt ds, if it is not 0 then in that case dt ds will be perpendicular to t, alright.

So, if this dt ds that the derivative of t with respect to s if it is not 0 then in that case it is perpendicular to t and therefore, it will be a normal at the point p. So, you have a tangent vector and if we differentiate the tangent vector and if dt ds is not equal to 0, then this dt ds will be actually perpendicular to the tangent vector t at the point p. So, what do we mean by this? Let me write this in terms of statement.

So, since the length of t is constant, its derivative dt ds if not 0, so if it is not 0 must be perpendicular to t, perpendicular to t, alright. And therefore, it will be a normal therefore, it will be normal to the curve at the point p. So, if it is not 0 then in that case this this dt ds will be perpendicular to the tangent to the tangent this t cap or unit tangent vector at the point p. And a directed line and a directed line through p, through p in the direction of in the direction of dt ds, in the direction of dt ds is called the principal normal to the curve at the point p.

So; that means, if dt ds is not 0 then it is perpendicular to the curve p to the curve at the point p and a directed line through p. So, if you have a directed line through p in the direction of dt ds, then that directed line will be called as a principal normal. So, and if n is the unit vector and if n is the unit vector so, we have a very we have a small formula. So, if n is the unit vector in the direction in the direction of the principal normal of the principal normal let us say n then we may write. So, then we may write dt ds is equal to kappa times n. So, this is basically kappa.

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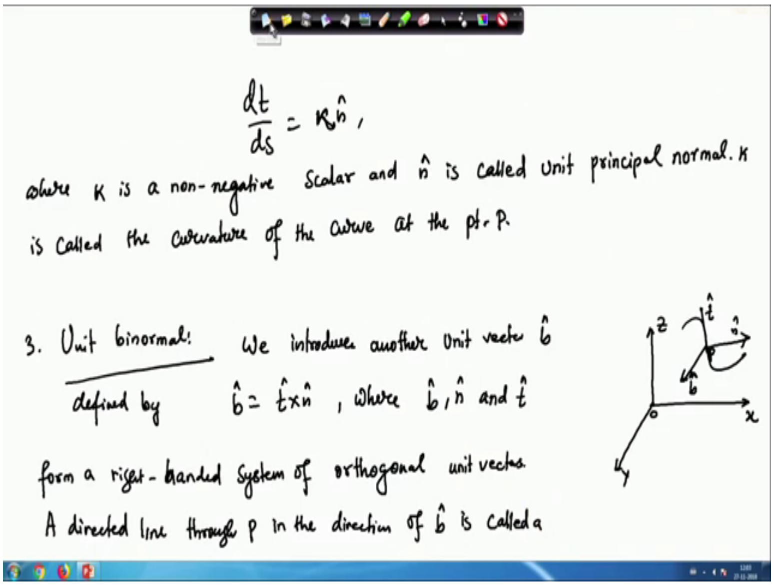

So, you see n is basically a vector or a unit vector in the direction of dt ds and this dt ds equals to kappa and this means that they are parallel to one another; that means, n is in the direction of dt ds and this n is called as the unit principal normal. So, here where K is a non-negative scalar and n is called unit normal a unit principle normal and kappa is called the curvature of the curve at the point p. So, this is not actually K, this is kappa but it is sort of like a curlier notation. So, I leave it up to you how do you want to denote this notation, but it is basically kappa. So, kappa is actually the curvature of the curve at the point p and dt ds is equals to kappa n is the required is the required a formula for dt ds and that is how it connects the unit principal normal with dt ds.

So, here we can see that the direction of the unit principal normal and dt ds are the same and if dt ds is perpendicular to the curve at the point p then in that case n is actually also perpendicular to the curve at the point p, and it is a unit vector, so it is basically unit principal normal alright. And this kappa is the curvature of the curve at the point p.

