Integral and Vector Calculus
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Lecture – 44
Application to Mechanics

Hello students, so, in the last class we were solving some examples based on level surfaces and directional derivative. So, in the level surfaces we sort of tried to cover a few examples where we can calculate the normal tangent plane and things like that. So, I did try to show you 1 or 2 more 1 or 2 examples and the rest of the problems can be solved in the similar fashion.

So, you may try to look into some books which I have listed in the references and I try to solve some examples. We will also try to include some problems in your assignment sheet. So, that you can be able to practice; practice them and we also solved few examples on directional derivatives.

So, today I will solve one more example on directional derivative just to make the concept clear and then we will move on to our next topic which is basically the Application of Vector Calculus in applied mathematics or in Mechanics. So, that would also be an interesting thing to study in this context all right.

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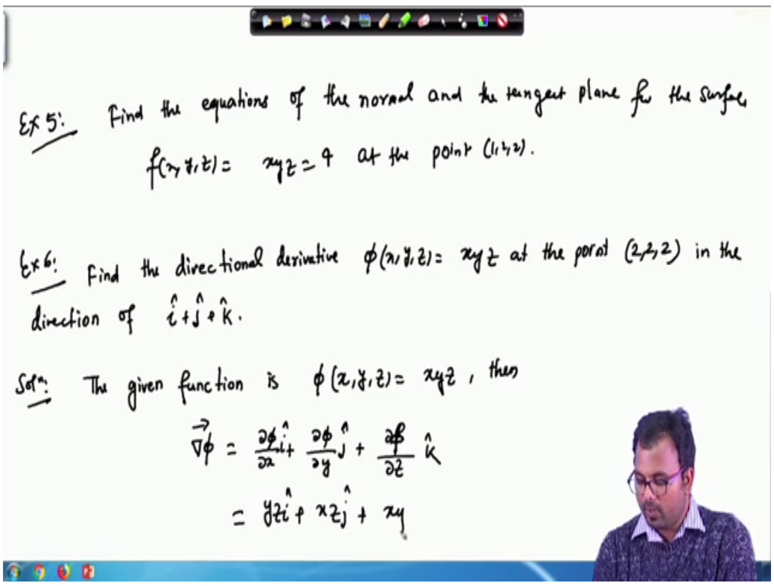

So, let us start with our very last example on the directional derivative chapter. So, find the directional derivative; directional derivative for the function phi x y z equals to x y z at the point 2 2 2 in the direction of; in the direction of i plus j plus k all right. So, the solution. So, first of all we know that when we are given a scalar function phi x y z or f x y z. So, the very first thing that we do is to calculate the gradient of phi or gradient of f.

So, the given function is phi x y z, then gradient of phi is del phi del x times i del phi del y times j and del phi del phi del z times k. We are used to write f, so, that is why sort of accidentally wrote f, but its phi actually. Now del phi del x is y z i del phi del y is x z j and del f del z is x y k all right.

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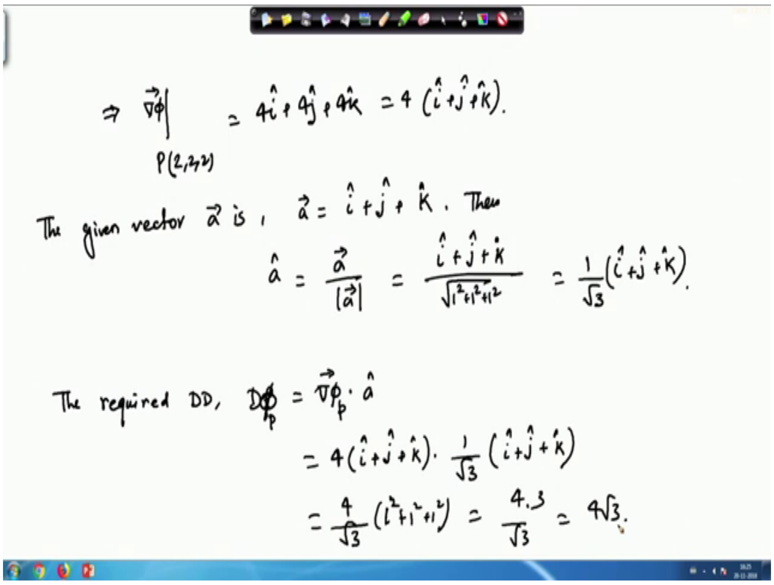

So, then the gradient of phi at the point p the gradient of phi at the point P which is 2 2 2 equals to 4 i plus 4 j plus 4 k, so, I take 4 common and then this will be i plus j plus k all right. Now, the given direction vector the given vector or I also prefer to call it as direction vector, because we have to calculate the directional derivative along the direction of this vector ok. So, the given vector a is equals to what do we have? We have i plus j plus k.

