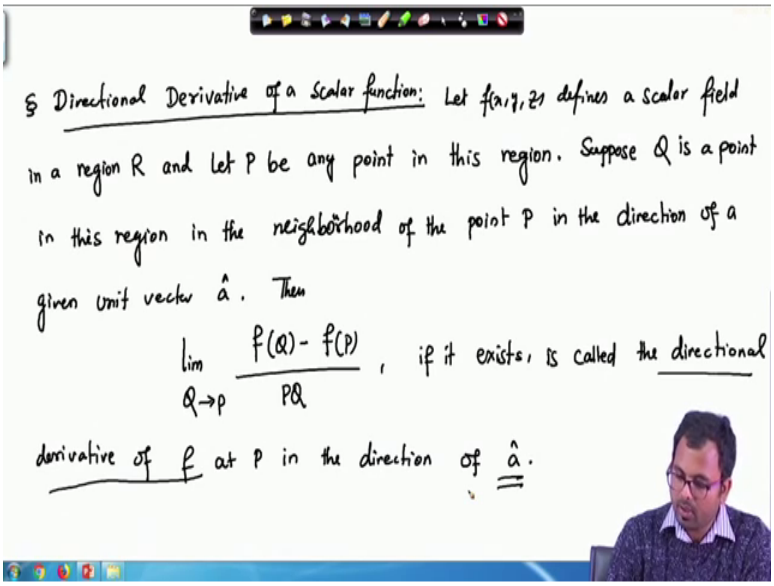
Integral and Vector Calculus
Prof. Hari Shankar Mahato
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 41
Directional Derivatives (Concept & Few Results)

Hello, students. So, in the previous class, we introduced the concept of level surfaces and we also saw two results that when you have a level surface on a region R, then at every point of R only one and only one level surface and that can pass through that point that that point in our and we also saw another result which says that gradient of a function is perpendicular to the surface f x, y, z equals to c.

So, now that we have those two basic results today we will start with the concepts of directional derivative and we will try to work out few examples based on directional derivative. So, we will start with a formal definition what do they actually mean.

(Refer Slide Time: 01:06)



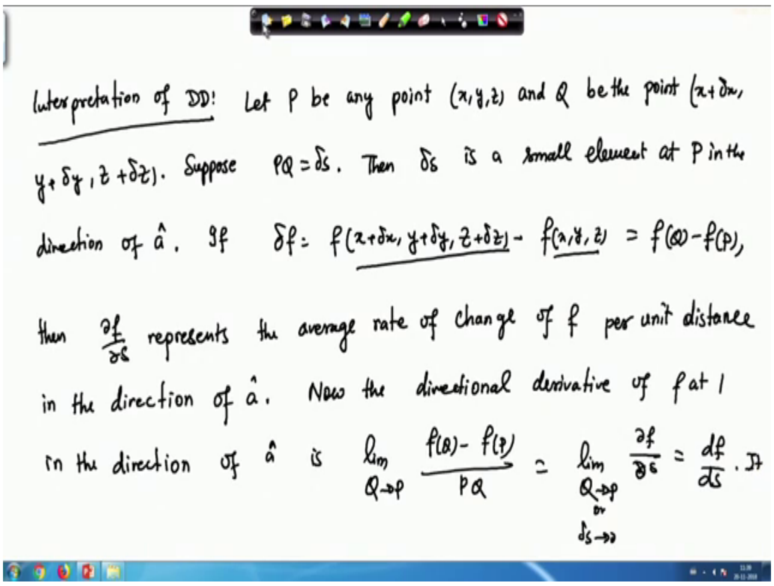
So, directional derivative of a scalar function, alright. So, the formal definition goes like this let f x, y, z defines a scalar field in a region R and let P be any point in this region and suppose Q is a point in this region in the neighbourhood in the neighbour hood in the neighbourhood bour in the neighbourhood of the of the point P of the point P in the direction of a given unit vector.

