Integral and Vector Calculus
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Lecture – 04
Theorems on Riemann integration

Hello students, so up until last class we saw the Riemann integrable functions, and how they are connected with the Riemannian sum, and how Riemannian sum is connected with definite integral. And we also looked into some properties of Riemann integrable function. Today, we will list few more properties of Riemann integrable functions, and then we go to fundamental theorem of integral calculus, and what do we mean by anti derivatives and primitive.

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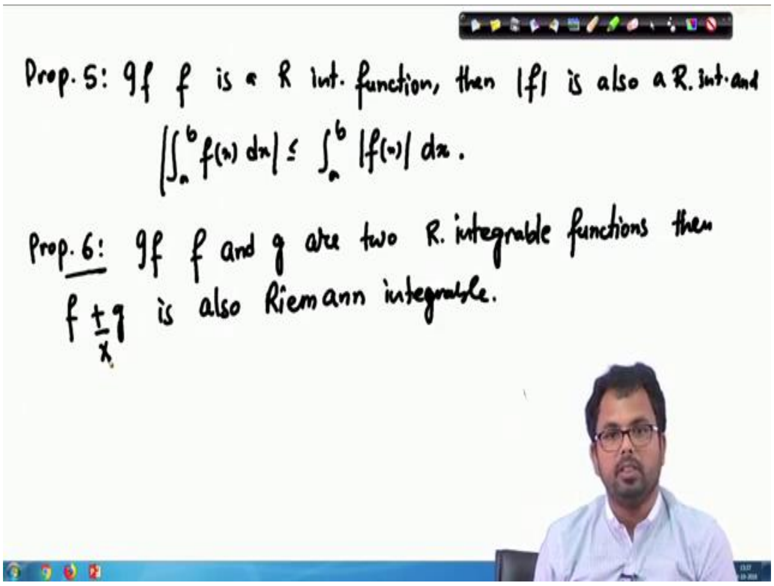

So, as I was mentioning about the properties, so another property, property 5 of Riemann integrability is if f is a Riemann integrable function, then $|f|$ is also a Riemann integrable function. And we have $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$. So, and so that means the mod of function f is also Riemann integrable, and it satisfies this inequality.

Third sorry the 6th property in this regard is its little bit how to say towards the direction of what we have already studied in limit and limit continuity and differentiability of two functions. So, we know that if f and g are both how to say continuous, then sum of the continuous functions would also be continuous or how to say if you are calculating the limit then in that case if $\lim f$ and $\lim g$ exist than the sum of their limits or difference of their limits or product of their limits also exist.

So, here in this case also we have the similar theorem. So, if f and g are two Riemann integrable functions, then sum of these two functions is also Riemann integrable. The difference of these two functions would also be Riemann integrable. And the product of these two functions would also be Riemann integrable. For the quotient you need to have how to say the nonzero condition on the function g that g has to be nonzero throughout that interval [a,b], and then we can talk about this Riemann integrability.

So, it is pretty much along the same line of limit and continuity and how to say properties of sum of two functions or difference of two functions and things like that. And again for the proof of this theorem I would recommend you to look into the books which I recommended. Next in Riemann integrable functions, we have something called let me go to a new page.

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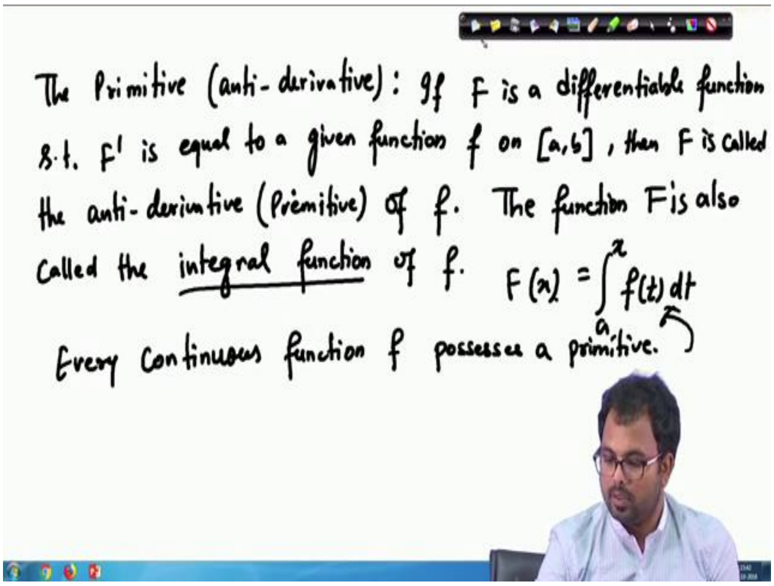

The primitive, we sometimes also call it as anti-derivative. So, the definition is if f is a differentiable function, such that $F' = f$ on the closed interval [a,b], then the capital function F is called the anti derivative or the primitive whichever you prefer primitive of the function f. And sometimes we write it, and this function capital F is also called the integral function of f.

