Integral and Vector Calculus
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Lecture – 42
Directional Derivative (Concept & Few Results) (Contd.)

Hello students. So, in the previous class we started with examples actually based on level surfaces and how do we calculate the normal on a given surface and also we introduced the concepts of directional derivative.

So, today we will practice few examples based on a Directional Derivative and afterwards if time permits then we move to our next topic. So, to start with let me start with one example on directional derivative.

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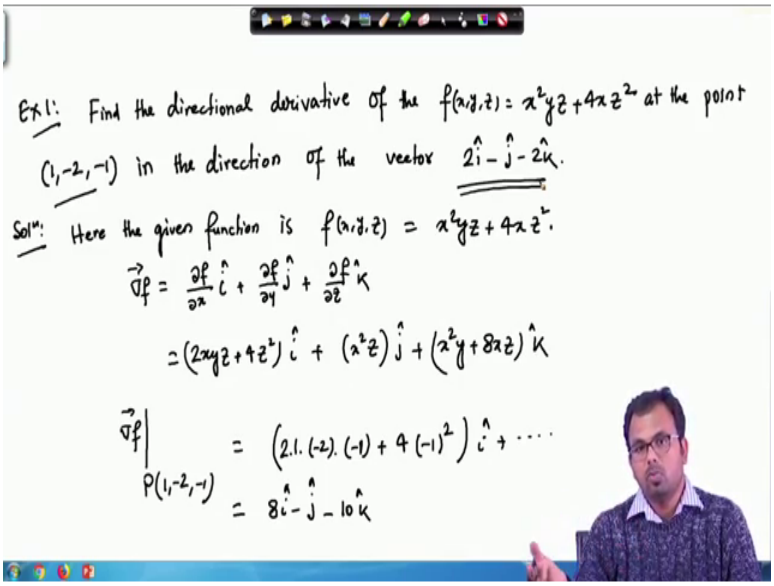

So, today our example one; so, the example goes like this find the directional derivative. So, find the directional derivative of the scalar function f x, y, z. So, we have a function in 3 dimension geometry this function is x square y z plus 4 x z square at the point. So, as we know we always have to calculate the directional derivative at a certain point and at the same time we also need to have a unit vector towards in the direction of which we will calculate the directional derivative.

So, at the point what is the point here? The point is 1 minus 2 minus 1 in the direction. So, this is our direction of the vector 2 i minus j minus 2 k alright; i j k are the unit vectors. So, let me adjust my chair first alright; so, here; so now we are going to solve this.

So, here the given scalar function the given function is f x y z equals to x square y z plus 4 x z square. And in the previous class we saw the direction derivative of a given scalar function at a point P is given as gradient of that function at the point p, times the unit vector in the direction of which we are calculating the directional derivative. So, here first of all we need to calculate the gradient of this function at the point p; so, this is our point p.

So, we will calculate gradient of f which is a vector quantity and gradient of f would be del del x of f; del del y of f and del del z of f and then k. So, if I do partial derivative with respect to x then this will be 2 x y z plus 4 z square times i and then del del y. So, this will be x square z the second term does not contain any y; so the derivative of second term with respect to y will be 0.

So, here it will be j and del del z of f. So, this will be x square y plus 8 x z times k alright; so that is our gradient of f. But we need to calculate the gradient of the function at the point P because that is where we are calculating the directional derivative. So, what is the value of gradient of f at the point p? So, this is one of the notations to write gradient of f at the point P and the point P is 1 minus 2 minus 1. So, the value is we substitute all these things the 2 times 1 times minus 2. So, this is very simple to do and I am pretty sure you can be able to do this simple calculation ah; i plus you substitute the value for the rest of the variables here in i n j in j and k.

So, ultimately if you calculate then at the end you will be able to obtain 8 i minus j minus 10 k. So, that is the gradient of the function f at the point P and it is being calculated for substituting x equals to 1, y is equals to minus 2 and z equals to minus 1 in this in this expression here alright; so that is our gradient of f.

Now, in order to calculate the directional derivative we need the unit vector. So, here in this case we have the given vector, but is it a unit vector? It is; obviously, not because if it is a unit vector then the modulus of this vector should be 1. So, that is what we mean by unit vector that the mod of that vector is equals to 1, but in this case it is certainly not. So, for this vector we will obtain a unit vector and we will see how we do that.