So, this is very important we also we always have to have a unit vector when we are calculating the directional derivative of a given unit vector a cap. Then, limit Q tends to P f Q minus f P divided by PQ if it exists is called the directional derivative of f at the point at the point P in the direction of a cap in the direction of a cap.

So, basically what it means is you have a you have a point P on a on a on a surface f x, y, z it goes to C in a region R and then you have another point let us say Q in the neighbourhood of the point P. So, when P when Q approaches to P then if this limit exists then the then this limit is actually called as the directional derivative in the direction of this unit vector a cap. So, this point Q which we have chosen so, this point Q is of course, is a point in the neighbourhood is of course, a point in the neighbourhood of the point P, but it is also in the direction of a unit vector a cap.

So, it is a point in the direction of in the direction of a unit vector a cap and then we calculate this limit and if this limit exists, then we say that the then we say that this limit is actually the directional derivative is the directional derivative of the function f at the point P in the direction of a cap, alright.

(Refer Slide Time: 05:32)

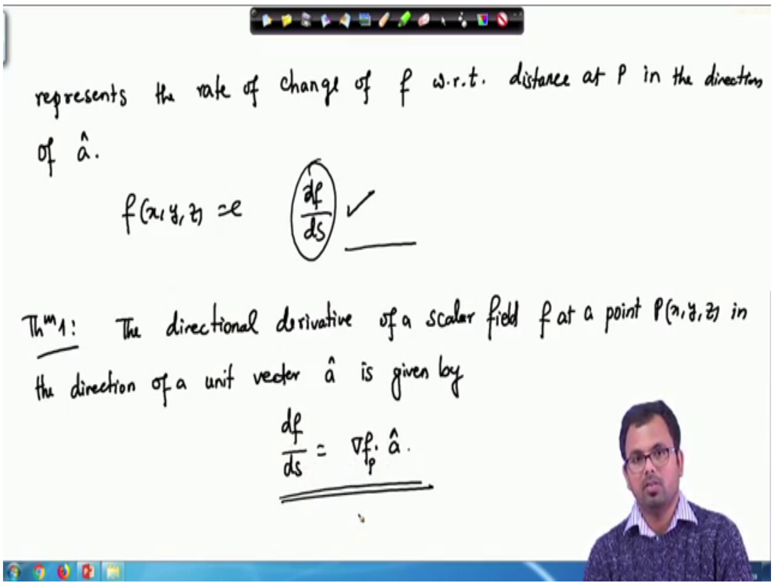


So, basically what it means is what it means is so, physical meaning or what it means is interpretation we can write interpretation of directional derivative. So, I write it as DD. So, interpretation is let P be any point. So, what we are saying is let P be any point x, y, z and Q be the point x plus delta x, y plus delta y and z plus delta z and suppose, PQ which is a very small arc length is delta S. So, it is a very small arc length and we assume that the length is delta S alright.

So, suppose PQ is equals to delta S then delta S is a small element is a small element at P in the direction with a small element at P in the direction of a cap and if delta f is equals to f of x plus delta x y plus delta y and z plus delta z minus f x, y, z is equals to. So, this is our point Q this is our point P. So, f Q minus f P then del f by del S represents the average rate of change the average rate of change of f per unit distance in the direction of a cap.

And, now, the directional derivative of f at the point P in the direction of a cap is basically limit Q goes to P f Q minus f P divided by PQ is equals to basically limit Q goes to P or delta s goes to 0 or we can write delta S goes to 0 del f by del S. So, this is nothing, but our df dS because when delta S goes to 0, this whole thing we will converge to df dS.

(Refer Slide Time: 09:04)



And, it represents so, basically it represents the rate of change of the rate of change of f with respect to distance at P in the direction of a cap which is a unit vector. So, what it does what it means physically is that we have two neighbouring points on the surface f x, y, z equals to C and then that is small increment in the function f or in the in from the point P to Q is denoted by delta f and delta S. So, that a small increment in the function f we write it as delta f which is basically f Q minus f P and the small element which is this PQ is basically delta S.

