Integral and Vector Calculus
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Lecture – 39
Divergence & Curl important Identities

Hello students. So, in the previous class we were solving some examples on Divergence and a Curl of a vector function. Today we will continue with those examples and we will also address some important in some way they are important because you have say let say dot product of two vectors functions and when you are charging gradient onto them, how the formula would look like or if you have one scalar function one vector function and you are charging divergence on them, then how the formula would look like things like that. Because the not only in vector calculus, but these formulas will also be used or would be of some use in your how to say mechanics and in other branches of applied mathematics.

So, it is always nice to know these formulas and today we would try to derive at least one or two of them. So, today we will start with our first example and then we will move on to those formulas so, just to recapitulation.

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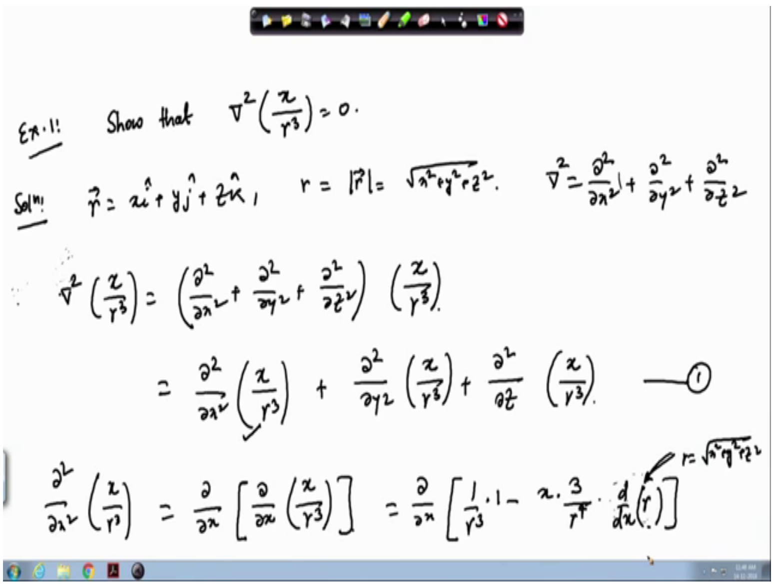

So, example one for today show that or find the value. So, show that Laplacian of x by r cube is equals to 0. So, remember our r is this one xi yj and zk. So, this is our r and when we say small r, then it is basically mod of r and that is nothing, but x square plus y square plus z square. So, in vector calculus or in vector algebra whatever you are studying its always how to say common to know that r is basically this xyz vector and mod of r is basically x square plus y square plus z square that is the length of this vector r. So, just keep in mind all right.

Now this nabla nabla double square; it means that double s square not double square nabla square is basically your Laplacian or Laplace operator. So, this is called as Laplace operator and we know that Laplace operator we know that this Laplace operator is nothing, but del square by del x square plus del square by del y square and del square by del z square; that means, partial derivative of with respect to x twice partial derivative with respect to y twice and partial derivative with respect to z twice. And on this Laplace operator, we always charge a scalar function because Laplace operator itself is a scalar operator in a way yeah.

So, that is why here you have a scalar function x by r cube; r cube is a scalar function, x is a scalar function all right hm. So, let us start with calculating our del square x by del x square. So, basically what we have isthe Laplacian of. So, what we have is so what we have is here Laplacian of x by r cube ; that means, we have to calculate del del x square plus del square y by del y square plus del square z y del z square of x by r cube all right.

So that means, we will charge this on every op how to say partial derivative. So, we have del square y by del del square by del y square x by r cube and del square by del z square x by r cube all right. So, let us first calculate this one. So, we can call it as equation 1. So, del square by del x square of x by r cube. So, this is nothing, but del del x of del del x of x by r cube all right. So, when we calculate del del x of x by r cube. So, this will be del del x of x by r cube. So, we treat it as a product of 2 functions. So, one by r cube times a derivative of the second function which is one minus x and then derivative of the first function.