So, then our a cap would be vector a divided by mod of a and then this will be i plus j plus k and then square root of 1 square plus 1 square plus 1 square. So, ultimately 1 by square root of 3 I plus j plus k all right.

So, the required direction derivative; so, the required directional derivative or we can write it as Df at the point P is equals to gradient of phi sorry not Df in this case it is D phi, excuse me. So, this is the gradient of phi at the point p dot product with a cap. So, what is my gradient of phi at the point p? Its 4 times i plus j plus k dot product with 1 by square root of 3 i plus j plus k and now this will be 4 by square root of 3 and this will be i square, so, 1 square plus 1 square plus 1 square.

Basically, the dot product and then it will be 3. So, 4 times 3 divided by square root of 3. So, it is ultimately 4 is square root of 3. So, that is the required directional derivative of the given scalar function phi in this case in the direction of the vector i plus j plus k. And we just followed the traditional how to say method to calculate the directional derivative and this is how we obtain the directional derivative of the function phi all right.

So, I will stop with the examples on directional derivative, because we have we I have tried to cover as many examples as possible. And now we move on to our next topic, which are basically an application or some applications not an application, but some applications of vector calculus in mechanics in a way and in elementary differential geometry and in mechanics.

So, we now learn; we will now learn the concepts of a tangent normal binormal, there is a very nice formula called as Serret Frenet formula that shows that how you connect in vector calculus the normal, the tangent and the binormal on of a curve at a certain point P. So, we will now move on to those topics and today we will start with elementary how to say difference as geometric concepts and then we move to those tangent normal and binormal concepts all right.

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So, basically so, we will start with today with this Serret; tangent, normal, binormal, Serret-Frenet formula all right. So, this is what we start with today. So, before we start with tangent, normal and binormal we give some basic definitions and how to say a idea of what do we mean by a parametric representation of a curve, how do we define a curve in a space and then we slowly move on to these topics all right. So, let me go back to my notepad ok.

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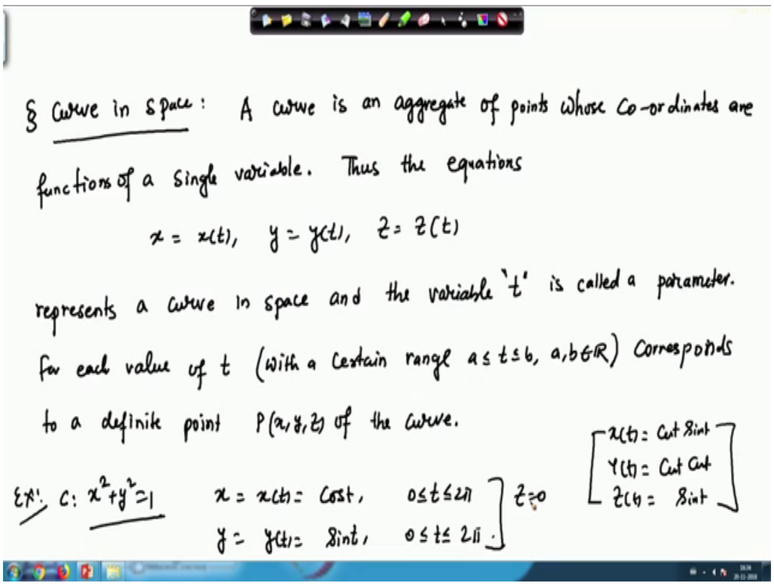

So, now, what do we mean by a curve in space? So, a curve in space; so, a curve a formal definition goes like this; basically, a curve is an aggregate; is an aggregate of points whose co-ordinates are functions of a single variable.

