**Integral and Vector Calculus**
**Prof. Hari Shankar Mahato**
**Department of Mathematics**
**Indian Institute of Technology, Kharagpur**

**Lecture – 38**
**Divergence & Curl Examples**

Hello students. So, in the last class we introduced the concepts of a gradient of a scalar function and Divergence and Curl of a vector function. We also worked out few examples on a gradient of a scalar function; how do you calculate it and then we gave definitions of divergence and curl of a vector function.

Today we will practice few more examples, but basically we will start with the examples on divergence and the curl of a vector function and we will see how we calculate them and there are some results which are sort of important from these two these two concepts. So, let us start with our with our a how to say examples on divergence and curl.

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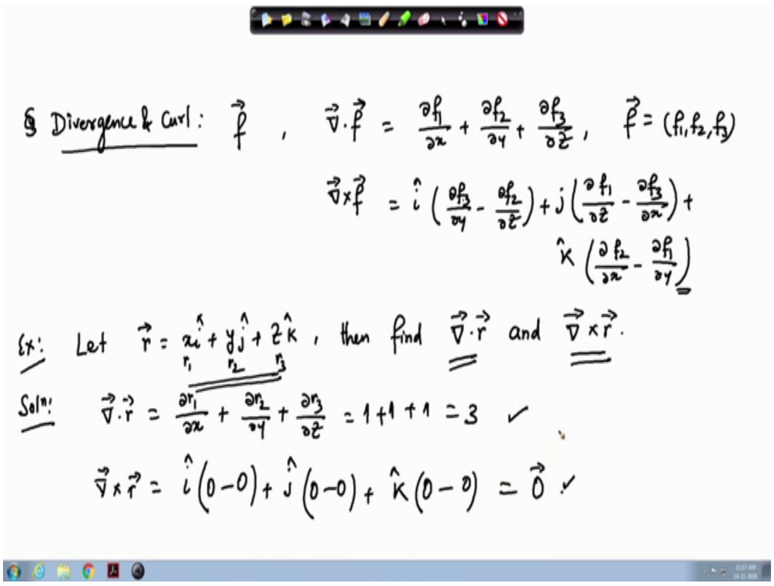


So, we yesterday we introduced that if f is a vector function, then the divergence of f is basically defined as a gradient of I sorry this nabla dot f and it is basically del f 1 del x plus del f 2 del y plus del f 3 del z where f is a vector function and it has three components f 1 f 2 and f 3 and each f 1, f 2, f 3 are functions of xy and z all right.

Similarly the curl of f is defined as nabla cross product with f and we have calculated via the determinant what it would look like. So, in short I can write the formula. So, the formula would go like this i times del f 3 by del y minus del f 2 by del z, then j times del f 1 by del z minus del f 3 by del x plus k times del f 2 by del x minus del f 1 by del y. You can be able to calculate with that with the help of that determinant. So, this is basically our how to say curl of a vector function f.

Now motivated from these 2, this how to say formulas or definitions, we will first start with our example one. So, let r is equals to x i plus yj plus zk. So, when I write i cap so; that means, it is a unit vector all right and then find divergence of r and curl of r right. So, let us try to find. So, solution let us try to find divergence of r. So, divergence of r would go like this we know that from the formula r has 1 components; so, r 1, r 2, r 3. So, I can be able to write this formula del r 1 del x plus del r 2 del y plus del r 3 del z now r 1 is basically our x. So, del del x of x is 1 plus del del of y is also 1 and del del z of z is also 1. So, ultimately divergence of r is equals to 3 if r is given as this vector here.

Now, we will calculate the curl of r. So, if I calculate the curl of r then it will be, i times and del r 3. So, we just go through this formula here. So, del r 3 by del y so, this is basically our r 3. So, I can write r 1 r 2 and r 3 just for our clarification, you really do not have to write r 1 r 2 r 3 at the bottom of these 3 variables. If you really want to, then you can say let r is equals to this where r 1 is equals to x r 2 equals to y and r 3 cost result that you can do.

So, now, our r 3 is basically z so, del del y of z is 0 minus our r 2 is y. So, del del z of y is again 0 plus j. So, del our r 1 is x. So, del del z of x is 0 our r 3 is z and del del x of z is 0 plus k times our r 2 is y. So, del del x of y is 0 minus del del y of f 1. So, a r 1. So, r 1 is x so, this is again 0. So, 0 times i 0 times j 0 times k.

So, ultimately we will get a 0 vector. So that means, if r is equals to this vector here than in that case our divergence and curl would be given in this fashion. So, divergence is 3 and curl is 0. So, that is how we basically calculate this a divergence and curl let us let us consider a few more examples.

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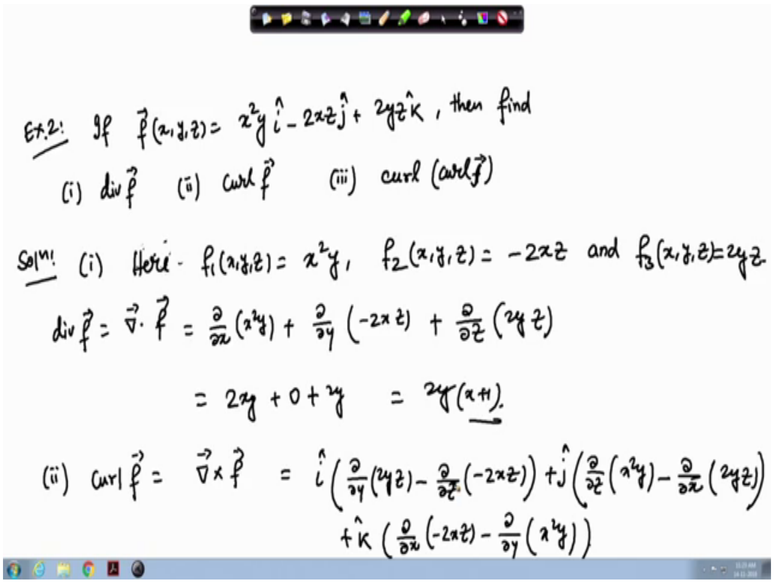