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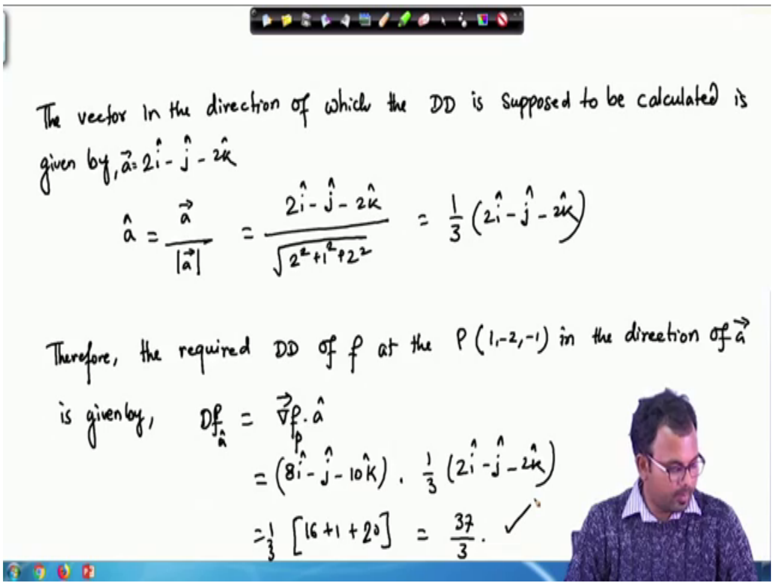

So, the vector the vector in the direction of which in the direction of which the directional derivative D D is supposed to be calculated; supposed to be calculated is given by. So, what is the vector? We have to i minus j minus 2 k; so, I call it as let us say a. So, this is my unit vector this is my normal this, this how to say a normal vectors not that normal. So, this is basically the given vector do not focus on normal or anything else. So, this is the given vector from here I have to obtain a unit vector.

So, the unit vector will be obtained by dividing the vector a; by dividing the vector a with its magnitude right. So, the given vector is 2 i minus j minus 2 k and when we divide it by its magnitude it will be 2 square plus 1 square plus 2 square; so, it will be 9.

So, we take 1 by 3 out and then it is 2 i minus j minus 2 k; now it is a unit vectors. If we take the how to say magnitude of this a cap then this will yield 1; so; that means, this is a unit vector. Therefore, therefore, the required the required directional derivative of f at P; what is our P? P is 1 minus 2 minus 1 in the direction of a cap; in the direction of a cap is given by directional derivative D f at a cap is equals to gradient of f dot a cap.

Here instead of a cap; I will write a because we are calculating basically in the direction of a. So, it does not matter whether we write cap a; it is just that if we write a here the in the formula we should be careful that we always write the unit vector because here we have a unit vector always in the formula. So, the directional derivative I have chosen a notation D f although the notation does not play a big role here, you can ignore the notation and use your own notation for the directional derivative; this is not universal in a way this is just my notation to signify that the directional derivative of the function f in the direction of a cap alright.

So, now here we have gradient of f at the point P dot a cap now gradient of f at the point P is given by 8 i minus j minus 10 k. So, 8 i minus j minus 10 k dot a cap; a cap is 1 by 3 2 i minus j minus 2 k. Now, we calculate this dot product and this will be sixteen plus 1 plus 20 isn't it? Yes.

So, we will ultimately obtain 37 by 3; so directional derivative of the function f at the point P in the direction of this vector a is given as 37 by 3. So, of course, it is a scalar quantity and its calculated in this fashion alright. So, this was our first example we will practice few I mean like 2 or 3 more examples just to make the concept clear.

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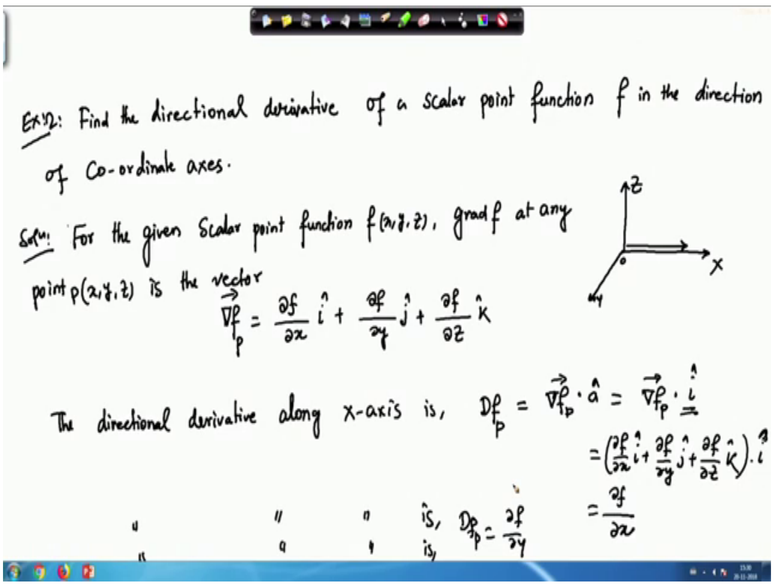

Next let us consider an example find the directional derivative; directional derivative of a scalar point function; scalar point function f in the direction of coordinate axes; so, this is our example 2. So, it is also a very interesting example we have to calculate the directional derivative of a function f in the direction of coordinate axes. So; that means, if I draw my coordinate axes; so, this is X axis this is my Y axis and this is Z axis right.

