Integral and Vector Calculus
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Lecture – 45
Equation of Tangent, Unit Tangent Vector

Hello, students. So, up until last class we looked into the topics of tangents and we also worked out examples based on a gradient divergence curve, we calculated directional derivative and things like that. And in the last class, we started looking into more like differential geometry aspect of vector calculus. So, like tangent, normal, by normal and then we will look into the Serret-Frenet formula.

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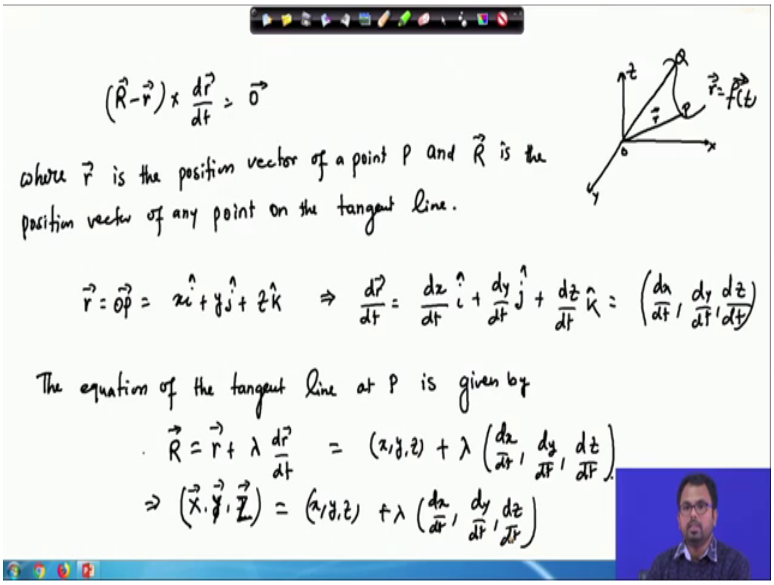

So, in the last class we derived a formula for the tangent as R minus small r dot product with dot sorry cross product with dr dt. So, cross product with dr dt is equals to a 0 vector where our r is the position vector; is the position vector of point P.

So, if we have a curve like; if you have a curve like let us say in three-dimensional geometry this is x-axis, this is our y-axis, this is z-axis, that is origin. So, if you have a curve like this. So, let us say this is my point P and this is my point Q and the position vector of the point P is r. So, r is the position vector of a point P and the capital R is the position vector of any point and capital R is the position vector of any point on the tangent line on the tangent line and the equation of the curve is of course, given by r is equals to f t.

So, if you want to calculate r is equals to f t. So, if you want to calculate the equation of the; equation of the tangent line then in that case we calculate the giver for the given curve we calculate dr dt and then taking the difference of capital R which is the position vector of any arbitrary point with small r and then cross product for dr dt will give us the required equation of the tangent to that curve r is equals to f t.

Now, we can also express this equation in terms of a Cartesian coordinate system. So, we are familiar with the equation of the tangent in three dimensional geometry to a surface and so, we can actually express this equation in terms of Cartesian coordinate system as well. So, how do we do that?. So, we can write r as position vector of OP as x i; so, I have told you at very beginning that when we choose r, r is always x, y, z, so, x i plus y plus z that came.

So, let us write that and then from here I can calculate dr dt as dx dt. So, x, y and z are the functions of t and i dy dt j plus dz dt k or I can also write dx dt comma dy dt, comma dz dt. So, of course, if you can either it is up to you whether you write in terms of i, j and k or you write in terms of as a triplet, alright.

So, the equation of the tangent line; of the tangent line at the point P is given by. So, if you remember the last in the last class we derived something like R capital R is equals to small r plus lambda times dr dt. So, small r is basically our x, y z plus dr d lambda times dr dt is dx dt dy dt and dz dt. So, from here I can write capital R as x, y, z. So, I can write capital R as a capital X capital Y and capital Z, alright equals to small x, y, z plus lambda dx dt, dy dt and dz dt.

Now, from here I will take this triplet on the left hand side and then we will equate the coefficients of i, j and k from both sides.

