Integral and Vector Calculus
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Lecture - 43
Directional Derivatives, Level Surfaces

Hello students. So, today we will again continue with our examples on Directional Derivative. It is a very interesting topic. So, that is why, I am trying to cover as many examples as possible motivated from Directional Derivative and Level Surfaces, tangent planes and things like that. So today, we will continue with the example which we left off last time. So, in the last class, we had a problem on directional derivative which was this one.

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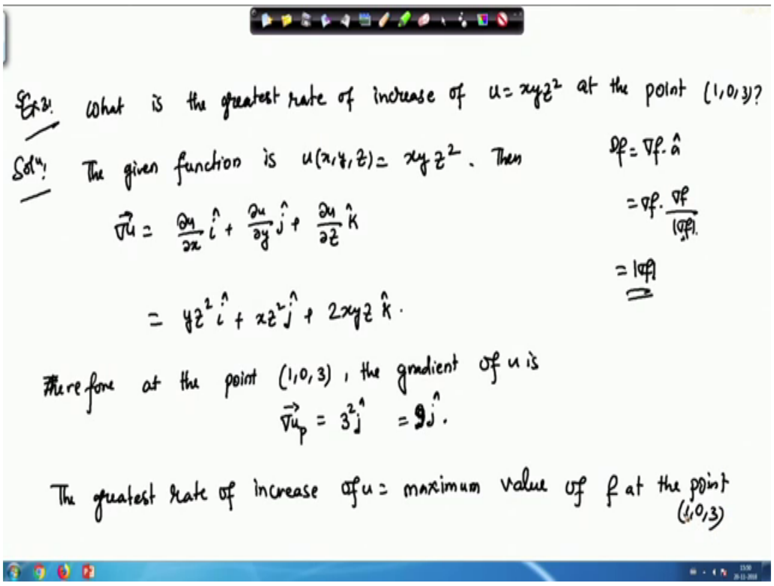

So, what is the greatest rate of change for the function u x y z equals to x y z square at the point 1, 0, 3, all right. So, let us start with this problem here. So, the given function, the given scalar function or the givens function, simply is u x, y, z equals to x y z square. So, then here, then gradient of u is equals to del u del x times i del u del y times j and del u del z times k. So, what is del u del x? It is y z square times i and del y del u del y would be x z square j and the del u del z is 2 x y z k, all right. So, that is our gradient of u. And now, at the point therefore, therefore, therefore, at the point 1, 0, 3, the gradient of u is we can write gradient of u at the point p is equals to y z square. So, the j component would be z I component would be 0 because y is 0 j component would be 3 square and this k component would again be 0. So, basically 9 j, all right so that is our so basically 9 j, so that is our gradient of u at the point p. Now, we have to calculate the greatest rate of change, the greatest rate of change our rate of increase of u; that means, we have to calculate the maximum value.

So, we have to calculate the maximum value of f at the point, at the point 1, 0, 3. So, greatest rate of change or greatest rate of increase is nothing but the maximum value of d f d s because d f d s is the rate of change of f all right. So, this greatest rate of increase is nothing but the maximum value of f at the point 1, 0 and 3.

Now, as I was speaking in the previous class that the maximum value of d f d s will be attained if we consider the vector a as gradient of f. So, if we are calculating the directional derivative along the direction of gradient, then the I mean that is the basically the maximum rate of change or maximum value of d f d s will be attained.

