Integral and Vector Calculus
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Lecture – 40
Level Surface Relevant Theorems

Hello students. So, in the last class, we started with the examples on divergence, gradient and curl. So, up until last class we looked into three important operators in vector calculus. And they are divergence of a scalar function, sorry divergence of a vector function gradient of a scalar function, and a curl of a vector function. And we also worked out few examples, where we saw how we can calculate these three operators.

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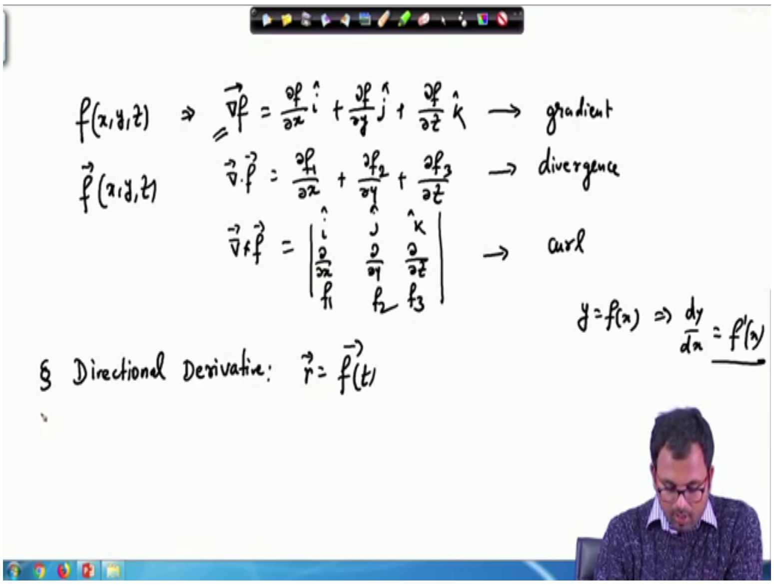

Like if you if you were given a scalar function let us say f, so then in that case how we can calculate. So, if we are given a scalar function f, which is a function of let us say three scalar variables f x, y, z, then basically the divergence of this scalar function f is denoted by sorry gradient of the scalar function will be denoted by this nabla of f. And this nabla is a vector quantity. So, we put a vector sign here.

And this is basically del f del x i plus del f del y j plus del f del z k. So, this is a vector quantity this gradient. And if we are given a vector function, if we are given a vector function, then we know that since we are charging this nabla, which is a vector quantity on to a vector function. So, then in that case there can be two possible operations. So, either it can be a dot product or it can be a cross product.

So, if it is a dot product, then this is called divergence, and it is nothing but del f del x plus del f del y plus del f del z. Some people also prefer to write it as del f del x 1, del f del x 2, del f del x 3, it is depending on the fact that whether you are choosing your coordinate accesses as x, y, z or x 1, x 2, x 3, it really does not make any difference. And if we charge let us say cross product, then this can be calculated as i j and k. So, we have to calculate this determinant del del x del del y and del del z. And so this one is f 2 f 3 f 3, and then this one is f 1, f 2, f 3.

So, of course a vector function has three components, so we write it in this way. And this is basically divergence this is our divergence, and this is our gradient, and this one is actually our curl of a vector function f. So, here I was supposed to write three components of the vector function f. So, f 1, f 2, and f 3. And here we also use the same components f 1, f 2, f 3.

So, we worked out few examples based on these three operators, and we also calculated somehow to say a vector function whether it is solenoidal or irrotational. So, we did some examples like that as well. Now, using this gradient of a scalar function. There are two new topics that we are going to introduce at the moment.

So, the first one is actually called it is called directional derivative. So, we will start with today directional derivative. So, in case of scalar function, we calculated the derivative of a given scalar function. So, for a scalar function let us say y is equals to f x if it is differentiable, then we just differentiate d y d x, and we write the derivative of this scalar function as f dash x.