So, delta f by delta S when delta S goes to 0 is actually the average rate of change of f per unit distance. So, how much how much the function is changing how it is changing per unit distance is given by del f del S in the direction of the vector a cap.

So, we always have a unit vector along which we are calculating the rate of change of the function f and the directional derivative of the function f at P at this point P is basically when limit Q goes to P we write f f Q minus f P by PQ which is basically our del f by del S and at the point P we have this we have this delta S this surface element is that a small element at the point P in the direction of a cap. So, that is what we are calculating here. So, this is basically our rate of change of the function f per unit distance in the direction of a cap at the point P.

So, this is represent the rate of change of the function f with respect to distance or per unit distance at the point P in the direction of a cap. So, a cap is a unit vector at the point P and this df dS is actually denoting our rate of change of the function f. So, this is what we mean physically by the directional derivative. So, at a certain point you have a unit vector and you need to calculate the rate of change of that function along that unit vector. So, that is that is what simple and an in simple words it mean.

So, you have a unit vector at a point P on a surface f x, y, z equals to C. So, your directional derivative is actually the rate of change of that function f x, y, z with respect to distance of course, in the direction of that unit vector at the point P. So, or whichever point it is where you are calculating the directional derivative and this is what we mean physically or the physical interpretation of the direction and derivative.

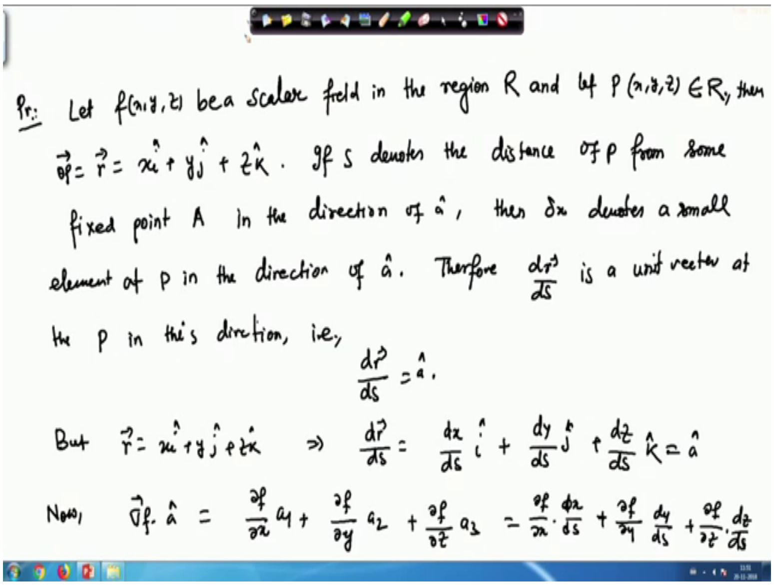
Now, that we have stated df dS is basically that rate of change of f with respect to distance how do we calculate the directional derivative do we really have to differentiate f x, y, z equals to c. So, do we really have to differentiate f x, y, z equals to c as del f del S or do we have to calculate df dS to calculate the directional derivative or is there any other formula to calculate the directional derivative so, that we will now prove, in terms of a small theorem. So, whether we calculate this thing or whether there is some other tool that will help us calculate the directional derivative. So, that we are going to the there some that is something we are going to see now.

So, let me put a small theorem here. So, today it is theorem 1, and it says that the directional derivative the directional derivative of a scalar field f at a point P x, y, z in the direction of a unit vector a cap is given by df dS is equals to gradient of f at the point P times a cap. So, we have to calculate the gradient of the function at the point face. So, I can put a P here and that basically says that the gradient of the function has to be calculated at the point P times the unit vector a cap.

So, that means, instead of calculating df dS we can just using this theorem the directional derivative of the function f at the point P in the direction of a is given by gradient of f at the point P dot product with a cap. So, this df dS is equals to this. So, this df dS is equals to gradient of f at the point P times a cap.