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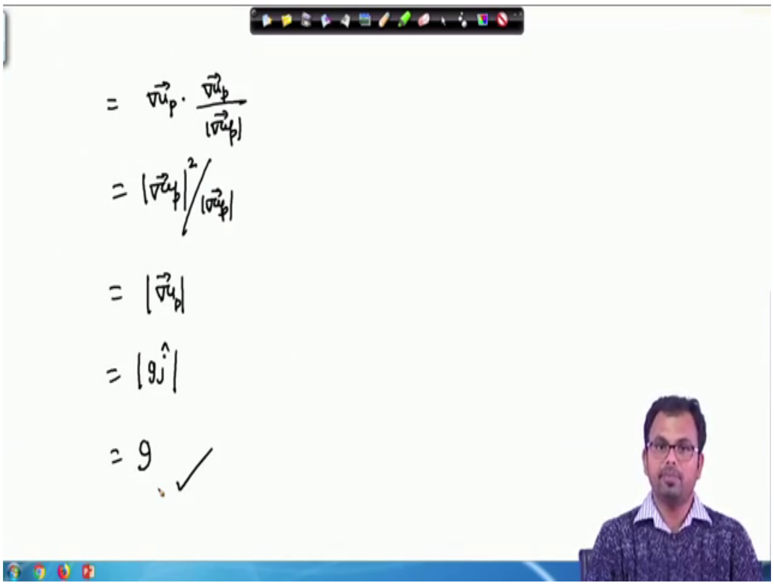

So; that means, here, here what we will have is we will have gradient of u at the point p dot product with gradient of u at the point p divided by gradient of u at the point p. So, if we consider the vector a cap as gradient of u by mod of gradient of u, so that is where a maximum value would be attained. And here, we can take the dot product. So, ultimately we will be left with gradient of u at the point p mod because this will be mod of gradient of u at the point p divided by gradient of u at the point p and one of them we will get cancel out. So, we will have basically gradient of u, I am, so I am just avoiding p now.

So, we will basically obtain gradient of u at the point p and this value is nothing but mod of 9 j right. We have already calculated this value here. So, I substitute gradient of u at the point p as let me write it has 9 j and this mod is nothing but 9 because it will be square root of 0 square plus 1 square plus again 0 square and instead of the second component is basically 9 square, it is not 1 square. So, the second component is 9 square. So, square root out of square root, it will become again line and the rest of the term will be one. So, ultimately that is our required maximum or greatest rate of increase for the function f at the point p equals to 1, 0, 3.

So, this is the required answer. So, this is a very vital thing to know that in which direction the maximum rate of change is occurring and that is the direction of actually gradient of f. So, if instead of any other vector, if you consider the vector a as gradient of f, then along that then along the direction of gradient of f, the maximum rate of change or the maximum value of d f d s would be at the end.

So, you do not have to calculate any other thing if it is says that calculate the directional derivative along the direction of maximum rate of change. So, all you have to do is calculate the gradient of the function at the point p and then take it is mod and that will be the required answer, like in this case what we have obtained all right.

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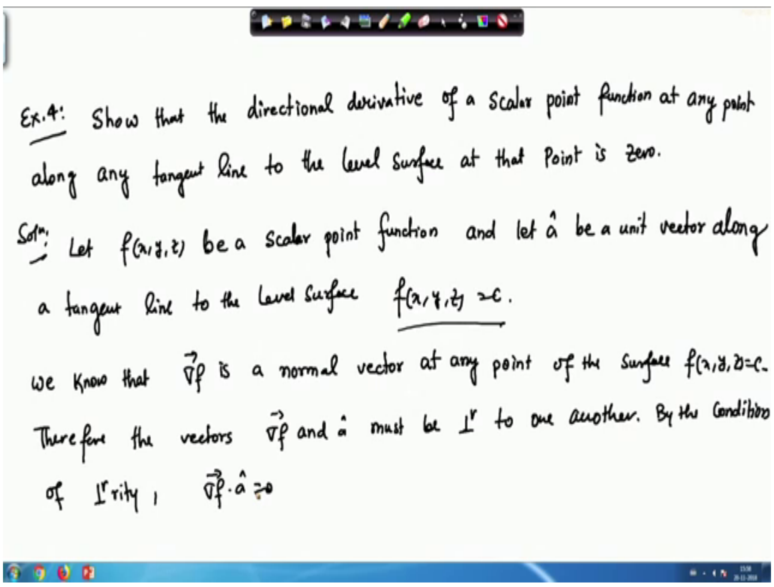

Now, let us consider an another example. I have lost the track. So, I am just writing example 4, all right. Now, in this case, show that the directional derivative, directional derivative of a scalar point function at any point along any tangent line to the level surface at that point is 0.