So, we have basically learnt about tangent vector, unit tangent vector and unit principal normal, alright. Now, what we can do, we can write the unit binormal. So, we know tangent vector, we know normal and now we learn about unit by normal. So, we introduce another unit vector b cap defined by defined by b cap equals to t cap cross n cap where b cap n cap and t cap form a right handed, a right handed system of orthogonal unit vectors. So, that means, if you have this is x, this is y and this is z, and let us say this is my curve then in that case and if this is the point p let us say.

So, I can write as this as I am not very good at drawing. So, this could be our t, this could be our n, and let us say b is in this direction so, b t; and b, t and n. So, that is actually our how to say, our required this right handed orthogonal unit vector. So, this is how b, t and n are connected, now to define it formally I write a directed line a directed line through p through p in the direction of b is called a binormal to the curve at the point p; and we.

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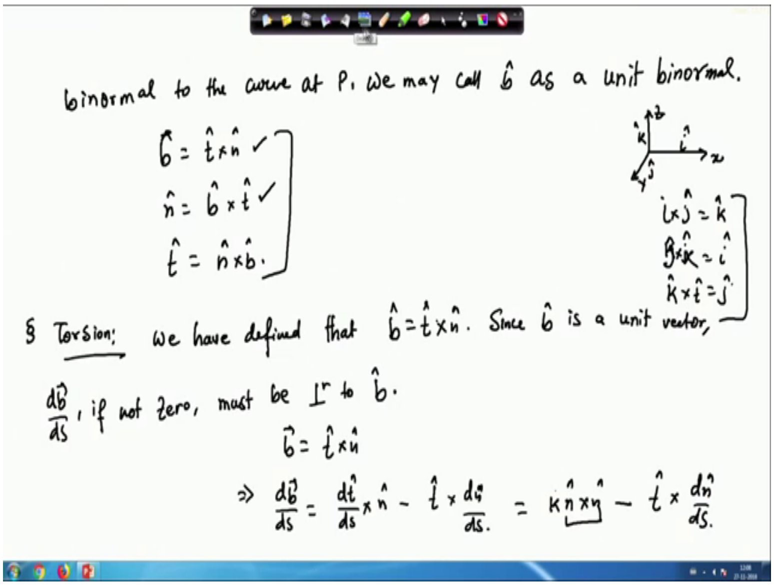

So, we may call b as a unit binormal. So, we may call b as a unit by normal. So, b is a unit vector and it is in the direction of; it is in the direction of. So, any any directed any directed line through the point p in the direction of b is called as the binormal and then b is called as a unit binormal or we call b as a unit binormal.

So that means, when we have this coordinate system y, x and z, and if we have i in this direction, j in this direction, and k in this direction then we know that i cross j equals to k, k cross j equals to i sorry j cross k equals to i, and k cross i equals to j. So, this we know for i, j and k. Similarly, since b, t, n, n forming they form how to say a right handed system for unit vectors we can be able to write b equals to b cap equals to t cross n then t cross, then n equals to b cross t and t equals to n cross b, alright. So, this is also true for these 3 vectors I can take b, n, n or I can take t, b and n then in that case it will yield a similar formula.

So, this is how normal unit normal and ten sorry binormal normal or unit binormal, unit principle normal and unit tangent vector are connected with one another. And this is a very important formula to remember because I mean sometimes you might be given tangent and normal and then you are asked to calculate binormal. So, we use the first formula. Similarly, you could be given I do not know binormal and tangent and you will be asked to calculate the normal, so we used it, we use this formula, alright.

So, now they are called and they are also called as moving tetrahedral. So, trihedral sorry not tetrahedral. So, they are called as moving trihedral. So, they are 3 in number, so they are called as trihedral, alright. And now there is something called torsion we will introduce the concept of for torsion now. So, what is torsion? So, we have tangent normal and binormal, now we will introduce the concept of torsion. So, an another topic.