So, what do we mean by this? So, it means that we can write the function f, this small f, let me use a different notation. So, we can write this function small f as let us say $\int_a^x f(t)dt$. So, the capital F(x) can be expressed as the integral of the function small f. And if you take the derivative on both sides, then in that case the right hands that are the left hand side will be F'(x) which is equals to the function f(x); and that capital F based on this definition is called as the anti-derivative or the integral function of this small function f.

And if you have, but you see that in order to write this expression, we need to have some special conditions on this function small f and capital F. We cannot just simply write every function capital F as this integral form. So, the very first criteria in this regard is every continuous function. So, every continuous function f possesses a primitive that means, if f is a continuous function, then it can be written as this integral form. So, the very first criteria we need to have for the function small f is that it needs to be continuous. And then you can be able to write the capital F as the integral form of this small f.

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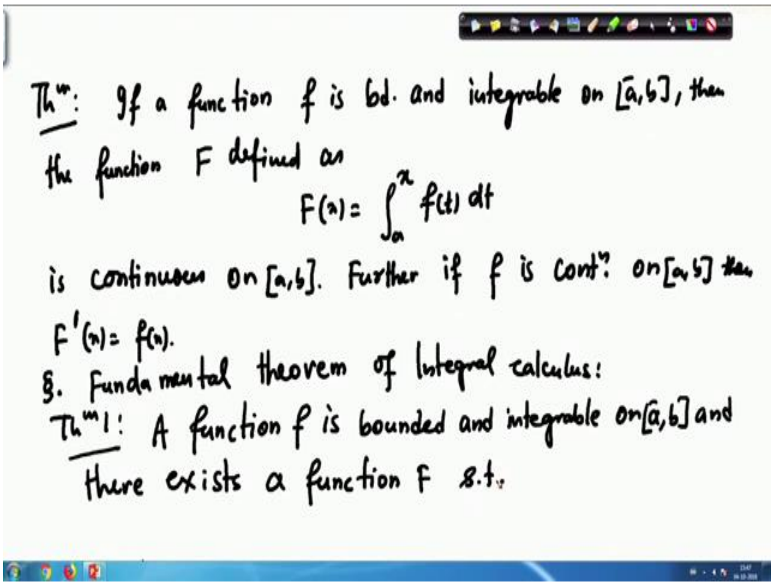

Now, we can put this in a small theorem. So, the theorem goes like this. So, the first theorem is, if a function f is bounded and integrable on [a,b], then the function F defined as capital $F(x) = \int_a^x f(t)dt$ is continuous on [a,b]. And if further if f is continuous on the closed interval [a,b], then capital F is the primitive or you can write $F'(x) =$

$f(x)$.So, if you have the bounded function f small f, then in that case this continuity criteria is pretty much clear because in order to show the continuous function you take F(x) at a certain point you take F(x)-F(c)=$\int_a^x f(t)dt - \int_a^c f(t)dt$ and from there f(t) is bounded. So, it will be pretty much obvious to so the continuity of the function F, a capital F on this closed interval [a,b]. However, if your small function f is continuous then in that case your capital F is actually the anti-derivative or the primitive of this small f(x). And from this theorem in a way or from this definition in a way, the first the fundamental theorem of integral calculus is motivated.

So, from here we can state our very important theorem of Riemann integration which is basically fundamental theorem of integral calculus.