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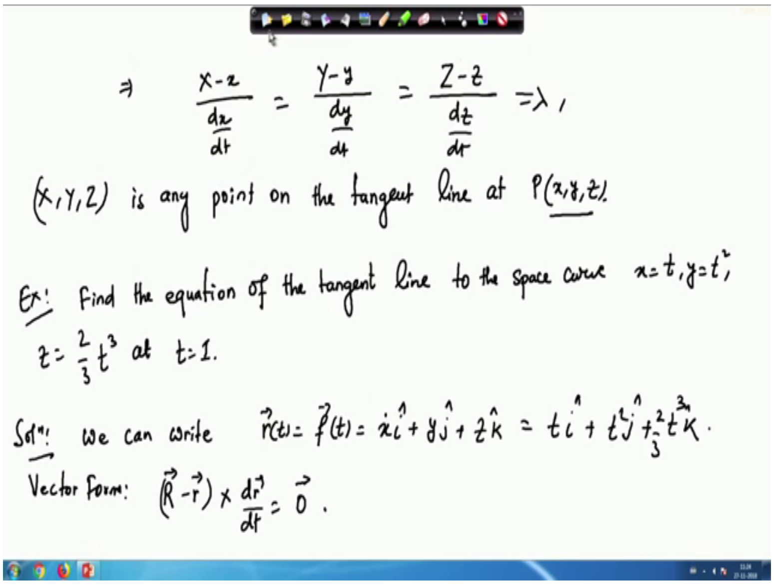

So, ultimately we will obtain X minus small x by dx dt Y minus small y by dy dt and Z minus sorry small z. So, this should be capital Z minus small z by dz dt equals to lambda and here capital X, capital Y and capital Z is any point; is any point on the tangent line; on the tangent line at small p x, y, z.

So, this is the required equation of the tangent line in terms of Cartesian coordinate system and you may have seen this type of equation in three-dimensional geometry for the equation of a line in 3D. So, this is just an another form in terms of a Cartesian coordinate system and for the vector form we know its capital R minus small r cross product with dr dt.

So, now let us work out an example where we see how we can calculate the tangent line at a given point. So, example find the equation; find the equation of the tangent line; of the tangent line to the space curve x equals to t, y equals to t square and z equals to 2 by 3 t cube or we can write r is equal to or do not write or we will do something about it at the point t equals to 1, right. So, we have to find the equation of the tangent line to the space curve given by these equations at the point t equals to 1.

So, first of all we can write, so, we can write r as of course, its a function of t equals to f t equals to x t i, y t j and z t k. So, instead of x i can write or let us write x, y, and z and now I can substitute x as t i, y as t square j and z as 2 by 3 t to the power k right. And now that we have this equation of the curve in terms of vector as a vector form I can calculate the; I can calculate the equation of the tangent at the point P or at a point when where t equals to 1, first in terms of the vector and the second we will calculate as a Cartesian form.

So, for the vector form capital R minus small r cross dr dt must be 0. So, that is the vector form equation for the tangent line. So, first of all we need to calculate dr dt. So, what is our dr dt?

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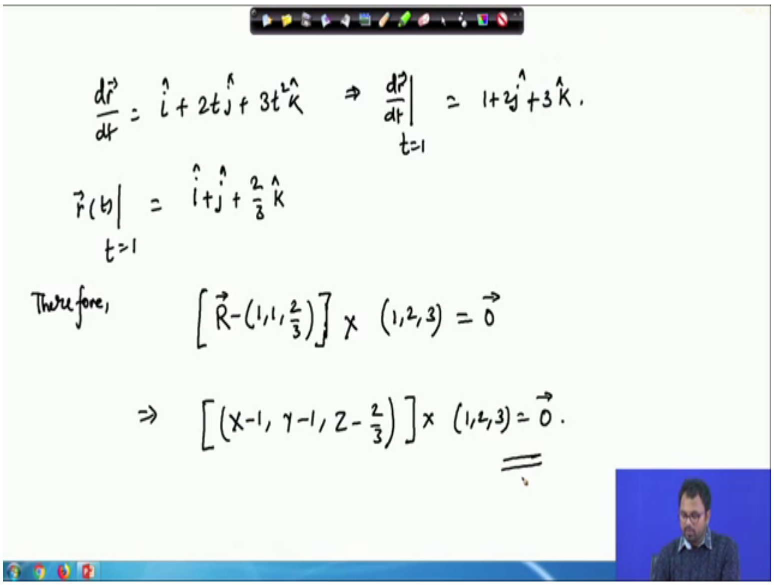