So, we have to calculate directional directional derivative in the direction of coordinate axess; that means, in the direction of X axis in the direction of Y axis and in the direction of Z axis. So, when we are calculating in the direction of X axis the directional derivative. So, then in that case what will be our unit vector.

So, if we are calculating the directional derivative in the direction of X axis; that means, the unit vector is basically the vector along this direction is actually i cap because X axis is perpendicular to Y Z plane and the unit vector for this for this X axis in a way is actually i cap. So, i cap is the unit vector in the direction of X axis and our a cap previously in the formula is actually i cap. So, i is the unit vector which is perpendicular to Y Z plane and it is along the direction of X axis and therefore, we use it as a normal in this case alright.

So, let us see; what is our direction derivative in the direction of X axis first. So, we have to calculate along the coordinate axes; that means, we have to calculate the direction derivative along the X axis, along the Y axis and along the Z axis. So, we have to calculate the directional derivative along all these 3 axes alright; so, let us see.

The given for the given let me write for the given a scalar function or scalar point function scalar point function f x y z gradient of f at any point x y z is the vector in del f del x del f del y and del f del z alright. So, that is our gradient of f and at any point P; so that point P is x y z. So, I can write it as P and then I can write P out here.

So, this is a gradient of the function f at any point P x y z and therefore, the directional derivative or the directional derivative; the directional derivative along X axis is; so, I will write D f at the point P is equals to gradient of f at the point P dot a cap and a cap is the unit vector along x axis. So, along X axis the unit vector is given as i cap alright. So, we will write gradient of f at P dot i cap.

Now, if we take the dot product of del f del x i plus del f del y j plus del f del z k with i; then in that case only first term we will survive the rest of the term will be 0 because the component of j and k is 0 in case of this vector. So, here in this in case of this vector; j and k components are 0. So, when we take the dot product only del f del x we will survive alright.

So, del f del x times i plus del f del y times j and del f del z times k dot product with i. So, if I take the dot product then this will yield del f del x. So, that is the direction derivative of the function f at any point P in the direction of X axis alright. Similarly the directional derivative of the function f at the point P along Y axis is we will proceed in the similar fashion and this will be basically del f del y. And similarly the directional derivative along Z axis for the function f at the point P is del f del z; so, D f D d f at P is del f del z.

So, basically in this case we got 3 directional derivative and each one of them are directed along three different; I mean along three different directions. So, the first one is directed directional derivative along X axis the second one is directional derivative along Y axis and third one is directional derivative along Z axis. And you just have to know that i j and k are the 3 unit vectors along X, Y and Z axes respectively and rest of the things are very easy.

So, this is another interesting example. So, you might be given a function f has x is square y plus 3 y square z and then you will be asked to calculate the directional derivative along X axis. So, you just have to calculate these gradients and based on that you just take the dot product. And if you are asked to calculate just along the X axis then you just calculate del f del x and that will be a required answer at a point P. So, this is an interesting example which I wanted to show; we will move on to our next example; example 3.

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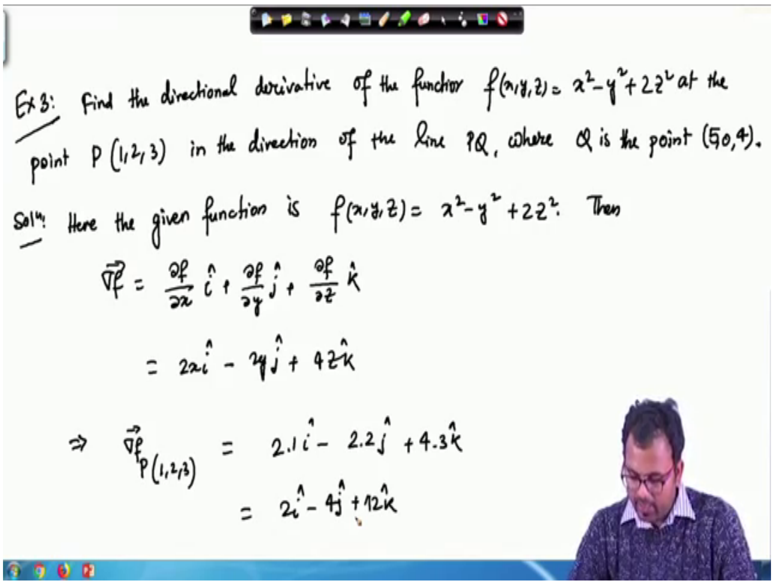

So, find the directional derivative of the function f x y z equals to x square minus y square plus 2 z square at the point; at the point P 1 2 3 in the direction of the line P Q; where Q is the point 5 0 and 4 alright.