So, this will be minus 3 by 3 by r to the power 4 because minus will come at the front and then it will become r to the power minus 3 minus 1. So, this and then r cube would be differentiation of r cube and then, this will be x by r right yes because r itself is x square plus y square plus z square. So that means, first of all it will become r to the power 4 and then the differentiation of r would be 1 by square root of x square plus y square plus z square. So, what we are doing is here let me let me expand this thing a little bit so just to make things clear to the you. So, then it will become then it will become sorry, then it will become d d x of d d x of 1 by r all right.

Because first we differentiate it with differentiated with respect to r and then it became then sorry then it d d x of r it will become d d x of r. So, it will become d d x of r. So, first we differentiate it with respect to r and now we are differentiating r with respect to x in a way and our r is if you look our r is x square plus y square plus z square. So, when we differentiate with respect to x, it will be 1 by 2 square root of x square plus y square plus z square times 2 x because then you differentiate with x square with respect to x.

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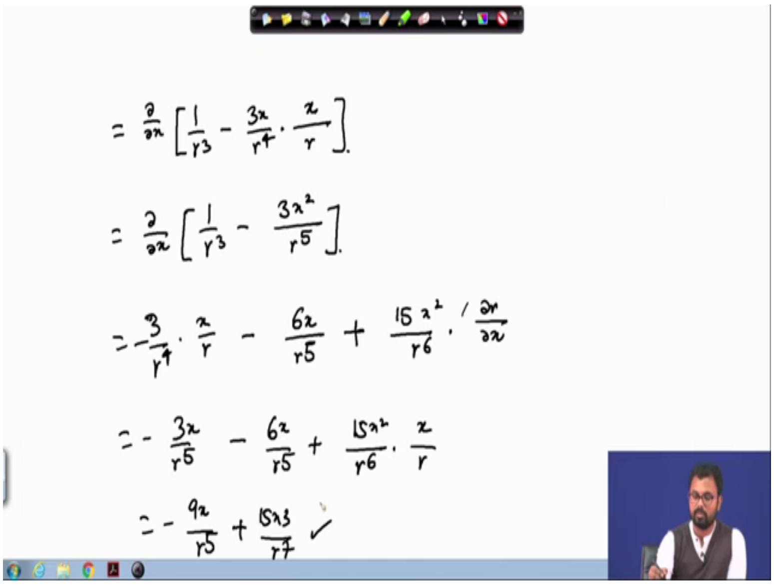

So, this whole thing this whole thing will turn into very nicely. This whole thing will turn into the del del x of 1 by r cube minus 3 x by r to the power four and then two will get cancel and then we will have x by x square, square root of x square plus y square plus z square which is again our r.

So, I wrote r it is very simple to see and it is also very simple to calculate. I am sure you can be able to do that. So, we have del del x of 1 by r cube minus 3 x square by r to the power 5 all right. Now we again differentiate with respect to x this expression. So, this one would be again 1 by sorry minus 3 by r to the power 4 and then it will become x by r like this term became. So, this is similar as doing del del x of one by r to the power 3. So, this whole thing will yield the first term will yield this thing minus 3 by r to the power 4 times x by r minus this will become 6 x r to the power 5 and then minus this will this will turn into a plus and then we will have 15 x square.

So, we will have 15 x square by r to the power 6 and then this will become del r del x Now again del r del x would be x by r. So, I can write minus 3 r to the power 5 3 x r to the power 5 minus 6 x r to the power 5 plus 15 x square by r to the power 6 times del r del x would be x by r all right. So, ultimately we will have minus of 9 x r to the power 5 and this one would be r to the power 7; so, 15 x cube r to the power 7. So, that is just the first component. Now similarly we can calculate del square y by del del y del square by del y square.