So, if we like to calculate dr dt then it will be i because the derivative of t is one then it will be 2tj and then 3t square k. So, from here dr dt at t equals to 1 will be 1 plus 2j plus 3k, right. So, that is the tangent and at the point t equals to 1, r t will be; so, r t at t equals to 1; at t equals to 1 will be it is 1 plus sorry 1. So, this is i plus j plus 2 by 3 K. Therefore, the required vector equation will be capital R so, capital R minus the vector. So, this is 1, 1, 2 by 3 cross product with. So, let us write the cross product here cross product with dr dt. So, dr dt is 1, 2 and 3 equals to 0 vector.

So, if we can be if you want we can write this capital R as a capital X, capital Y, capital J. So, this is basically capital X minus 1 capital Y minus 1 capital Z minus 2 by 3 and cross product with 1, 2 and 3 equals to 0 vector. So, this is the required equation of the; equation of the tangent line at the point P to the space curve given by that equation r t equals to; r t equals to this equation and that is basically cross product with the tangent dr dt equals to the 0 vector. So, this is how we calculate the equation of the tangent line in terms of the in the vector form.

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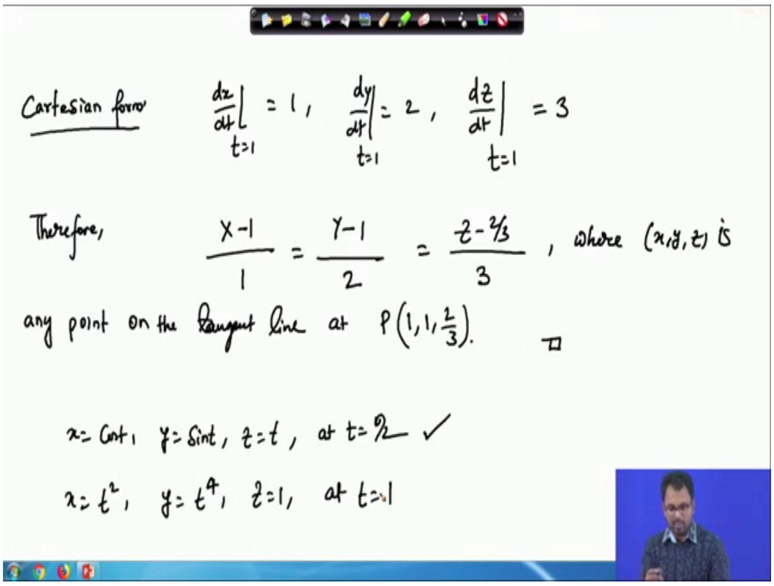

Now, we can also calculate it in Cartesian form. So, Cartesian form would be Cartesian form. So, in the Cartesian form we have to substitute t equals to 1 in dr dt, then from there dx dt at t equals to 1 would be 1, dy dt at t equals to 1 would be 2 and dz dt at t equals to 1 would be 3, right and therefore, the required therefore, the required equation would be therefore, the required equation would be X minus; X minus small x small x is as the point P.

So, that is 1 divided by dx dt which is again one and then we have Y minus small y, so, that is again 1 which is divided by dy dt which is 2 and Z minus 2 by 3 divided by 3 and where x, y, z is any point is any point on the tangent line on the tangent line at P 1, 1, 2 by 3.

So, at t equals to 1 the point P is given by 1, 1, 2 by 3 and that X, Y, Z is any point on the tangent line at the point this. So, basically which was X Y Z as 1, 1 2 by 3 and that is the required and then we substitute for X Y Z here. So, this is the required equation of the tangent plane at the point P and of course, this is in the Cartesian form. So, either you can express it in terms of the vector form or in terms of the Cartesian form that is up to you and yeah similarly we can find a tangent equation of the tangent line for any arbitrary curve. We can have a curve something like x equals to. So, this example ends here ah, but we can have something like x equals to cos t, y equals to sin t and z equals to t and let us say we want to find the equation of the tangent line at the point t equals to pi by 2.