So, here we are asked to calculate the directional derivative of the function f which is given at the point P. But instead of giving a vector to in the direction of which we are supposed to calculate the directional derivative we are given an another point. And the vector which you obtain by joining P to Q along which we have to calculate the directional derivative.

Now if you are familiar with vector algebra if you have a if you have two points; let us say P and Q then the vector P Q can be obtained by writing P Q as O Q minus OP. So, you take the distance from the origin and then you take the; for Q you take the distance for P from the origin and then you subtract them and that will give you the required vector P Q or P to Q. So, that is some that is exactly what we are going to do here.

So, first of all in order to calculate the directional derivative we need to calculate the gradient of the given function; so, that is where we start from. So, here the given function is f x, y, z equals to x square minus y square plus 2 z square. Then del f del x i del f del y j and del f del z k. So, when I take del f del x then it will be 2 x i then minus of 2 y j and then 4 z k. So, that is the required gradient of the function f at the point P and at in general the gradient of the function is given by this one.

Now, we have to calculate the gradient of the function at the point P. So, the point P is 1, 2, 3. So; that means, we have 2 times one times i minus 2 times 2 times j plus 4 times 3 k. So, this will be 2 i minus 4 j plus 12 k alright. So, this is the required gradient of the function f at the point P. Now, we have to calculate the direction derivative along the direction of P to Q.

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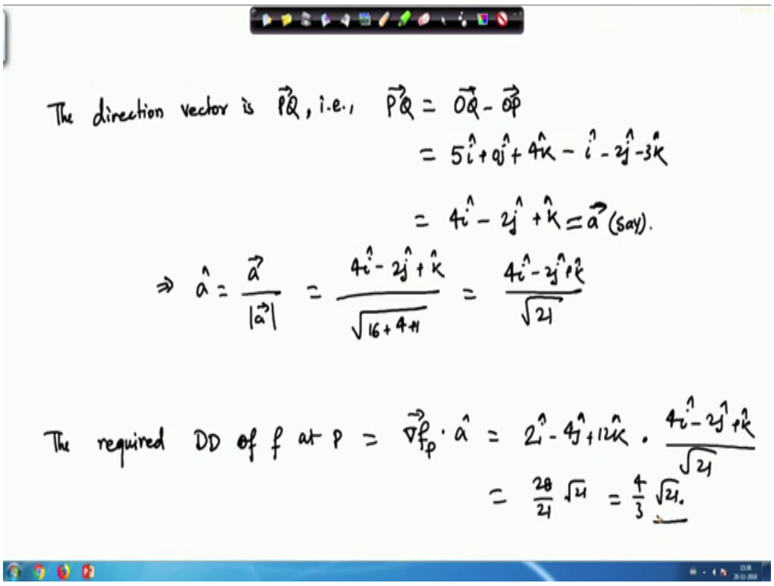

So, the vector or the direction vector the direction vector is P to Q alright. So, P to Q can be given that is let me write this in a clear manner. So, that is P to Q is equals to O Q minus OP. So, O Q is what is that point 5 0 4; so, we have 5 i plus 0 j plus 4 k minus i minus 2 j minus 3 k. So, this will be 4 i minus of 2 j plus k alright. So, along direction along this vector P to Q we have to calculate the directional derivative, I will call it as let us say a vector. So, say P Q is our vector a alright

So, from here I can write to say. So, from here a cap is a by mod of a. So, this will be 4 i minus 2 j plus k divided by 16 plus 4 plus 1. So, this is basically 4 i minus 2 j plus k divided by square root of 21 alright. And now the required directional derivative of f at the point P is given by gradient of f at the point P dot product with a cap.