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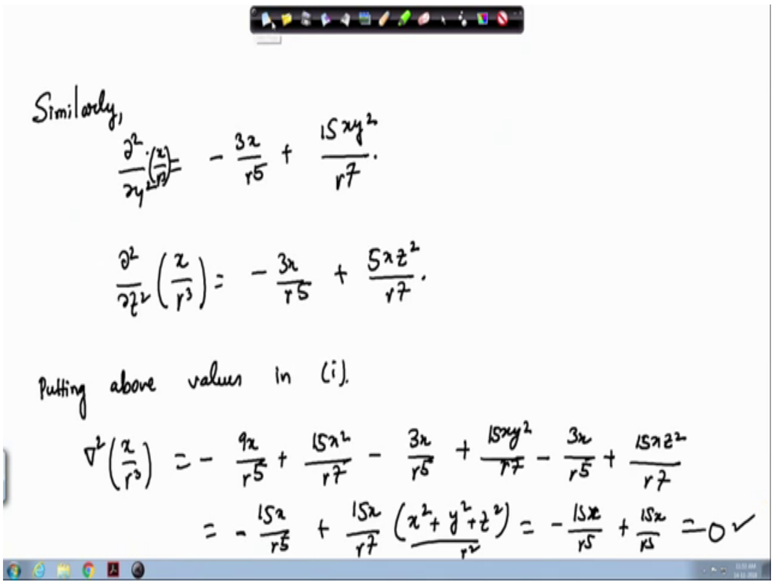

So, similarly we can calculate similarly del square y by del y square. If you calculate, then it will actually result into minus of 3 x by r to the power 5 plus 15 x y square r to the power 7 and sorry here it was that given vector function x by r cube and del square y by del del z square del square del square by del z square of x by r cube, then this will result into minus of 3 x r to the power 5 and r to the power 5 plus 5 x z square r to the power 7 all right.

So, now, let us let us let us add all these how to say all these del square del y and del z term. So, putting above values above values in 1 so, then our Laplacian of x by r cube would result into minus of 9 x by r to the power 5 plus 15 x square by r to the power 7 minus 3 x by r to the power 5 plus 15 x y square by r to the power 7 minus 3 x by r to the power 5 plus 15 x z square by r to the power 7.

So, ultimately this will result into minus of 15 x by r to the power 5 plus from here, I can take 15 x common and I can also take r to the power 7 common and then this will result into x square plus y square plus z square. Now x square plus y square plus z square is r square from here. So, from here we know that x square plus y square plus z square is r square. So, let me write r square then 1 r square would get cancel and then this will result into 15 x by r to the power 5 plus 15 x by r to the power 5 because r square 1 r square will get adjusted here.

So, we will have this thing this is equal to 0. So, they and this is what we needed to prove right. So, this is what we needed to prove and therefore, we have the answer. So, it is all about doing partial derivative carefully and I am pretty sure you can be able to solve these type of problems all right. Now I think we have covered enough examples on calculation of gradient divergence and curl of several types of vector and scalar functions and now we move on to some vector identities all right.

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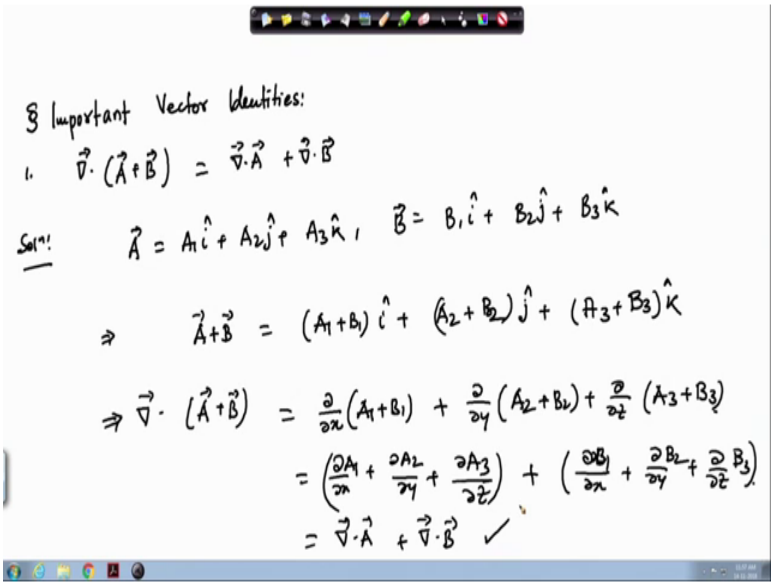