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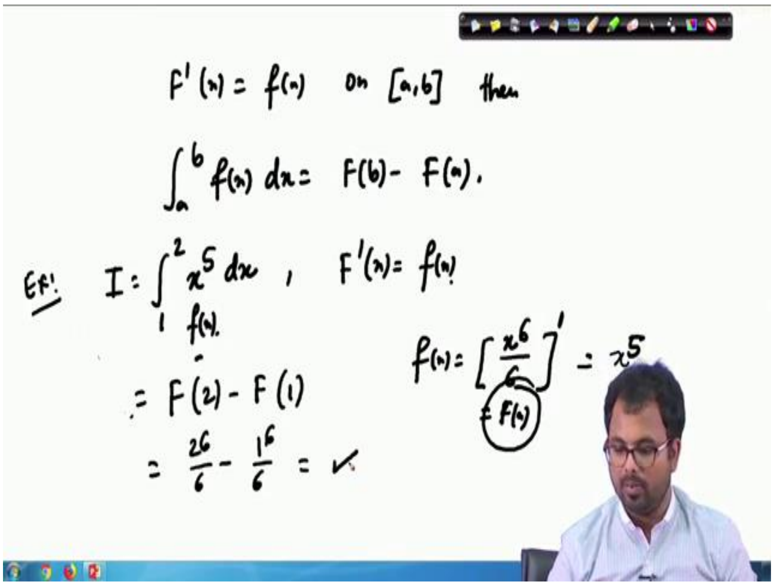

So, fundamental theorem of integral calculus says that let us say I write it as theorem. So, I can write it in terms of theorem. The theorem says that a function f is bounded and integrable on the closed interval [a,b], and there exists a function F such that $F'(x) = f(x)$ on the closed interval [a,b], then we can be able to write $\int_a^b f(x)dx = F(b) - F(a)$. So, that means, if we have a function small f(x) which is basically bounded and Riemann integrable on the closed interval [a,b], and if we have $F'(x) = f(x)$ then in that case $\int_a^b f(x)dx = F(b) - F(a)$

So, this is also in a way our Newtonian integral. So, Newtonian integral is also about finding an anti-derivative or finding how does a primitive of the function, so that you can be able to write it as a how to say integral of that function. So, what I mean is, so if we have something like $I = \int_1^2 x^5 dx$, then this is basically my small f(x). And I need to find a function capital F such that $F'(x) = f(x)$. So, this is all about finding an anti-derivative. And if we can be able to find a $F'(x)$ of this type such that $F'(x) = f(x)$, then in that case the integral would be nothing but the difference of that capital F at the point 2 minus the value of that capital F at the point 1.

So, in this case, it is very easy to find out this $F'(x)$. So, here our $F'(x)$ would be $\frac{x^6}{6}$. So, this is our $F'(x)$. And if I take the derivative of that $F'(x)$, then we will obtain x^5 and this x^5 is our small f (x) here and you see this is our capital F(x). And if I this is our capital F(x), then it is basically $\frac{2^6}{6} - \frac{1^6}{6}$ and that will be the answer.

So, based on this how to say fundamental theorem of integral calculus, we can see that how a function and its anti-derivative are connected. And it is how to say one of the important theorems in integral calculus. And it is also very interesting, and I hope you were able to understand this. I would also try to give you some examples. So, first let me state that two last theorems of this Riemann integrable functions before we jump into any example.

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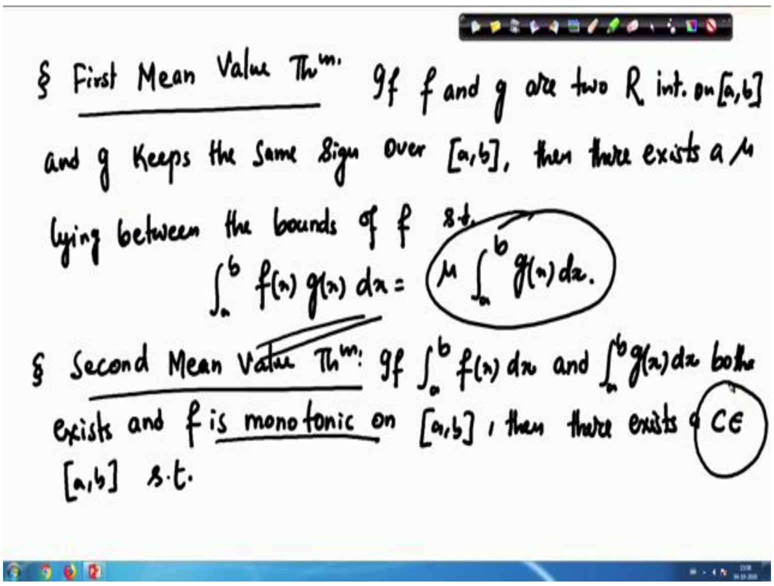

So, the first theorem is; the first theorem which I am talking about is first mean value theorem. So, the first mean value theorem says that if f and g are so it is more or less like the mean value theorem which we studied in differential calculus.