So, this is a very small result and we will try to prove that and see how this thing this df dS equals to this gradient of f at P times a cap.

(Refer Slide Time: 14:45)



So, the proof or the solution. So, let f x, y, z be a scalar field. So, let f x, y, z be a scalar field in the region R and let P x, y, z be any arbitrary point in the region R. So, P x, y, z belongs to R. So, then I can write OP or the position vector of the point P is equals to x i y j and set k, alright.

Now, if S denotes the distance of P of P from some fixed point from some fixed point A in the direction in the direction of a cap then delta x denotes a small element denotes a small element at P in the direction of a cap and therefore, dr dS is a unit vector at the point P in this direction that is dr dS is equals to a cap.

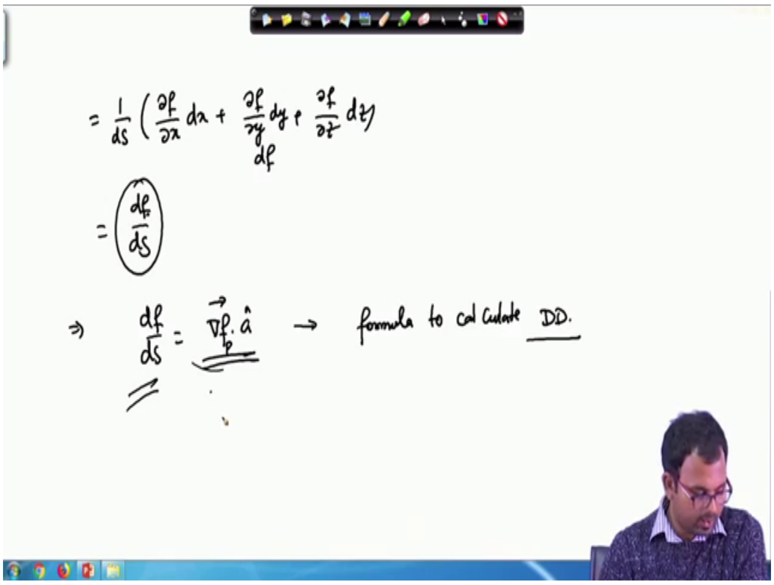
So, what we are doing is we are assuming the distance of P from any fixed point A in the direction of a cap. So, in the direction of a cap we assume any arbitrary point let us say A, and S denotes the distance and then in that case a small element or a small increment in the point P in the direction of a, then that small increment is denoted by delta x. So, if we are moving along the surface and from P to let us say Q or P to a we are going then that small increment is basically our delta x then in the direction and then in the direction of a of course,.

So, therefore, dr dS basically the rate of change of R with respect to the distance is a unit vector in the direction of this in the direction of this unit vector a cap. So, this basically dr dS is a unit vector at the point P in the direction of a cap, alright. So, but our r is equals to x i plus y j plus z k. So, from here our dr dS will be dx dS times i dy dS times j and dz dS times k alright.

So, now our gradient of f times a cap, what is this? So, this one will be del f del x times a 1 and then del f del y times a 2 and del f del z times a 3 right. So, this a actually. So, this a 1, a 2, a 3 will be actually we can write it now. So, dr dS is basically dx dS from here dy dS and dz dS. Now, dr dS is equals to a cap. So, that means, if I write this one as a cap, so, the components of a cap will be a 1, a 2, a 3.

So, a 1 equals to dx dS, a 2 equals to dy dS and a 3 equals to dz dx. So, I am substituting them here. So, del f del x times dx dS del and this one will be dx dS and this one will be del f del y times dy dS and del f del z times dz ds. So, since dz and del z they are a small element. So, we can cancel this del x the del this del x del x. So, then this will be df and del f del S del f del y and then del f. So, del f del S del f del S and then this one again will be del f del S.