So, let us move on to some vector identities or important let us say important vector identities ok. So, the first identity is and we assume that all these vector functions have can have continuous partial derivatives.

So, these are all vector functions of variables x y and z and they all have continuous partial derivatives so, that we can at least talk about their gradient, divergence and curl. So, I am not going to write all those things, I will just write the vector identities. So, suppose we have A and B as the 2 vector functions, then divergence of A plus B. So, A and B are 2 vector functions then divergence of A plus B would be divergence of A plus divergence of B, I wrote already this result.

I do not remember, but I think I may have written this and of course, doing this is not complicated. So, we basically do so divergence of the sum is equal to sum of the divergence. So, from here we charge the divergence on both sides and then we separate divergence for a divergence of for B and then that will give you the required answer all right.

So, here this one proving this one is not complicated; let me how to say show you at least one of them. So, A is a vector function. So, A has 3 components A1 i A 2 j and A 3 k. And B has 3 components so, B is equals to B 1 i plus B 2 j plus B 3 k. So, when we talk about A plus B so; that means, we will sum that component. So, A 1 plus B 1 times i A 2 plus B 2 times j and A 3 plus B 3 times k. And now when we charge the divergence on both sides, then this will become divergence of A plus B is equal to del del x of A 1 plus B 1 del del y of because it is a it is it usually I mean it always yields a scalar functions or del del y of A 2 plus B 2 and del del z of A 3 plus B 3.

So, we can separate the term for A 1 at one place del del x of B 1 del del y of B 2 and del del z of B 3. So now, this is nothing, but our divergence of A right and this is nothing, but our divergence of B and this is what we needed to prove. So, divergence of the sum is equals to sum of the divergence this result this result follows.

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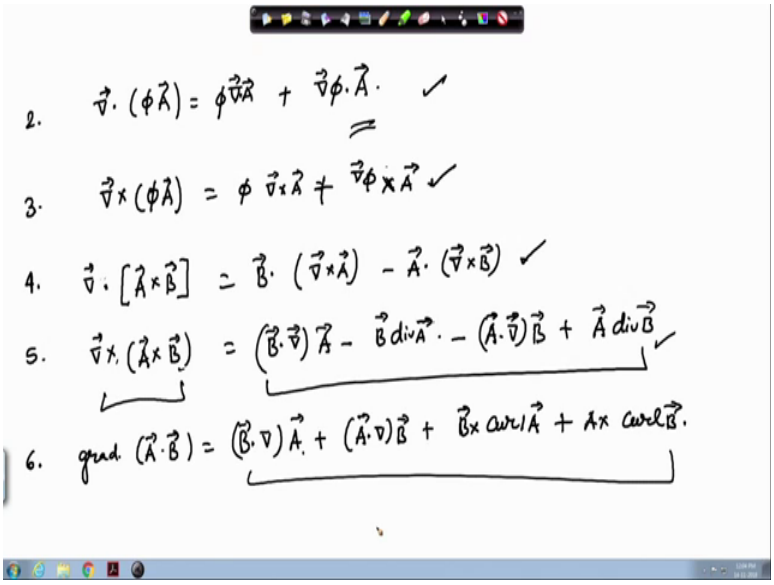

Now second result is let us say we have divergence of phi A, then this one will be divergence of A times phi plus divergence of phi times A. Now since we had a dot product here, we can write A dot A dot gradient of phi or gradient of phi dot A that is all fine, but if it was if it were a curl here, then this formula cannot be how to say played with.