So, why I am introducing all these things is because now I am slowly moving towards I Serret Frenet formula. So, we have these 3, now we will introduce the concept of torsion. So, in the above relation in the above relation we have defined that b is equals to t cross n. So, we have defined that, we have defined that b equals to t cross n, right and since b is a unit vector, b is a unit vector db ds if not 0 must be perpendicular, must be perpendicular to b. So, it is a unit vector; that means, its magnitude is 1, but if it is not 0, if it is not a 0 vector then in that case db ds the derivative of b with respect to arc length must be perpendicular to b, alright.

And here if I write b equals to t cross n and if I differentiate both sides with respect to s, then db ds would be dt ds cross product with n minus t cross product with dn ds, right. Now, dt ds is kappa n, alright. So, I can write here kappa n, sorry kappa is a scalar quantity kappa n cross n minus t cross product with dn ds. Now, n cross n is a 0 is a 0 vector because they are the I mean a cross a is 0. So, this is basically a 0 vector. So, k times 0 vector is again a 0 vector, now we subtract it, you add it in minus of t cross dn dn, dn ds then it will remain the same.

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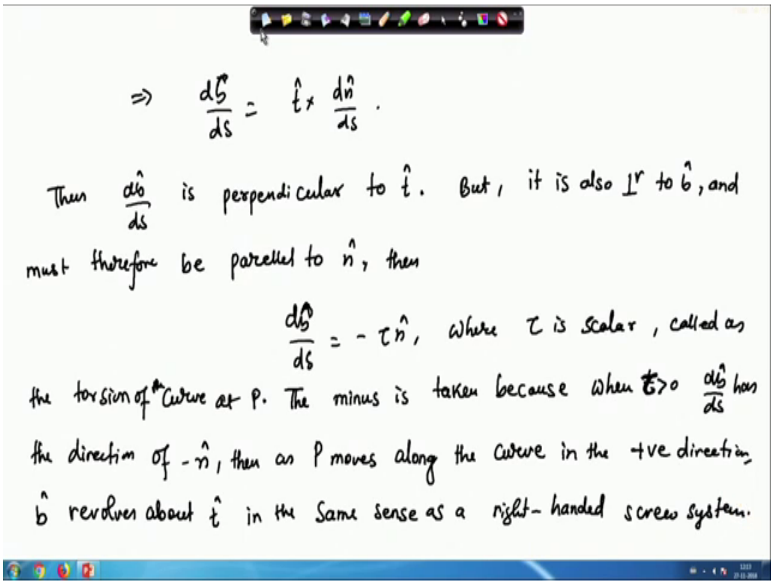

So, here we will obtain db ds equals to db ds equals to minus of t cross dn ds, alright. So, from here we can say that; so dt ds and why there is a minus sign. So, there should not be any minus sign. So, there is a plus yes. So, this is plus t cross product with dn ds and then we have this thing so, of course its plus.

Now, that means, from here we can write, so from here I can write this db ds is perpendicular to t because db ds equals to t cross dn ds. Now, db ds and dn ds they are along the same. So, dn this db ds and dn ds from here we can say that this db ds is actually perpendicular to this t cap from this formula. So, from this formula we can be able to say that db ds is perpendicular to t, but it is also perpendicular to b. So, it is also perpendicular to b, we have stated here.

So, it is perpendicular to b and it is perpendicular to t; that means, and must therefore, be parallel to n. So, you have a vector which is perpendicular to both b and t. So, if it is perpendicular to both b and t; that means, it must be along the direction of n because n is the vector which is perpendicular to t and b. So, the only possibility this db ds has is to become parallel with n. So, if it is parallel to n then I can write db ds equals to minus of tau n, and where tau is a scalar called as the torsion of the curve at the point p. And the minus sign in this case is taken for has, I mean this the minus sign has this purpose that I mean basically this minus sign is taken because, how do I write it in a nice word.

So, the minus sign is taken because when tau is positive sorry when tau is positive, so this is not t this is tau when tau is positive db ds has the direction of minus n. Then, as p moves along the curve in the positive direction, in the positive direction b revolves about t in the same sense as a right handed screw system, right handed screw system advancing in the direction of t.