So, thus the equation; thus the equations x equals to x t, y equals to y t and z equals to z t; represents a curve in space and the variable; and the variable t is called a parameter and for each value; for each value of t with a certain range. So, t has a certain range let us say, a less or equal to t less or equal to b, where a and b are both real numbers corresponds for each value of t there corresponds to a definite point P x y z of the curve right.

So, for example, if I want to write the how to say, let us say this is our curve in 2 dimensional space; here they are talking about 3 dimensional space I am just for the sake of explanation I am just taking this example. So, let us assume that our curve C is basically this circle. So, how do we write this curve in space using this formula? So, I can write x equals to x t. So, x t would be let us say cos t and y equals to y t and y t would be sin t and I can choose t between 0 to 2 pi all right.

So, if I choose t equals to 0, then we have x equals 2 cos 0 is 0 cos 0 is 1 and sin 0 is 0. So, basically 1 0; obviously, this 1 0 point lies on this circle, I can choose t equals to pi by 2 and then in that case cos pi by 2 is 0, sin pi by 2 is 1, then the point 0 and 1 lies on the circle and so on. So that means, for every value of t we get a unique point P. So, well not unique, but there exists for every value of t there corresponds a point P on the curve because for t equals to 2 pi we get the same point actually. So, for t equals to 2 pi and t equals to 0 we are getting the same point.

So, for every value of t, there corresponds a definite point. So, we must have a point on that curve and that point on the curve is obtained for that particular; for a certain particular for a certain value of the parameter t. So, for every t; so, t has a range and for every t will obtain a point on that given curve. And that curve is basically called as a curve in space. We can also have a sphere x square plus y square plus z square equals to 1 and then this equation would change. It would be cos t, sin t and then cos t.

So, cos t cos t; cos t sin t and then again, so, basically we have to use 2 different variables. So, then in that case it will be cos t and then some other variable and. So, for a sphere it will be cos t sin t and then we will have cos t cos t and then.

So, you basically you got the idea how; how you formulate. So, x t if it is a sphere then we will have cos t sin t and then y t would be cos t cos t and then z t would be just sin t. So, then in that case cos squared is so, that will be one. So, again for a sphere you can give the formula in this fashion. So, that is one way to define a curve in space, if the given curve is actually a sphere and the parametric. This is also called as the parametric representation is given by if it is a sphere, then it will be given in this fashion, if it is a circle then is given in this fashion all right.

Now, and if we want to have the circle as a curve in space, let us say in 3 d then we write it as x equals to cos t pi equals to sin t and we put z equals to 0. Now we have a 3 dimensional representation. So, this is again a curve in space where z equals to 0. So, these are all tricks basically. So, we have a 2 dimensional circle and if you want to represent it in a 3 dimensional sense then basically the z component is 0.

So, you just write z equals to 0 and x equals to cos t y equals to sin t and that is your curve in space whose z component is 0. So, that is how we write this curve in space all right. Now, that we have a curve in space, we can actually be able to write, we can actually be able to write. So, let me put it in a nice sentence.

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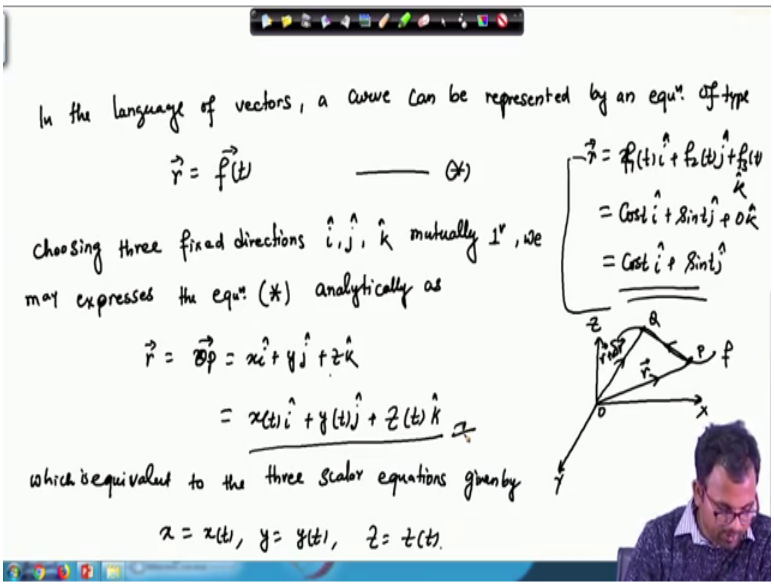