(Refer Slide Time: 20:38)



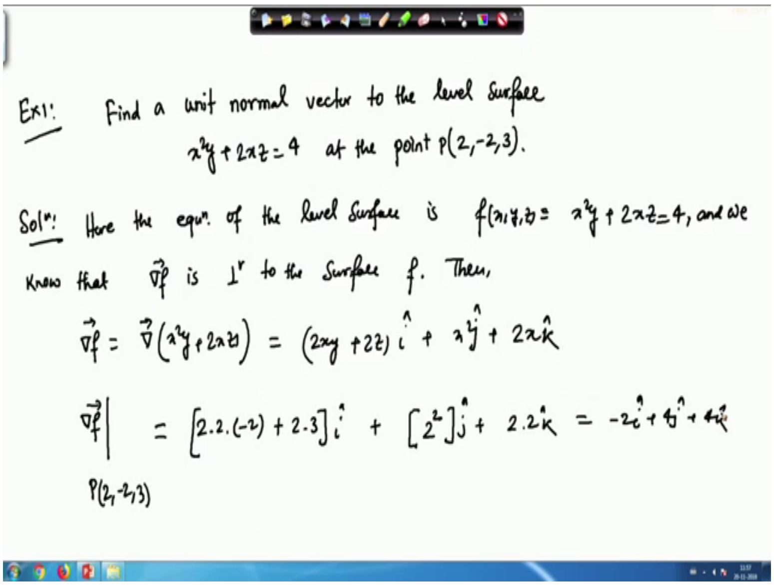
So, ultimately if I if I multiply so, if I so, ultimately so, what I am trying to say is that we can write it something like this 1 by dS and then what I am trying to say is del f del x times dx del f del y times dy and del f del z times dz. So, this is what we can do here and now this is nothing, but our df. So, that is from differential calculus this is our df. So, I can write it as df dS. Now, we know that df dS is the directional derivative at the point P in the direction of a. So, that is what physically it means from the definition of directional derivative.

So, this is our directional derivative of the function f at the point P in the direction of a. So, what we have is we have df dS is equals to gradient of f times a cap and if we want to calculate a directional derivative, then we can calculate the gradient of the function f at the point P, this is very important times the unit vector a. So, this is the required formula to calculate directional derivative, alright. So, and this is what we wanted to prove.

So, the direction derivative of a scalar field at the point P in the direction of a unit vector a cap is given in this fashion and just assuming some basic results from the differential calculus and not assuming, but using those results from the differential calculus we can be able to show that the direction derivative df dS is equals to gradient of f at the point P times a cap and this is what this is what is equal to our directional derivative, alright.

So, now we will try to work out few examples before we go to our next topic. So, let us calculate the directional derivative we will work out few examples on level surfaces, alright.

(Refer Slide Time: 22:47)



So, in the previous class we started with level surfaces and in today's class we started with directional derivative. So, we will work we will try to solve the examples on both of these two topics. So, the first one is first example find a unit normal find a unit normal vector to the level surface x square y plus 2 xz equals to 4 at the point 2, minus 2, and 3. So, our given level surfaces x square plus x square y plus 2x then equals to 4 and we have to calculate the unit normal vector at the point. So, let us put a P here at the point P 2, minus 2, and 3.