So, this is basically divergence of phi times vector A and this result proving this result would also be not difficult. So, we multiply every component of A with phi and then we just follow the product product rule of 2 scalar functions and th

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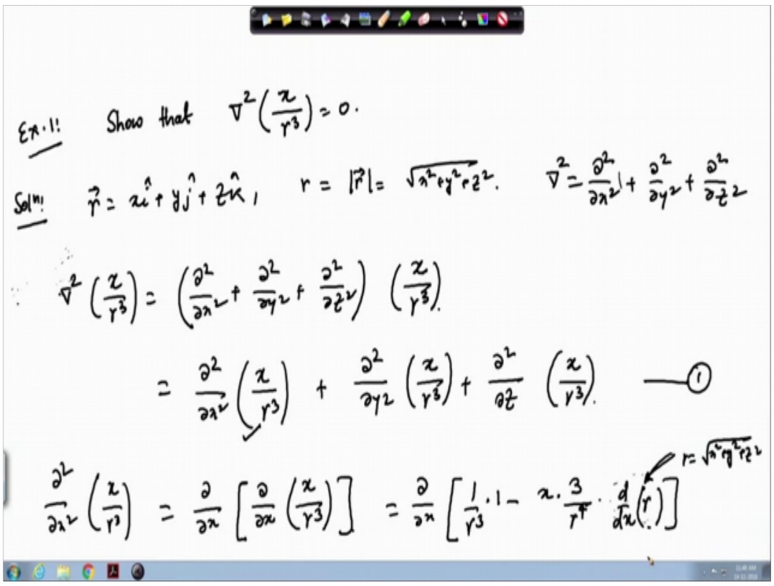

So, example one for today show that or find the value. So, show that Laplacian of x by r cube is equals to 0. So, remember remember our r is this one xi yj and zk. So, this is our r en you equate the terms. So, you keep the terms for phi at one place and term for A t one place, you take something common and that will give you the required result. So, provi g this one would also be not difficult and it is very straightforward. Third result is curl o now this is interesting; now curl of phi A. So, this can be written as phi times curl of A and then curl of phi sorry plus curl of phi as sor

So, phi times curl of A plus gradient of phi times curl of A. So, this is our required result and here I cannot write A times A curl A curl A a cross product with divergence of phi. So, that I cannot write because then in that case, I have to put A minus sign here. So, here I can write A dot gradient of phi, but here I cannot write A cross gradient of phi because as I said this formula cannot be played with all right.

Now next we have suppose instead of one scalar function if we have 2 vector functions. So, I have 2 vector functions, I do not have any scalar function. So, this can be proved easily this can be proved easily. Now when we have 2 vector functions and then in that case this will be B dot dot product with sorry.

So, when we have so let us start with the dot product. So, when we have dot here , I will come to the curl formula. So, when we have dot here, then B dot curl of A because ultimately we have to obtain a scalar function. So, since we are taking the divergence of A vector. So, it will the left hand side will always be a scalar. So, the right hand side should also be a scalar. So, this minus A dot curl of B so, the minute I take dot product it will become a scalar function here also it will become a scalar function. So, ultimately we are getting a scalar function. Proving this one is also not difficult here we basically calculate the curl of A the cross product of A and B.

So, that will be like i j k and then component of A, component of B you break that determinant and charge the gradient and then separate the terms basically and that will give you the required answer. So, since they are a little bit tedious, but obvious to do I am avoiding them; however, they are very important I mean it is very nicehow to say important to know these formulas, because they might be useful at some point all right. Next we have is curl of A cross B, then in that case this will be if we have curl of A cross B then this will be B dot divergence gradient.

So, then A minus B divergence of A sorry divergence of A and then minus of A dot nabla times B plus A divergence of B so; that means, here we are obtaining a scale vector quantity. So, A cross B is a vector and its curl is also vector.