So, in the language of in the language of vectors a curve can be represented; can be represented by an equation of type r equals to f t. So, basically the equation of the curve in space can be written as r equals to f t. So, let us take our circle case. So, for example, for that circle case I can write r is equals to f t and f has 3 components. So, I can write f 1 t i f 2 t j and f 3 t k. So, f 1 t is cos t i f 2 t is sin t j and f 3 t is 0. So, basically we have cos t i plus sin t j. This is actually a function of t and that is why we are writing f of t. So, f is a vector function of a scalar variability.

So, you see our circle can also be written as r is equals to f t. So, in the language of vectors actually, a curve can be represented by an equation r is equals to f t all right. Where r is equals to actually the vector the vector OP. So, let me draw a figure. So, if I draw a figure I can be able to write it as. So, this is my x axis, this is my y axis, this is origin this is z and suppose this is our curve f all right.

So, this is my curve f, and suppose this is the point P x y z this is my point Q x plus delta x y plus delta y and z plus delta z. So, I am going to make them worse and this one is another vector. So, there is our vector O Q, O P and then this is our vector P to Q all right. Now this is r and this is r plus delta r plus delta r all right.

Now, choosing a 3 fixed directions now, I am just trying to put it in a nice word. So, choosing 3 fixed; choosing 3 fixed directions i j and k mutually perpendicular to one another mutually perpendicular; that means, i is perpendicular to j j is perpendicular to k and then k is perpendicular to y. So, they are mutually perpendicular to one another and mutually perpendicular, we may be able to write, we may express the equation. Let us say, this is our equation star the equation star analytically as analytically as r equals to x.

So, r is basically this vector which is basically OP and OP is my x y z. So, x i plus y j plus z k, but since x y z are all functions of t. So, this is nothing, but x t i plus y t j plus z t k right. And this is which is equivalent which is equivalent to the 3 scalar equations given by x equals to x t y equals to y t and z equals to z t the equation from which we started originally. So, this is the scalar equation for a curve in space.

Now, for every point on that curve in space we are associating a vector, let us say P is any arbitrary point on that curve then from origin we are associating a vector OP and that is given as r; r is the position vector of a point P on that curve. And basically, r can be with the help of i j and k, r can be written as x i plus y j plus z k, but x y and z are all functions of t. So, we write x t i y t j and z t k and this is in a way how to say a way to write a curve in terms of vector and that curve is actually a curve in space.

So, just using this r is equals to f t, where f t has 3 components x t y t and z t, we are how to say transforming that scalar representation we are sort of finding an alternative way to write that scalar representation in a vector form. So, basically instead of writing x t x equals to x t y equals to y t and z equals to z t, we are just writing r is equals to which is a function of t x t i y t j of y t j plus z t k. And that represents that vector representation r t equals to that expression is same as the scalar equation.

So, that scalar equation and that vector equation both are the same thing it is just that its an alternative way to write the same curve in space. And based on that now we introduce several concepts like if we had a scalar equation x equals to x t y equals to y t and z goes to z t since it is a curve in space we can about tangent, we can talk about normal.

Similarly, for this vector representation the way we have did the vector representation that is x t i y t j and z t k. We can still be able to talk about tangent normal and some other things as well; we will see what are those things. So, its not completely different its the same thing, is just that the way we are expressing it using the vectors is slightly different, but it is convenient its very convenient to write the equation of a curve in space in a vector form than writing it in a scalar form. And we will see what are its benefits over the time. And now, as I was saying that we can actually be able to define the tangent and normal. So, we will start with the definition of this tangent how do we define the tangent for the curve in space given by this vector form all right.