So, what here says is that we have a scalar point function, let us say f and we want to calculate the directional derivative along any tangent line to the level surface. So, if we have any tangent line to the level surface and if we are calculating the directional derivative along that tangent line, then in that case, that directional derivative would be 0. So, this would be true for any kind of level surface f x y z is equals to c. If we calculate the direction derivative along the tangent line, then in that case it will be always 0.

So, let us see how we can prove that. So, we will start with, let f x, y, z be a scalar point function and let a cap be a unit vector along a tangent line to the level surface and the equation of the level surfaces f x, y, z equals to c. So, we assume that f x, y, z be there scalar point function and a cap with a unit vector along the direction of tangent line to the surface f x, y, z equals to c. So, this is our required level surface. So, this is up to this we assumed.

So, now we know that, we know that gradient of f is a normal vector at any point of the surface f x, y, z equals to c all right. Therefore, the vectors gradient of f and a cap must be perpendicular to one another. So, this is very simple. If a cap is the unit vector and if a cap is along the direction of tangent line, so we have a vector along the tangent line and a gradient of f.

Now, remember there is a directional derivative to calculate the directional derivative; we always need that gradient of f at a certain point p dot product with a cap where a cap is the unit vector in the direction in the direction of which we are calculating the directional derivative. Now, if a cap is along the direction of tangent line and since gradient of f is normal to this level surface f x, y, z equals to c, then in that case gradient of f must be normal or must be perpendicular to a cap because a cap is in is along the direction of tangent line and we have a perpendicular vector on the surface.

So, it is a perpendicular to the tangent line as well and therefore, it is perpendicular to that a cap as well. So, that is what we are saying. Therefore, the vectors gradient of f and a cap must be perpendicular to one another, but from the condition of perpendicularity by, the condition of perpendicularity of perpendicularity, we have gradient of f dot a cap must be 0 because they are perpendicular to one another, their dot product must be 0.

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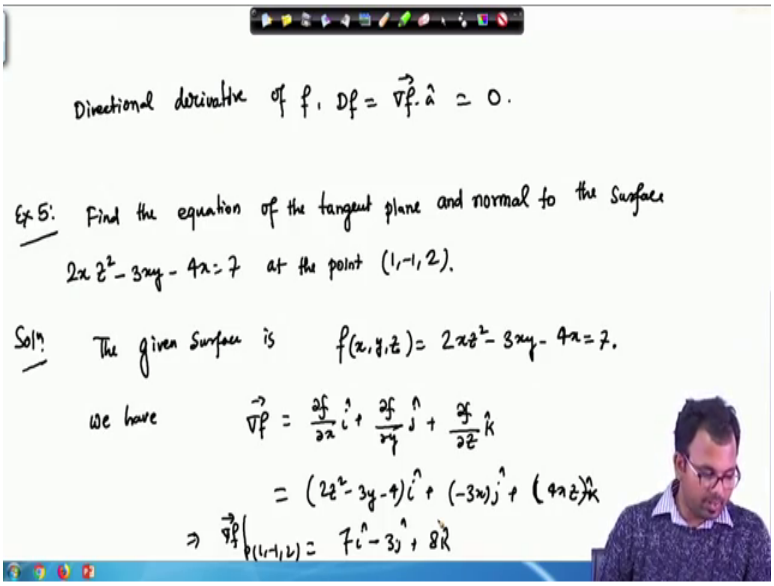

And if their dot product is 0, then the directional derivative directional derivative of f is given by D f at the point p is equals to gradient of f dot product with a cap and this is 0. And therefore, the directional derivative of the function f along any tangent line in the direction of any tangent line is always it goes to 0 because gradient of f and that vector along the tangent line are mainly perpendicular to each other and therefore, this is our required result. Next problem is motivated from the level surface and the tangent plane normal to the surface, all right.

So, let me write the problem. So, find the equation of the tangent plane and normal to the surface 2 x z square minus 3 x y minus 4 x equals to 7. So, the given equation of the surfaces, so we have to calculate the tangent plane and the normal for this surface. So, these now we are doing both types of examples there.