So, we will follow the similar steps and then use the same formula for the tangent line and you can be able to calculate the tangent line in the vector form or in the Cartesian form. You can also have something like x equals to t square, y equals to t the power 4 and z equals to 1 and you are asked to calculate the equation of the tangent line at t equals to let us say 1. So, just follow the similar formula what we did before and that will give you the required equation of the tangent line either in vector or in the Cartesian form.

So, the examples would not be that much complicated when you are asked to calculate the tangent line for a given curve it might involve some doing some derivative which may have some higher powers or something like that, but pretty sure you can be able to do it. So, now we move to our move to our how to say a little bit more into differential geometry aspect, but it is not too much differential geometry ah. It is somehow involving our tangents and normal and the Serret-Frenet formula. So, keeping those in mind we have to study these concepts. So, let us let me introduce that via a small statement.

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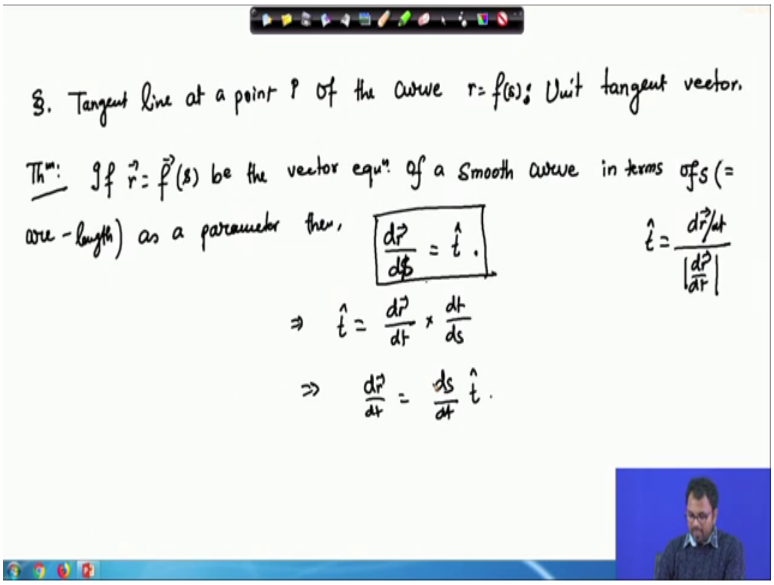

So, basically now we are going to study about tangent line at a point P of the curve r equals to f s or basically we are interested in unit tangent vector. So, s is the arc length basically. So, s that t is the parameter, but when we are using s its pretty much understood that s denotes the arc length. So, suppose you have a given equation of a curve in terms of arc length and from there how we define the unit tangent vector or first of all the tangent afterwards the unit tangent vector we will see that here alright.

So, I am going to state a very important theorem, but we will avoid the proof because the proof is slightly lengthy and it is out of the scope of this course, but interested readers they can look into the books that have suggested in the references for the proof, alright. So, the statement goes like this. So, if vector r equals to of course, vector f s be the vector equation of a smooth curve of a smooth curve in terms of; in terms of s which is basically arc length which is basically the arc length.

So, s as a parameter then dr ds equals to the unit tangent vector. So, remember dr dt is I mean in our previous class we learnt the dr dt is actually the tangent vector in a way. So, it with actually gives us how to say that I have the tangent vector on that curve at a certain point.

Now, if we want to calculate the unit tangent vector technically if you want to calculate the unit tangent vector then we have to divide dr dt with its magnitude right. So, dr dt denotes the tangent to the curve r is equals to f t. So, if we want to calculate the unit tangent vector we must divide it divide dr dt by its magnitude, but this theorem says that if we have an equation r is equals to f s of a smooth curve equation of a smooth curve then the unit tangent vector will be given by dr ds. So, if we differentiate r with respect to the arc length as a parameter then in that case the differentiation the first order differentiation will give us the unit tangent vector.