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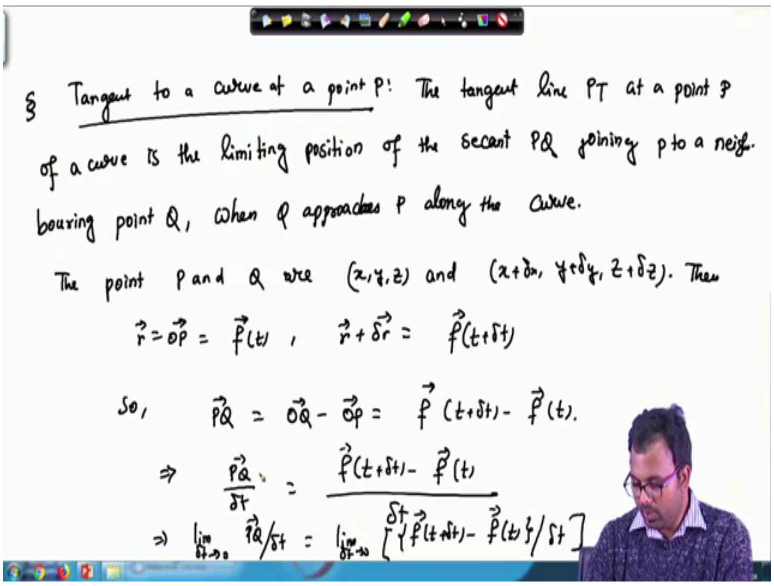

So, let us start, tangent to a curve at a point P. So, we refer to the same figure and when Q tends to P. So, when Q tends to P, then this will actually be how to say along the direction of a tangent to this curve all right. So, the formal definition would be the tangent line PT at a point P; at a point P of a curve is the limiting position; is the limiting position of the secant PQ joining P to a neighbouring point to a neighbouring point Q, when Q approaches P along the curve.

So, this is we know already from our previous topics. So, when it is basically the tangent is nothing but a limiting position. So; that means, when Q tends to P this will actually be along the along the direction of tangent or this is actually a tangent in a way not along the, but it is exactly the tangent. So, when Q approaches P, then it will not how to say go through this curve, it will actually be a tangent. So, it will actually be touching the curve and that is that will happen when we are making Q going to P. So, it is basically a limiting approach all right or a limiting case.

So; that means, how do we; how do we define the tangent? Basically, in terms of vector. So, we first saw that; so, how do we define the tangent? So, to define the tangent in terms of vector we take two points P and Q which we have already done. So, the points P and Q are basically x y z and x plus delta x y plus delta y and z plus delta z. So, this one is for the vector r and this one is r plus delta r all right.

So, now then our vector r is OP all right and OP is f t. So, that is how we are giving the equation and the point for the point Q r plus delta r is the position of the point P at a time t plus delta t. So, at time t it is simply x y z and at time t plus delta t it is now x plus delta x y plus delta y z plus delta z so, that is the point Q. And in terms of the function f, I can be able to write t plus delta t. So, at the time t plus delta t the point has moved along the curve to the point Q and it is given as this way all right.

So, our curve our line P to Q can be given by OQ minus OP and this is nothing but f t plus delta t minus f t all right and we divide by delta t. So, then this will imply PQ by delta t is equals to f dash sorry f of t plus delta t minus f t by delta t right. And now we take delta t goes to 0 limit delta t goes to 0. So, this is P to Q divided by delta t and then this is again limit delta t goes to 0 and this whole expression. So, f of t plus delta t minus f t both are vector and then both divided by delta t all right. So, we will have this thing here. Now, this quantity P to Q is nothing but that delta r right, so, P to Q is nothing but delta r.