So, it is very obvious that the right hand side should also be a vector all right. So, right hand side should also be a vector so; that means, the formula would read as B dot nabla. So, B dot nabla would become scalar however, when you charge a vector then it would become a vector. Similarly B dot B times divergence of A. So, divergence of A would be scalar and then you multiply by a vector. So, ultimately this is also vector. A dot nabla is a scalar, but in when you charge B and then in that case it will become a vector and divergence of B is a scalar, but multiplying with a vector A is again vector.

So; that means, from the scalar and vector point of view these 2 in these 2 qualities are settled. Now we have to prove this. So, what you can do? You can extract the components I mean you so, first calculate the curl of A and B and then whatever curl you get you charge nabla and then you get a certain expression, then similarly you calculate every term on the right hand side and show that they are same. So, of course, this will be a slightly complicated or may be time taking in a way, but it is not difficult. So, it is not difficult or it is not tough it is just tedious or too long to do, but if you remember the formula that is fine. So, this is an another formula from the divergence and curl point of view.

Our next formula is our next formula is gradient of so, our next formula is gradient of A dot B equals to B dot nabla A plus A dot nabla B plus B dot B cross curl of A plus A cross curl of B. So, gradient of A dot Bis equals to B dot nabla A. Let us put A vector sign and then plus A dot nabla B plus B cross curl of A and then A cross curl of B. Again this the result is not complicated; it is just that you have to calculate the left hand side and right hand side and show that they are same. So, of course, calculating the left hand side is simpler compared to the right hand side. So, you basically calculate A dot B.

So, it is A 1 B 1 A 2 B 2 A A 1B 1 plus A 2 B 2 plus A 3 B 3 and then you charge the gradient and then you obtain an expression and you do the similar thing on the left hand and the right hand side and show that they are same all right. This will also be a little bit tedious or long to do, but it is simple.

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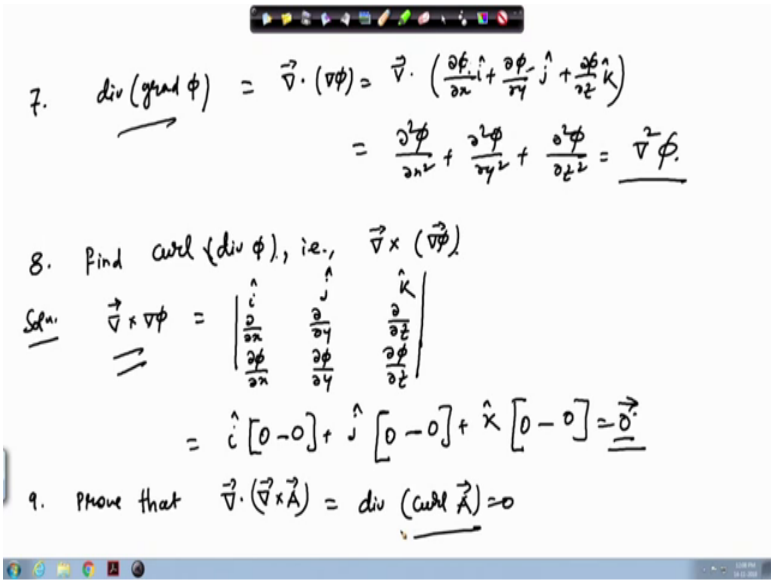

Our next formula isour next formula is divergence of gradient of phi so; that means, we have divergence of gradient of phi. And this is nothing, but divergence of gradient of phi is del phi one del x plus del phi 2 by del y i and sorry this one is deldel phi del x plus del phi del y plus del phi del z times k. And when you charge this divergence, then it will become del square phi by del x square del square phi by del y square and del square phi by del z square. So, ultimately this is Laplacian of phi.

So, basically divergence of gradient of phi is actually Laplacian of phi right; very simple to solve and now this is interesting find curl of divergence of phi or divergence of phi. So, the solution curl of divergence of phi; that means, we have to that is we have to find nabla cross product with divergence of phi. So, nabla cross product with divergence of phi is basically i j and k nabla is del del x del del y and del del z and divergence of phi would be del phi del x del phi del y and del phi del z.