So, in the differential calculus also you have a how to say a function differentiable function on an open interval (a,b) which is continuous on the closed interval [a,b], then in that case you can be able to find a point c in between the interval (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. So, here it is pretty much the same. So, we say that f and g are two Riemann integrable functions on the closed interval [a,b], and g keeps the same sign over [a,b], then there exists μ lying between the bounds of f such that the product of the integration $\int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx$.

So, this is the first mean value theorem we says that if you have two Riemann integrable functions f and g, and the function g keeps the same sign over the interval [a,b], then you have a value basically μ which is lying in between the upper bound and the lower bound of the function f. So, the function f is assumed to be bounded because it is Riemann integrable and this point μ can be in between anywhere in of the lower bound and upper bound, then the sum of then the product of the integral $\int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx$. And this is basically our first mean value theorem.

So, here you can see that instead of calculating the product of the integral, we have replaced the function value of the function f with its value μ. So, this μ is attained somewhere. So, μ is attained at any point on this interval [a,b]. So, it can be at I do not know $x = x_2$ or x_3 any such point in the closed interval [a,b] and that value μ is basically multiplied with the integral $\int_a^b g(x)dx$ and that will actually give us the value of the product of the integral on the left hand side.

So, this is in a way a mean value theorem. So, we are taking any value of the function f between the upper bound and lower bound. The next theorem in this regard is the second mean value theorem, which states that,So, the second mean value theorem says that if $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ both exists, and f is monotonic on the interval [a,b], then there exists a c in the closed interval [a,b] such that $\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx$.

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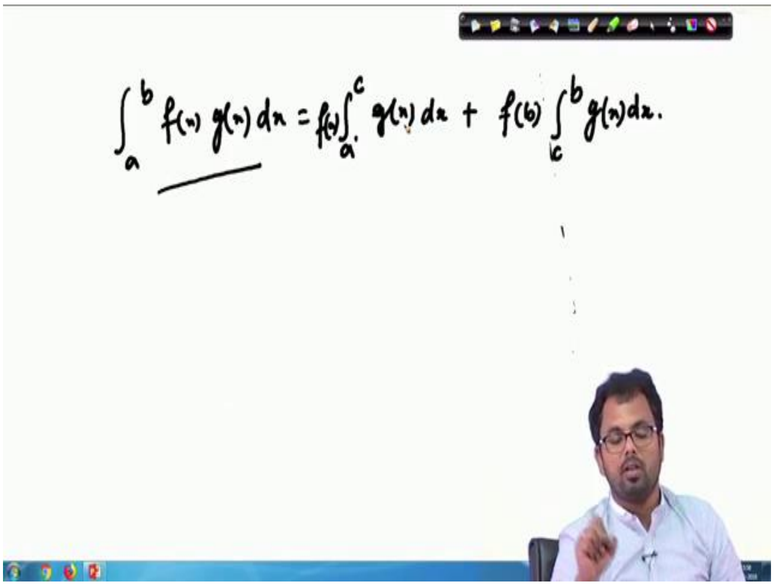

So, this is an another interesting theorem. So, we have both Riemann integrable function f and g so that these two definite integral exist. And the function f is actually monotonic on this closed interval [a,b], then in that case we have a point c lying between [a,b]. So, all we need is the monotonicity of the function f, then we can be able to find a function c in this closed interval [a,b] such that the product of the integral here $\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx$.

So; that means, in order to evaluate the product of the integral, we do not need to evaluate the product all we have to find the value of the function f at the endpoints. And then just evaluating the integral $\int_a^c g(x)dx$ and $\int_c^b g(x)dx$, we can be able to find the value of this definite integral on the left hand side. So, this is an another how to say important theorem in this regard. So, to the syllabus which I gave you earlier, and there we in the first lecture we were sort of planning to cover the partition, definition of Riemann integrable functions and some properties and then mean value theorems.

And so far we have covered the theoretical part. And since it is a very extensive topic I would recommend you to look into the books of (Refer Time: 23:48), S. K Mapa, also Santhinarayan for to know these things in detail. I will also try to work out two or three examples probably just to give you an idea and that what do we, how does it mean by these integrability conditions, and the app how to say one or two examples on fundamental theorem of integral calculus.

Also most of my lectures I will since I prefer to write it here and work out the examples or theorems here, but some of my lectures would also include a PowerPoint presentation, and I will try to make things clear in those a Power Point presentations as well. So, it will be a mixture of both this writing on the board as well as a PPT. And in the next lecture we will look into a few examples related to Riemann integrability and fundamental theorem of integral calculus. And today, we will stop our lecture here.

Thank you.