So, the examples motivated from directional derivative and examples motivated from level surfaces. So, I am trying to solve examples on both of these two topics one by one. So, the given surface is, so we have we can calculate first of all gradient of f. So, gradient of f is del f del x i del f del y j or del f del z k. So, now, del f del x is we will basically obtain 2 z square minus 3 y minus 4 times i, then del f del y is minus of 3 x times j and del f del z is 4 x z minus 0 minus 0. So, this is ultimately k and k is anyway 0. So, this is our gradient of f.

And now, we have to calculate the tangent plane and the normal at the point 1, minus 1 and 2. So, calculating normal is not difficult because we know that gradient of f is normal to the surface gradient is normal to the surface f x, y, z equals to c. So, basically the, this gradient will act as a normal for the given function f x, y, z. And gradient of f at the point p 1, minus 1, 2, we substitute the values here and ultimately will be able to obtain 7 i minus 3 j plus 8 k. So, this is the required directional derivative sorry the normal for the function f at the point 1, minus 1 and 2 all right, but calculating tangent plane is little bit how to say time taking.

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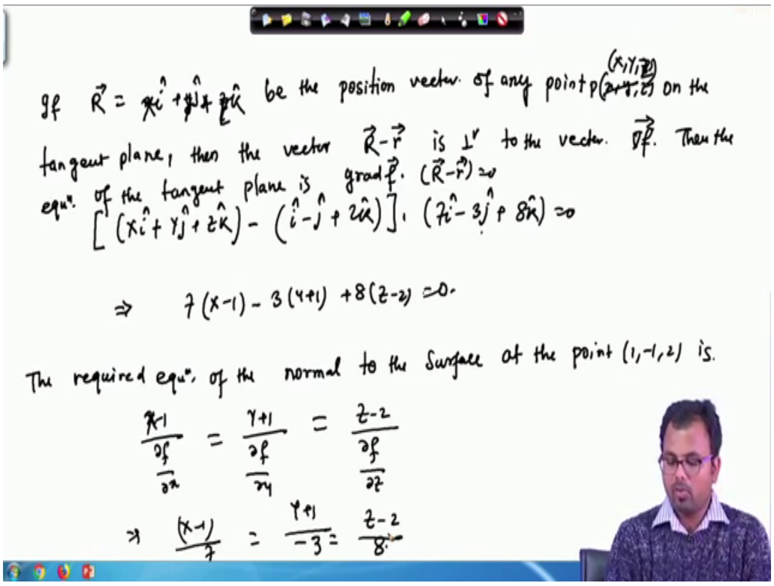

So, we now, calculate the tangent plane. So, if R equals to x i plus y j plus z k be the position vector, be the position vector of any point with a position vector of any point, let us say p x, y, z on the tangent plane on the tangent plane right. Then, the vector let me write it as capital. So, since we are using capital R, I will use capital X i, capital Y j and capital Z k. So, this one is also capital X, capital Y, sorry I did not pay attention here.

So, since I am using capital R, is better to use capital X, capital Y, capital Z, capital Z. So, then the vector capital R minus small r is perpendicular. So, this is from coordinate geometry perpendicular to the vector gradient of f all right. So, we have. So, we have basically a tangent. So, we have basically this vector R which is capital X i, capital Y j plus a capital Z k be the position vector of any point X, y, Z on the tangent plane and then the vector R minus capital R minus small r is perpendicular to the vector gradient of f because anything you have on the tangent plane will be by default perpendicular to grad a gradient of f because gradient of f is normal to the function f x, y, z equals to c. So, if you have a tangent plane on f x, y, z equals to c, then that tangent plane will always be perpendicular to the gradient of f. So, even if you take this difference, it will always be perpendicular all right.

So, now capital R is our x i plus y j plus z k all right and minus what is this small r? The small r is given as this point. So, we are calculating the tangent plane at a certain point p and that point is this one, 1, minus 1 plus 2. So, I can write i minus j plus 2 k right and ok. Here, I have then the equation of the tangent plane. So, this I, I forgot to write this.

So, the equation of the tangent plane is gradient of f is dot product with capital R minus small r equals to 0. So, this is the required equation of the gradient of the tangent plane. So, now, I am writing capital R minus small r is this one and dot product with the gradient of f is we have already calculated 7 i minus 3 j plus 7 i minus 3 j 7 i minus 3 j plus 8 k, just that gradient of f is this one and r minus capital R is this one because dot product is commutative.