Now, in case of vector functions, let us say we have r is equals to f t, when we differentiate this one we can talk about that is a direction, we can talk about a direction in which this vector function can be differentiable. So, whether it is differentiable along how to say x, y, x-axis, y-axis or z-axis or whether it is it is differentiable along any vector, which is not x, y or z-axis.

So, we can talk about its derivative along a along the along a certain vector actually, and that is derivative is called as the directional derivative. So, we will give the formal definition of directional derivative, but before that we will start with the definition of level surfaces. So, what do we mean by level surfaces, and we will try to see one or two results based on the level surfaces, before we move to the to the directional derivative all right.

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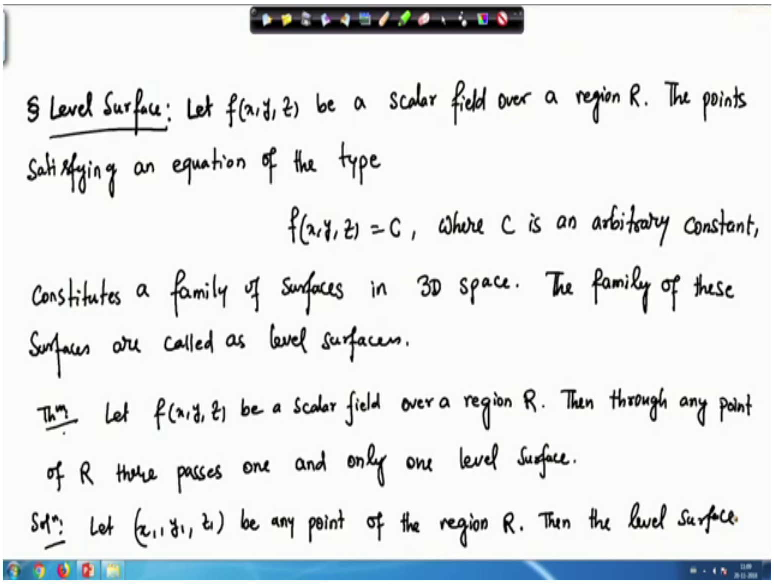

So, let us start with level surfaces. So, this is also in our syllabus, and I think we can learn a little bit about it. So, the formal definition goes like this. Let f x, y, z be a scalar field. So, we have defined the scalar field in our previous lecture, so or maybe not in previous, but one or two lectures before, so you can look into the definition of scalar field.

So, let f x, y, z be a scalar field over a region R, then the points the points satisfying the points satisfying an equation of the type f x, y, z equals to c, which is where c is an arbitrary constant is an arbitrary constant constitutes a family of surfaces in three-dimensional space. And the family of these surfaces; family of these surfaces are called as level surfaces.

So, for in so f is a scalar field in a region R, and the points which are the curl and at the points which are actually satisfying this equation f x, y, z equals to c, where c is can be any arbitrary constant. So, for every constant c will get a how to say will get a surface basically, so f x, y, z equals to let us say one, then it will give one surface, then f x, y, z equals to two, then it will give another surface.

So, the family of all these surfaces, they are called as the level surfaces. So, this and of course this family of surfaces, they are in three- dimensional space. So, because c is any arbitrary constant, we will get different types of surfaces actually and that constitutes a family, so that family is called as level surface. And that is that is our definition for the level surface.

Now, there is a very small result in this regard. So, the result goes like this. So, this is a small theorem. So, let f x, y, z be a scalar field be a scalar field over a region R over a region R, then through any point of R there passes one and only one level surface. So, it says that suppose you have a scalar field f x, y, z over a region R, then every point on this on this region R, there can be only one level surface that will pass through that point. So, there cannot be two level surfaces that will pass through the same point. So, we will prove that at any particular point on that surface R one and only one level surface will pass through that point.