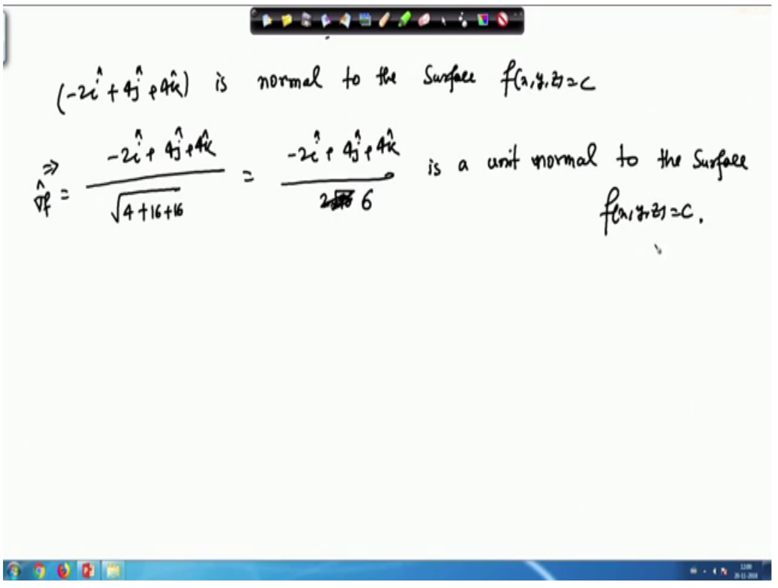
So, what we are going to do? First of all from the results on a level surfaces we know that gradient of f is a normal to this level surface. So, here it says that find a unit normal. So, we have to find a normal to this surface given here and then dividing it with it is magnitude will give us the unit normal, alright.

So, those two steps are clear. So, first of all we have to calculate its gradient. So, here the equation of the level surface is f x, y, z equals to x square y plus 2 xz equals to 4, alright and we know that gradient of f is perpendicular or normal to the surface f. Then, first of all we calculate the gradient of f gradient of f would be gradient of x square y plus 2x set of course, gradient of 4 will be 0. So, calculating this one is not difficult. So, del del x are

2 xy plus 2z times i well del del y would be x square j and del del z will be 2xk. So, this is the required gradient of f.

And, now, the gradient of f at the point P and P is 2, minus 2, and 3. So, if I substitute then it will be 2 times 2 minus 2 plus 2 times 3 times i plus x square which is 2 square times j and this one is 2 times 2, k. So, this will be ultimately an on minus 2 i plus 4 j plus 4 k and therefore, this gradient of f.

(Refer Slide Time: 26:41)



So, the gradient of f or minus of 2 i plus 4 j plus 4 k is normal, oh sorry, is normal or perpendicular to the surface is normal or perpendicular to the surface f x, y, z alright, but we have to find out a unit normal or unit vector perpendicular to this. So, to give the unit normal we write it as 4 j plus 4 k and then we divide it by its magnitude so; that means, minus to a square which is 4 plus 4 a square is 16 plus 16.

So, ultimately this will be minus 2 i plus 4 j plus 4 k and then this will be 32 plus 40. So, we can write 2 square root of 10 is a unit normal or unit vector a unit normal or unit vector perpendicular to the surface f x, y, z equals to c and f x, y, z is basically our x square y plus 2 xz equals to equals to 4. So, that c is 4.

So, this is the required this is the weight this one is a 16, 16, 32 plus 4, 36 ok. So, this has to be 6 it is not 40, it is a 36. So, it will be 6. So, this is the required unit normal perpendicular to the surface f x, y, z equals to c and you see we had a we had a equation of a level surface here in this example we had a question of a level surface.

So, we write we wrote this level surface as f x, y, z equals to c and then we had to calculate the gradient because gradient is a vector normal to the level normal to the surface f x, y, z equals to c and then we calculated the gradient at the point P and then we divided it with its magnitude and that gave was the unit normal or unit vector perpendicular to the surface x square y plus 2 xz equals to 4 at the point 2, minus 2 and 3 you also have to make sure whether this point lies on that level surface on that level surface or not because we can do all this calculation but, always make sure that this point lies there.

So, if I substitute there then this will be 4. So, minus 8 minus 8 and that will be 4 to also of course, this point lies on the level surface. So, before you calculate anything you also have to make sure the given point P lies on the level surface or not. So, here in this case it does lying on that level surface and therefore, the gradient of f is actually normal to that level surface at the point P and if you divide it with its magnitude then that is the unit normal; here I can write it as gradient of f cap, alright.

So, today we saw one example motivated from the level surface and unit normal. In our next class we will continue with the examples on level surfaces and directional derivative just to make the concepts clear and I look forward to you in your next class.

Thank you.