And therefore, from here what we can write is so, from here we can write that dr ds. So, from here this is basically the unit tangent vector and from here I can write unit tangent vector equals to dr dt times dt ds. So, basically dr dt is actually ds dt times unit tangent vector, alright. So, this is another result. So, that means, our dr dt the original how to set a tangent vector which was not a unit vector can be given by ds dt differentiation of this arc length s with respect to t times the unit tangent vector. So, this is a very how to say a nice result to remember that the differentiation of the curve with respect to the arc length is the unit tangent vector.

Now, from here this formula actually gives us the equation of the tangent vector where the equation of the curve is given in terms of s. So, here we see that the equation of the curve is given in terms of s and in this case and basically the tangent vector can be calculated at ds dt times the unit tangent vector t, alright. So, let us see how we can calculate this unit tangent vector capital unit tangent vector t cap. So, I can start with an example.

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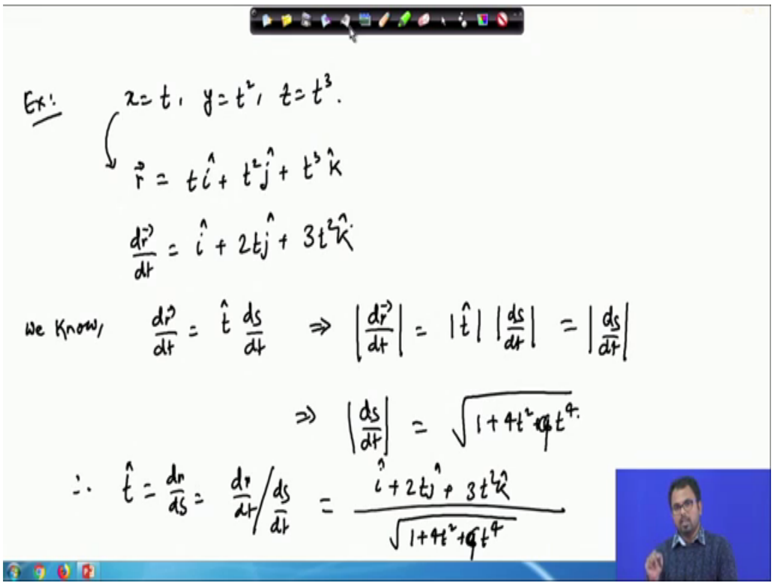

So, first of all suppose the given equation of the curve is x equals to t, y equals to t square and z equals to t cube. So, that is the given equation of the curve. We have to find the equation of a unit tangent vector. So, either we can calculate dr dt and then divide it with it is with its magnitude or we can use that previous formula. So, I can write from here I can write r is equals to t i, t square j plus t cube K. So, this I can always write because I can write x i, y j, z K.

Now, from here my dr dt is basically i plus 2tj plus 3t square K. So, that is my dr dt, alright and we know that; we know that from previous formula dr dt is basically t cap ds dt, alright. So, ds dt times t cap, so, that I have written here. Therefore, from here dr dt mod is equals to mod of t cap mod of ds dt is it not? Now, mod of t cap is one because t is a unit tangent vector. So, it is magnitude is always 1, so, we have mod of ds dt.

And therefore, mod of ds dt and therefore, from here mod of ds dt is nothing, but mod of dr dt and that is basically 1 plus 4t square plus 6t to the power 4, is it not? And now, I can write this here as t cap. So, the tangent vector is t cap dr ds and dr ds can be written as dr dt times divided by ds dt because this can be written as dt ds. I will bring it in the denominator and then it will become ds dt and if it is in denominator then ds dt is this here.

And therefore, dr dt would be ds dt is basically the mod or the positive value. So, it will always remain positive and dr dt would be i plus 2tj plus 3t square K and mod of ds dt; that means, taking only the positive value 4t square plus 6t to the power 4. So, this is our required how to say answer or the value of the unit tangent vector to the given curve this here. Of course, as I said we could have divided this weight it is not 6 its 9 actually yeah 9.

So, we could have divided this dr dt with its magnitude and that would have given us the unit tangent vector, but in case if you did not have t and if you had arc length as x equals to let us say some s, y equals to some s to the power 3 or something; so, that means, the given equation is in terms of the arc length and then we had to use a different formula to calculate the unit tangent vector. And if you have something in terms of s then basically the derivative of that r with respect to s would give you the unit tangent vector.