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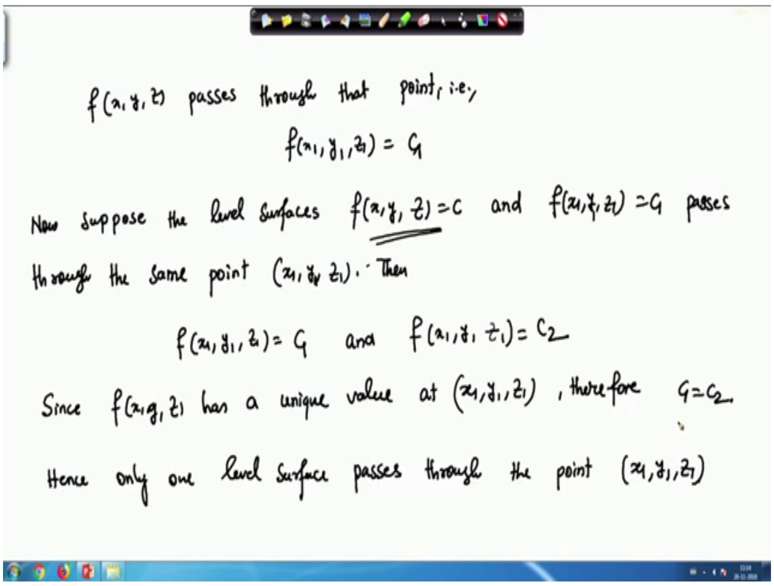

So, to begin with let x 1, y 1, z 1 be any point of the region R of the region R, then the level surface f x, y, z passes through that point passes through that point because f x, y, z is a level surface in R. So, passes through is a is a level surface on R. So, if we have a point x 1, y 1, z 1 in R, then f x, y, z equals to c will pass through that point, so that is that is this f x, y, z equals to c will satisfy the point x 1, y 1, z 1 will satisfy that equation, and this will be equals to some c 1 right.

Now, suppose the level surfaces level surfaces f x, y, z equals to c, and f of x 1, y 1, z 1 equals to c passes through the same point same point x 1, y 1, z 1. So, f x, y, z equals to c is the equation of one level surface. And f x 1, y 1, z 1 equals to c 1 is the equation of this second level surface all right or second surface.

Now, if we assume that both of these two level surfaces passes through the same point x 1, y 1, z 1, then they both satisfy that then the point x 1, y 1, z 1 must satisfy both the level surfaces, so that means when I substitute x 1, y 1, z 1 in f x, y, z equals to 3 x f x, y, z equals to c, then that equation will also hold true. And if I substitute, and x 1 f of x 1 y 1 z 1 equals to c 1, then that equation will also hold true. So, if both of these two level surfaces passes through this point x 1, y 1, z 1, then the point must satisfy these two equations.

So, what will happen what will happen is we will have f of x 1, y 1, and z 1 equals to so then f of x 1, y 1, z 1 equals to c 1 from here. And if it satisfy this equation, if it satisfy this equation, then it will be f x, y 1, z 1 and equals to c 2. We will get a different constant, because the level surfaces different. And so ideally I mean we should get a different constant. So, j and now we see that the equation since f x, y, z has a unique value at x 1, y 1, z 1, because the point is same.

So, if the point is same, and the equation of the level surface which was f x, y, z equals to c is same in a way, so that means the value at the point x 1, y 1, z 1 must also be same. So, we can write since f x, y, z as a unique value at x 1, y 1, and z 1, therefore the constants must be same. So, if the equation f x, y, z equals to c has a unique value at any point x 1, y 1, z 1, because it cannot have two different values, it cannot have c 1, and then it cannot have c 2, so because the equation of the levels level surface is f x, y, z equals to c is same.

So, if the point is also same, then it cannot yield two different values then in that case it is a contradiction. So, therefore, our constants must be same, so that means, c 1 must be equals to c 2. And this implies that and this implies that hence only one level surface passes through the point x 1, y 1, z 1. So, what I am trying to say is that you choose two different points say x 1 y 1 z 1 and x 2 y 2 z 2, and since those two points are in the region r we substitute those 2 points in the equation f x y z equals to c, so that means, we have according to this here we will have f x 1, f x 1 y 1 z 1 equals to c 1, and f x 2 y 2 z 2 equals to c 2 that is what we are trying to do here.