So, we can actually exchange them. And now, if I calculate this, so this will be 7 x minus 1 minus 3 y plus 1 plus 8 z minus 2 equals to 0. So, the required equation therefore, the required equation, so this is the equation of the tangent plane and we put we gave the normal vector. Now, we want to write the equation of that normal vector.

So, the required equation of the normal to the surface the normal to the surface at the point 1, minus 1, 2 is x minus 1 by del f del x is equals to y plus 1 by del f del y is equals to z minus 2 by del f del z and now del f del x is basically our 7 del f del y is minus 3 and z minus del f del z is 8. And this is so, basically these two so; this is that a tangent plane equation of the tangent plane, this is the equation of the normal and these two what we had to calculate.

So, calculating this tangent plane our normal vector and those are not complicated, it is just that we have to use some tricks from vector calculation and also from coordinate geometry too, so that we can be able to write the equation of the tangent plane and the question of the normal. So, this was an another example where we were supposed to calculate the tangent plane and the normal.

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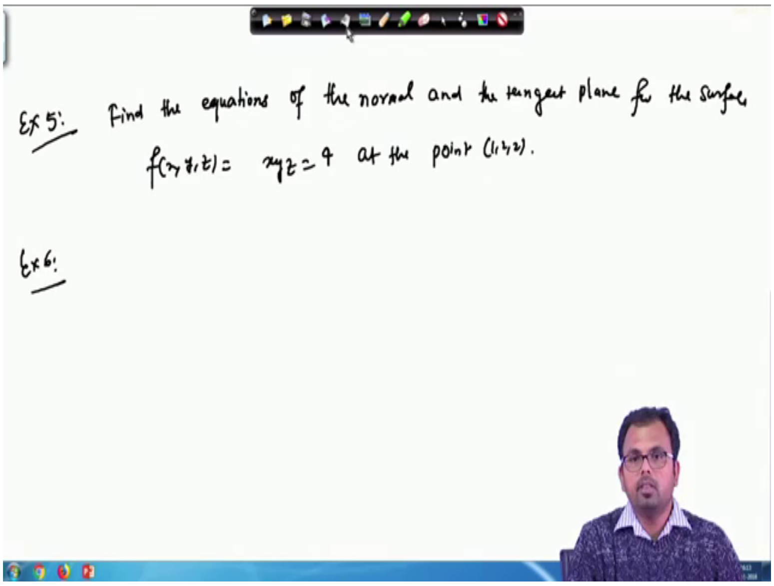

You can be asked I can write an another example, but I will of course, leave it as a task for the student. So, example 5, so find the equations of the normal, equations of the normal and the tangent plane question of the normal and the and the tangent plane for the surface f x, y, z equals to x, y, z equals to 4 at the point 1, 2, 2.

So, here in this example also, we have to calculate the equation of the normal and the tangent plane for the surface f x, y, z equals to x, y, z equals to 4 at the point 1, 2, 2. So, here in this case also, we will calculate the gradient and then we calculate the calculate the equation of the tangent plane and from gradient, we have to calculate and in such a way, so in the gradient itself we have to calculate the equation of the tangent plane in such a way that, we can be able to write this thing here gradient grad of f times r minus, capital R minus small r so that we can be able to obtain the equation of the tangent plane in this fashion.

And then, the equation of the normal to the surface at this point 1, minus 1, 2 can be given by this x minus 1 divided by del f del x y plus 1 divided by del f del y is z minus 2 divided by del f del z and just equate the coefficients for del f del x del f del y and del f del z and then divide them by 7, 3 7, minus 3 and 8 and that will give you the required equation of the required equation of the normal and the tangent plane is already given.

So, this is how we calculate the direction and this is how we calculate the now tangent and normal a tangent plane a normal of a given scalar function f at a certain point p. And in the next class, I will try to solve one more example on directional derivative before we move on next topic. So, today, I will stop here and I thank you for your attention and I look forward to your next class.

Thank you.