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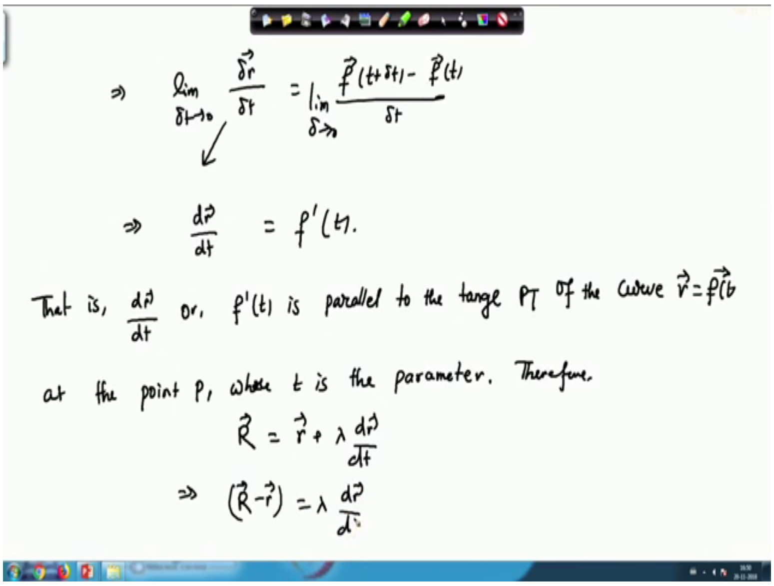

So, I can write limit delta t going to 0, delta r by delta t is equals to f of t plus delta t minus f t divided by delta t and then limit delta t goes to 0. So, this is nothing but our dr dt. So, when delta t goes to 0 delta r by delta t will go to dr dt. So, this is dr dt and this is nothing but the derivative of the vector function f. So, this we already know from the differentiation. So, this is the derivative of the vector function f.

So that means that is all that is dr dt or f dash t is parallel to the tangent PT right of the curve r is equals to f t at the point P, where t is the parameter. And therefore, capital R is equals to small r plus lambda times dr dt. So, capital R can be written as capital R minus small r is equals to lambda times dr dt, where.

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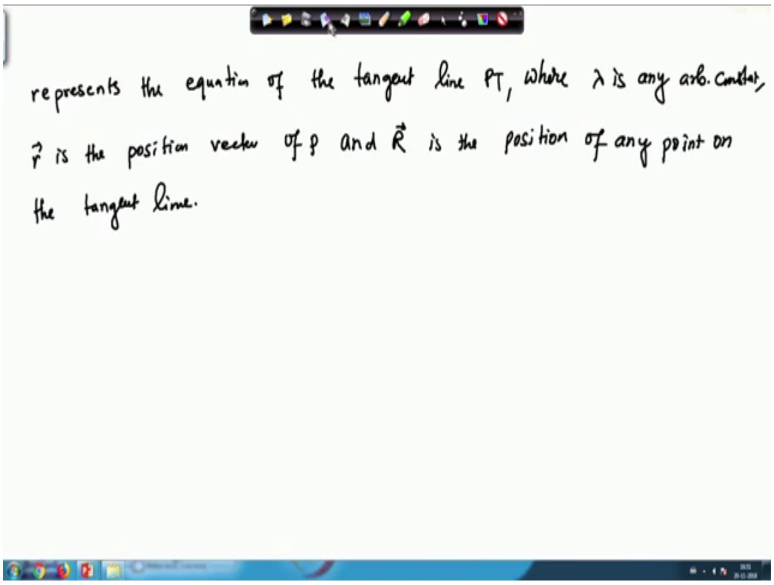

So, this equation represents the equation of the tangent line PT, where lambda is any arbitrary constant; any arbitrary constant; arbitrary constant r is the position vector is the position vector; position vector of P and capital R is the position vector of any point on the tangent line.

So; that means, in this equation we have shown that dr dt is basically parallel to the tangent PT of the curve r is equals to f t. So that means, the position vector of any point on the tangent plane or on the tangent line capital is denoted by capital R and this is basically small r plus lambda times del r dr dt, So; that means, we can be able to write the equation of the tangent line as r minus capital R minus is small r is equal to lambda times dr dt.

So, this is basically saying that the tangent is parallel to dr dt and this capital and small r are basically nothing but this is the position vector of the point P. And that is the position vector of any point on the tangent plane and dr dt is basically the derivative of the vector function r with respect to t and lambda is a constant.

So, this is how we give the equation of a tangent line at a point P for the curve r is equals to f t. So, today we will stop at here, in the next class we will work out few examples that how we calculate the tangent line although we saw in the previous chapter on level surfaces how we calculated how we calculate the tangent line. So, perhaps we will do one more example. And then we will introduce the concepts of normal and binormal of a vector function and then we will try to derive this Serret Frenet formula.

So thank you for attention and I look forward to your next class.