But if it passes through the same point, then we will have f x y 1 z 1 y 1 f x 1 y 1 z 1 equals to c 1, and f x 1 y 1 z 1 equals to c 2. But the points the level surface f x, y, z must have a unique value because the point is same. So, if you assume the same point, so the same level surface that is passing through the point cannot yield to different values. So, c 1 cannot be different from c 2. And therefore, the only possibility we have is that c 1 must be equals to c 2.

And if you have c 1 equals to c 2, that means, that the level surface every level surface can pass through point in r only once I mean they cannot how to say only one level surface can pass through the point x 1 y 1 z 1, so that means, when you have a certain point in the region r only and only one, one and only level surface the will pass through that that particular point and this is this is what we wanted to prove all right.

So, there cannot be two level surfaces that will pass through this point x 1, y 1, z 1. Similarly, you can choose any arbitrary point, so since x 1 y 1 z 1 was chosen arbitrarily this result will true for all the points in r all right. So, these type of results are also quite common in level surfaces.

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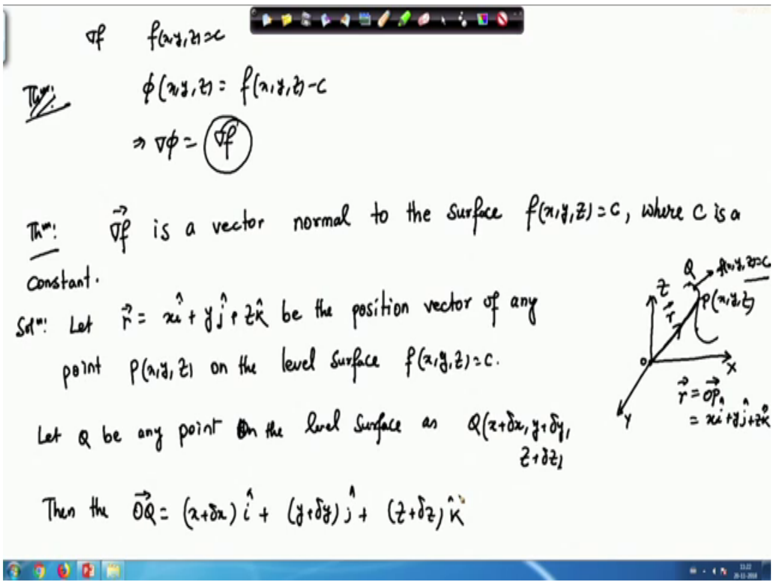

Now, another theorem which is a little bit more interesting; so, you remember we calculated the gradient of a scalar function. So, gradient of a scalar function we usually write in this way, but what does it mean actually what is how what it does to a surface let us say f x, y, z equals to c. So, when we calculate the gradient of this of this surface, that means we can simply write it as phi let us say we can write it as we can write it as phi x, y, z equals to f x, y, z minus c, and then calculating gradient of phi is like calculating gradient of f because gradient of c will be 0.

So, what it this gradient of f mean here I mean to this surface f x, y, z equals to c and that actually is its quite how to say interesting or important to know that this gradient of f is actually normal to the surface f x, y, z equals to c. And there this is also one of the important results in vector calculus that what do we actually mean by a gradient of a scalar function. And it actually means that a gradient of the scalar function is normal to the surface f x, y, z or that is scalar function equals to the equals to constant. So, we will try to prove that result. So, let me state that theorem. So, gradient of f is a vector normal to the surface f x, y, z equals to c or proof, we can write also as proof. So, gradient of f is a vector normal to the surface f x, y, z equals to c, where c is a constant all right.