So, if you break this determinant then it will be i times del del y of del phi del z which will be 0 minus del del z of del phi del y again 0, then j times del del z of del phi del x 0 del del z del del x of del phi del z is again 0 and k times del del x of del phi del y is 0 minus del del y of del phi del x is 0. So, ultimately a 0 vector so; that means, curl of divergence of phi regardless of what type of function phi is this will always be 0 all right. So, it is always a 0 vector. Similarly we can show that similarly we can show that or prove that we can prove that divergence of curl of A; so, similarly we can prove the divergence of curl of A.

Let us write in words divergence curl A is also equals to 0. So, first we calculate the curl and then when you take the divergence you will always get del del x of del A to del y minus del A 3 del z. So, basically you will differentiate with respect to a variable for which you will end up getting a constant which you end up getting a 0 partial derivative in a way.

So, this can also be proved very easily and here in this case, you will actually obtain the similar type of partial derivatives. So, here in this case you do not obtain how to say 0 partial derivative, you basically you obtain partial derivatives which are same and therefore, they will cancel out. So, it will be done like that and the final result is the final result is curl of curl of A.

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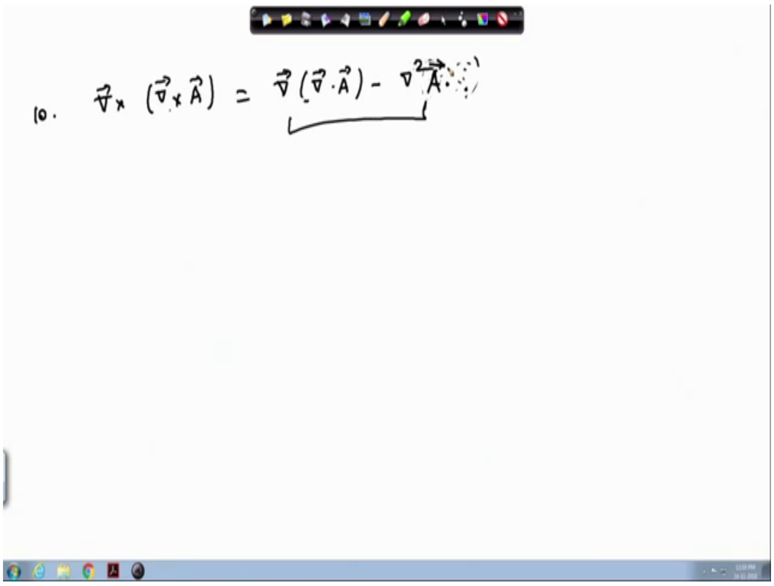

So, you have curl of curl of A then this will be gradient of divergence of A minus Laplacian of all right. So, this is also quite easy to prove. So, first we calculate the left hand side and then we calculate the right hand side and we show that they are same. So, let us see if it makes sense or not. So, we have curl of; so, we have curl of curl of A so; that means, this will result into this will result into a vector and here, it will also give us a vector and this one would also give us a vector because here we have A vector and ultimately we will obtain this result all right.

Yes. So, this is basically it is better to write this as this way. So, it is not exactly a Laplacian. So, we write it as this way. So, then you have a scalar, then this will become a vector and then you have a scalar then this again becomes a vector. So, this and now I get it. So, we can write it as Laplacian yes. So, we can write it as Laplacian of a vector A.

So, this is the required how to sayresult and you just calculate the left hand side and right hand side and show them. They are same and that this would complete our require our how to say collection of ten identities there are some more results we will not get into them and in the next class we will probably practice few few more examples maybe a couple of more on say gradient divergence and curl of a vector function and then we move on to our next topic which is directional derivative of a vector function.

So, I will stop here for today and I look forward to you in our next class.

Thank you.