So, gradient of f at the point P is 2 i minus 4 j plus 12 k. So, this is 2 i minus 4 j plus 12 k dot product with 4 i minus 2 j plus k square root of 21. So, whatever you get up to the simplification. So, here you have just have to take the dot product and ultimately we will be able to obtain 28 by 21 times square root of 21. So, this is ultimately 4 by 3 square root of 21. So, that is the required directional derivative of constant f in the direction of P to Q

So, here the direction vector let us say is not given and we had to calculate the direction vector by just; I mean how to say writing P Q as O Q minus O P you just substitute the value, take the difference and that is a required direction vector. And from there we have to calculate the unit vector should be divided it by its magnitude and that gave us the give us a cap. And therefore, the required directional derivative of f at the point P is the gradient of f at P dot product with a cap and that is your required answer alright.

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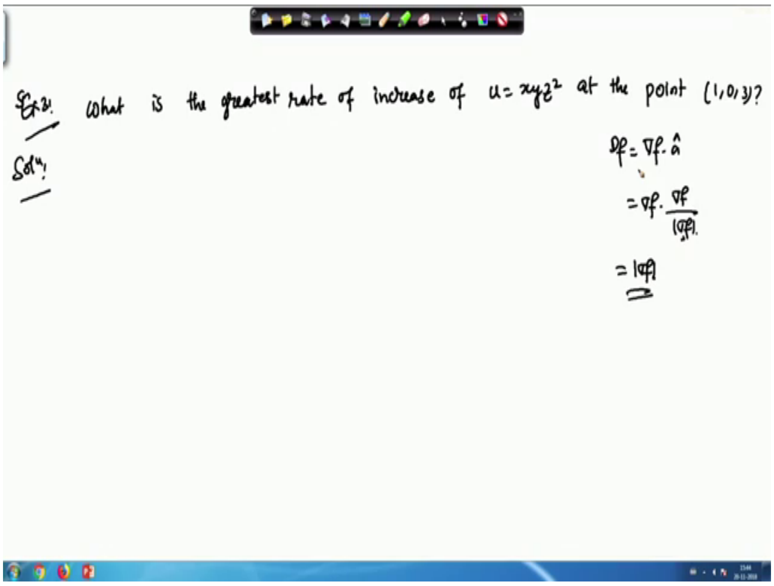

Now, we will calculate another example so, what is; so we know that directional derivatives is the rate of change of function f with respect to the; with respect to the; with respect to the distance in the direction of a unit vector.

So, D f D s; so D f D s is the directional derivative or the rate of change of the function f with respect to the distance or length in the direction of a vector a cap or a whatever we would like to say basically a cap So, from that formula the directional derivative is nothing, but gradient of f address at a certain point P dot product with a cap alright.

So, this directional derivative it will be maximum in what direction I mean; what is the maximum value of this directional derivative and in which direction would it be maximum? So, the thing is the maximum value would be attained if I mean if we consider this a cap as a unit vector. So, it is actually a unit vector, but if let us say if that is equals to a gradient of f by gradient a mod of gradient of f; that means, its maximum along the direction of along the direction of gradient basically.

So, instead of considering a is equals to any arbitrary vector a vector if we consider a is equals to gradient of f. So, that is what I am trying to say that if we consider a is equals to gradient of f then in that case a cap is gradient of f by gradient a norm of m mod of gradient of f and when we take the dot product then it is will be basically mod of gradient of f.

So; that means, the maximum value maximum value is attained when the directional derivative is equals to the gradient of f mod of gradient of f. So, that is the maximum value. So, what I am trying to say is that if we have D f equals to gradient of f a cap and the a cap if I write a cap. So, a cap is basically a unit vector along which we are calculating the direction derivatives of a cap is equals to gradient of f; then in that case this will be mod of gradient of f and there is the this is the maximum value for the directional derivative.

So, in the direction of gradient of f the maximum value is attained. So, basically the maximum value of the directional derivative is attained in the direction of gradient of f. So, instead of taking any vector a if you take gradient of f itself alright. So, you have a given surface if you take a as the gradient of f and here instead of any a if as a cap; if we take gradient of f by mod of gradient of f, then we will basically b w left with mod of gradient of f as the directional derivative.

And that is the maximum value of the directional derivative or maximum rate of change of the function f given as a scalar function. So, this is a very vital result that in the calculator direction derivative in the maximum rate of in the direction of maximum rate of change. So, that direction is basically the direction of the gradient of f and this is the message I am trying to convey.

So, the direction of maximum rate of change is basically the gradient of f; so, along which the function is how to say has the greatest or attains the maximum rate of change. And that can be basically the directional derivative along the direction of maximum rate of change; it is given by mod of gradient of f alright.

So, we will do the same thing here; so basically we will try to solve this example in our next class. And we will stop here for today and we will continue with the same example in our next class. We will try to solve this and then we will continue with some more examples on directional derivative.

So, I thank you for your attention for today and I will see you in the next class.