So, let us see. So, let r equals to our vector x i. So, always as I have told you whenever you write r in vector calculus, it is always the point x, y, z. So, x i y j plus z k be the position vector, be the position vector of any point p x, y, z on the level surface of x sorry level surface it is not a vector function. So, level surface f x, y, z equals to c. So, that means, if we have let us say x-axis, y-axis and z-axis and let us say if this is our level surface. So, and this is our point P. Of course, the level surface it is not looking like a level surface it has to be a surface, I drew it like a curve, but it is not a curve. We have to draw it in a proper manner that, so that it looks like a surface.

Since, I am not good at drawing, I just drew like that. But remember this has to be a surface all right. And now this is so this P, so this is basically our vector. And this is our vector r and this point this vector r is equals to if you write vector r is equals to vector OP, and it can be written as x i y j z k all right ok. So, P x, y be, z be any point on the level surface. So, this is my equation of the level surface f x, y, z equals to c all right.

And now I assume another point let us say Q here, I assume another point Q, and let Q be any point. So, Q be any point on the point on the level surface as I can write x plus delta x. So, it is a small increment. So, I can I can assume at no somewhere either here or here it is up to it is up to you. So, I assume let us say here as a point Q delta y and z plus delta z. So, it says it is a, it is a neighboring point on the level surface f x, y, z equals to c. And it is a neighboring point to the point P. Then the position vector then OQ or the position vector of Q can be given by x plus delta x i y plus delta y j plus z plus delta z k.

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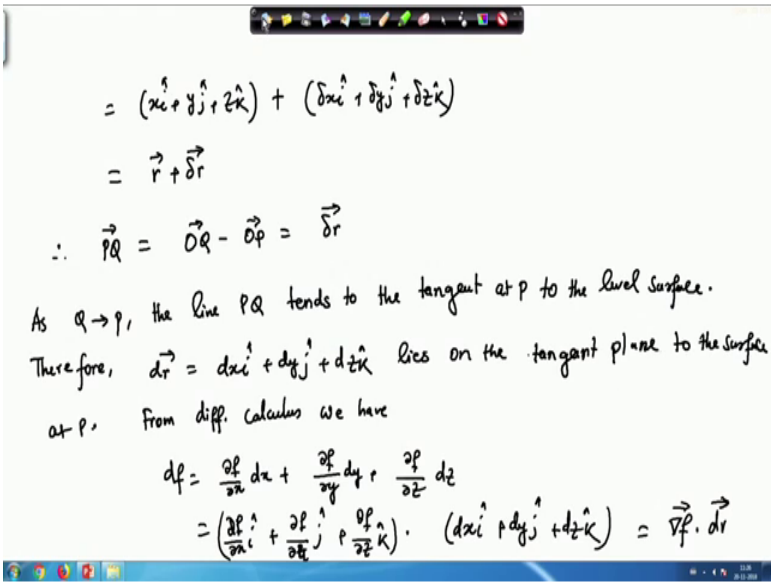

So, I can write it as x i plus y j plus z k, and then another bracket delta x i delta y j and delta z k. So, I write x i plus y j plus z k as r and I write this delta x i delta y j and delta z k as delta r. So, both of them are vector quantity. And this small increment is denoted by delta r all right. So, now P to Q this vector, so P to Q this vector is OQ minus OP and OQ minus OP will be delta r only, because the r minus r will be zero. So, we are left with only delta r.

Now, as Q tends to P, the line PQ tends to the tangent at P right to the level surface at P to the level surface. Therefore, dr equals to d x i d y j plus d z k lies on the tangent plane tangent plane to the surface at P. So, if we make P goes if we make Q goes to P, if we make Q goes to P, then in that case the line PQ will actually be a tangent to this level surface at the point P all right, so that is what we have written here.

So, as Q goes to P, the line PQ tends to tangent at the point P to the level surface. And therefore, this dr is equals to d x i d y j plus d z k lies on the tangent plane. So, when Q goes to P, this dr delta r will tends to dr and this del x i will tend to d x say and so on, and that will actually lies on the that will actually lie on the tangent plane to the surface at the point p. And therefore, from the differential calculus we know that so from differential calculus from differential calculus we have. So, what do we have from differential calculus, we have d f equals to del f by del x times dx and del f by del y times dy and del f by del z times dz.