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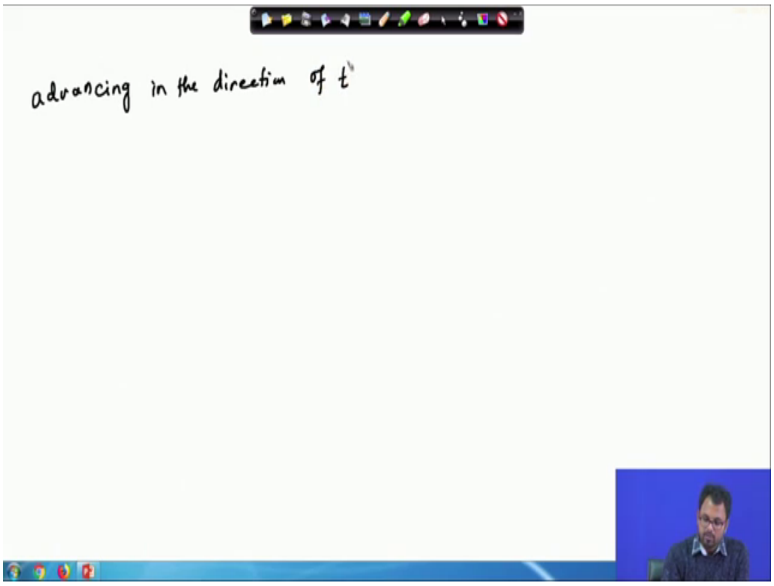

So, it is taken purposely because in the direction of t it is taken purposely, so that it maintains that perpendicularity condition that if you have db ds as minus of n. So, then in that case it will be along the curve in the positive, so it will then as the point p moves along the curve then b will revolves about t in the same sense as a right handed screw system. So, it will then form a right handed screw system like we stated earlier. So, that is why we have taken a minus sign, alright. So, we have minus db ds is equals to minus of tau n and therefore, to summarize we have derived several types of formula.

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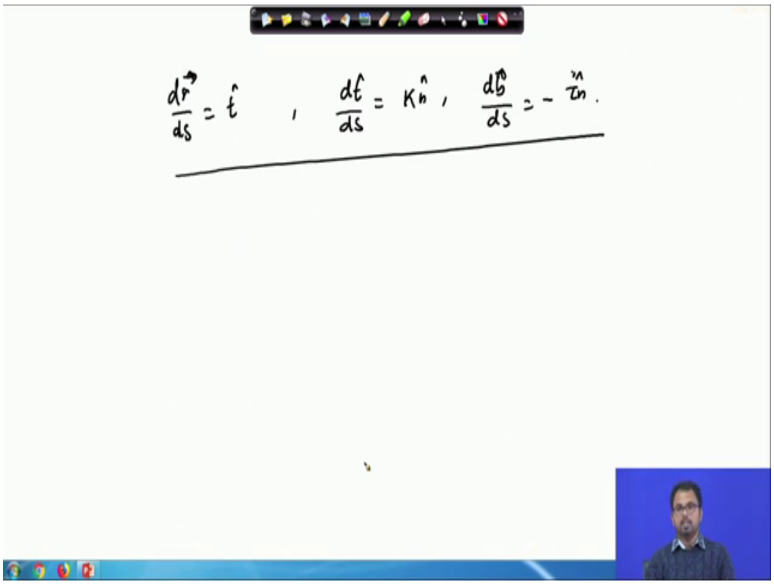

So, first of all we have derived dr ds equals to, dr ds equals to t cap, alright, then we derived and dr ds equals 2 t cap, then we derived dt ds equals to kappa n, and then we derived db ds equals to minus of tau n, alright tau n. So, these are the 3 important formulas that we derived today. In the next class we will finally, be able to derive this Serret Frenet formula. We will also work out few examples just to see why we need these formulas. So, today we will stop here, and I thank you for your attention and then we will continue with our same topic in the next class.

Thank you.