So, the unit tangent vector the curve and the arc length is related with that relation that t cap equals to dr ds and its also a very important relation to remember alright.

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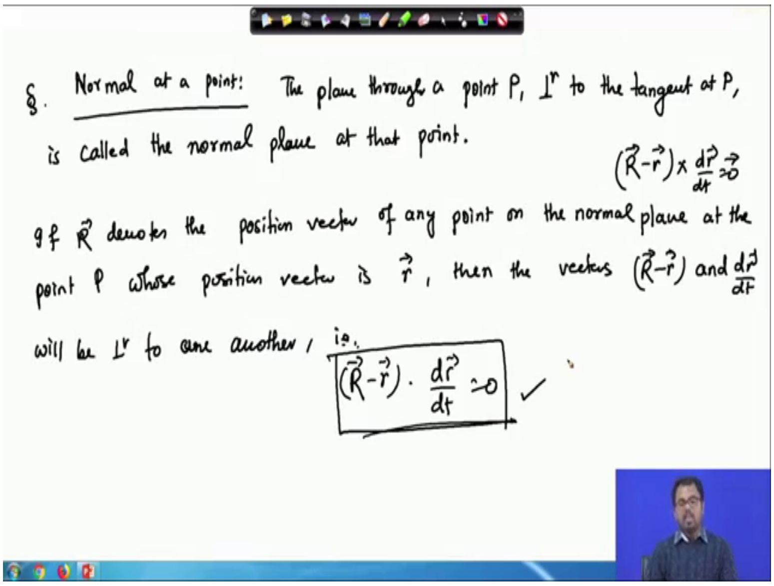

And now, that we have introduced the concept of how to say in a unit tangent vector we can move to our next topic which is normal at a point. So, normal or normal plane at a point so, normal or normal plane at a point; so, what is the definition? The definition is the plane through a point through a point P the plane through a point P perpendicular to the tangent at P is called the normal plane at that point, alright. So, usually that is what we mean by normal.

So, you have a curve and then you have a tangent at a certain point p. So, a line which is perpendicular to the tangent at that particular point is called as the normal to the curve at that point. So, usually a normal plane is like a generalization of the normal to a curve. So, if you have a plane that is passing through a point P and if it is perpendicular to the tangent plane at the point P, then that plane is called as a normal plane to the to the given surface or in this case to the given curve at that particular point P alright.

So, let us see how we can express its equation so, we know that; so, we know that the equation of the tangent plane is given by capital R minus small r. So, this we know capital R minus a small r dr dt equals to 0. So, this means that these two vectors are parallel, right. So, dr dt and capital R minus small r are parallel to one another and therefore, their cross product is 0. So, if I write capital R minus small r dot product with dr dt equals to 0, then capital R minus small r is perpendicular to dr dt and that will actually help us to obtain the equation of the normal plane.

So, if R denotes if R denotes the position vector the position vector of any point on the normal and the normal plane at the point P whose position vector is small r position vector is small r then the vector then the vectors capital R minus a small r and dr dt will be perpendicular to one another; that is capital R minus small r dot product with dr dt equals to 0.

So, dr dt is basically the tangent vector and if we choose capital R as any point on the normal which is normal at a point P then in that case capital R minus small r will be perpendicular to dr dt. Obviously and therefore, that equation capital R minus small r dot product with dr dt is basically the equation of the normal plane because dr dt and capital R minus small r are perpendicular to one another and that is actually what normal plane mean that a plane which is perpendicular to tangent vector is actually a normal plane at a point P to that curve. So, this equation is actually the equation of that normal plane at a point P to the curve r is equals to f t, alright. So, this is the equation of our normal plane.

So, in today's lecture we saw how we derive the equation of a tangent plane at a point P we worked out few examples we also gave the concept of a unit tangent vector and we derived the normal at a point P to a given curve. In the next class we will start working with some examples and I will also try to introduce the concept of unit normal and if time permits the concept of binomial. So, I will stop here for today and I will continue with our next topic in our next class.

Thank you.