So, this is basically if I want to write it, then this can be written as gradient our del f del x times i plus del f del y times j plus del f del del f del y times j plus del f del z times k dot product with d x i plus d y j plus d z k.

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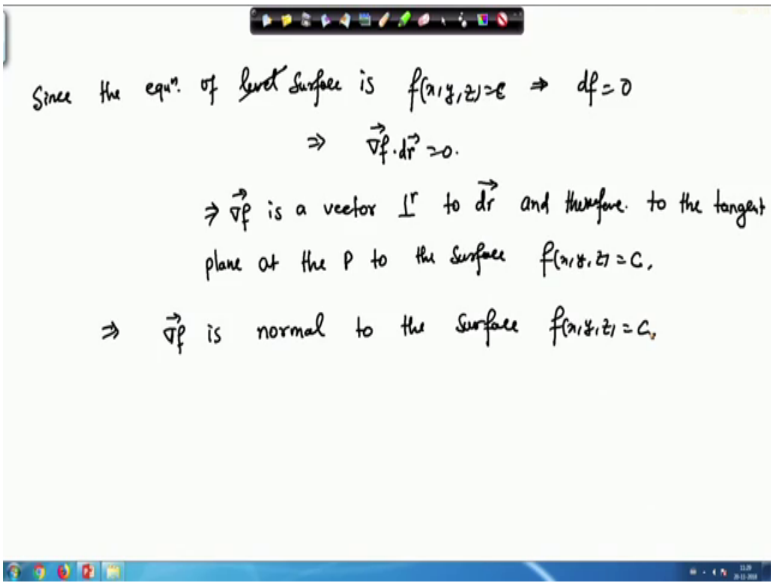

So, I can write this as, so I can write it here. So, this can be written as gradient of f times d r right. And since and since the equation of the level surface or of the surface simply we do not have to write levels. So, of the surface is f x, y, z equals to c. So, from here d f is equals to 0 right, it is a differential of this f x, y, z equals to 0 because right hand side is a constant. And from here this d f can be written as delta f sorry del f or nabla f dot r equals to 0, and this one is d r. So, if the dot product of 2 vectors as 0; then, they must be perpendicular to one another.

So, from here I can write this gradient of f is a vector perpendicular to d r perpendicular to d r. And if it is perpendicular to d r, then this d r if we look here, so this d r we draw the conclusion that d r is a is actually lying on the tangent plane, so that means, you have gradient of f that is perpendicular to d r, but since d r lies on the tangent plane and then this gradient of f must be perpendicular to the tangent plane or to the to the to the surface f x, y, z equals to c.

So, this delta f sorry del f or gradient of f will actually be perpendicular to d r, and hence and therefore, and therefore, perpendicular to the tangent plane to the tangent plane at the point P to the surface f x, y, z equals to c. And this implies that gradient of f is normal to the surface f x, y, z equals to c.

So, here we prove that gradient of a function grad f or del f is actually perpendicular or normal to the surface f x, y, z equals to c. So, you remember in the beginning when I introduced the concepts of gradient of a function I mentioned that the gradient of a function has a physical meaning. And this is what the physical meaning is that if you have a scalar function or a surface given as f x, y, z equals to c, then when you calculate the gradient as grad f then in that case that gradient of f is a vector perpendicular to the surface f x, y, z equals to c. And this results prove that this result proves that that how to say theorem.

So, we will stop here for in this class. And I will try to introduce the concepts of directional derivative in our next class, and then we work out few examples. So, today's class was a bit more theoretical as I told you right in the beginning that I will introduce the theory wherever it is necessary and wherever it is interesting and so that is what I try to do in the in today's class. And in the next class, we will start with some examples on directional derivative; we will introduce the concepts of directional derivative. And hopefully you will find it interesting.

So, thank you for attention and I look forward to you in your next class.