So, just so we can be thorough in this thing. So, example let us say 2 and the example states that if our vector function f x, y, z is equals to x square y times i minus 2 x z times j plus 2 y z times k then find first divergence of f, then curl of f and then third one is curl of curl of f. So, the thing is here curl is basically curl of a vector function is again a vector quantity.

So, we can talk about its curl again. So, unless the quantity is a vector function is na unless the quantity is not a vector function, we cannot talk about its curl. But as soon as soon as the quantity becomes or a function becomes a vector function will not becomes if it is a vector function, then in that case we can always talk about its curl and that is what is that is what is happening in this case.

F is a vector function, but once we charge the curl it again the else a vector function and therefore, we can again charge curl on it which will again give us a vector function and we can again charge curl on it. So, as soon as long as the function or the I mean we have a vector function, we can always talk about its curl. However, in case of divergence, we cannot charge divergence again because the divergence of f would yield of scalar function

So, once we get a scalar function we can talk about its gradient, but we cannot talk about its divergence. So, what could happen is, when we charge divergence on a vector function, it would give us a scalar function and then we charge a gradient. And once we charge the gradient, then it would become a vector function and then we can again charge gradient as a divergence and then it will again yield a scalar function.

So, divergence gradient divergence gradient and so, on; however, in case of curl every time we can charge a curl on it. So, these are some small aspects of vector calculus and I am pretty show you may have got an idea that, how to deal with these operators all right. So, let us start calculating the first how to say divergence of f.

So, solution divergence of f or let me start by writing so just start with writing here. So, I am just writing here f 1 x, y, z equals to x square y f 2 x, y, z equals to minus 2 x z and f 3 x y z equals to 2 y z all right. So, I just wrote the 3 components just for our how to say seek. Now we have divergence of f, I can write it in terms of notation as well. So, this is nabla dot f and now in divergence, we can go back to the formula its del f 1 del x, del f 2 del y, del f 3 del z.

So, del del x of f 1 f 1 is x square y plus del del y of f 2 f 2 is minus of 2 x z and del del z of f 3 which is basically 2 y z. So, if I do del del x, then this is basically 2 xy and then here we do not have any y variables. So, the partial derivative would be 0 and here it would yield del del z so, 2 y. So, if I take 2 y common and this will be x plus 1 and that is the required divergence of f all right. So, this is what we obtained as a divergence of x divergence of f.

Now, we can calculate the curl of f. So, curl of f I can write it as nabla cross f. I prefer to write in terms of there is operators instead of writing in words. Now we have i times lets go back to the formula del f 3 del y minus del f 2 del z. So, del f 3 del y. So, what is our f 3? f 3 is 2 y z minus del del z of in the formula del f 2 del z. So, what is our f 2? f 2 is minus of 2 x z plus j times we have del f 1 del z ; we have del f 1 del z.

So, del del z of f 1 which is x square y minus del del x of del del x of f 3. So, del del x of f 3 is basically 2 y z all right and then k times del del x k times del del x of f 2. So, del del x of f 2 is minus of 2 x z minus del del y of del del y of f 1. So, del del y of f 1 is x square y all right. Now doing this partial derivatives is very simple and I leave that task up to the students.

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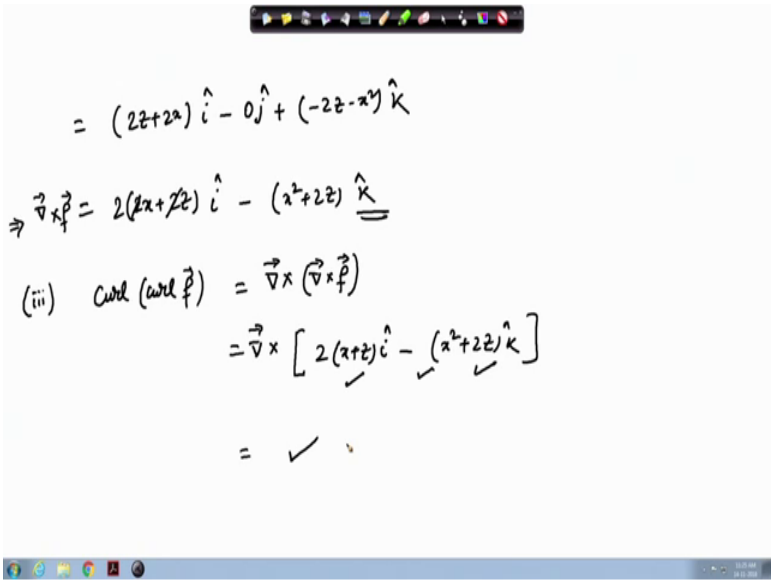

So, here at the end you will basically obtain your answer as 2 z plus 2 x times i minus 0 component of j plus minus of 2 z minus x square times k. So, ultimately we will obtain 2 x plus 2 z or we can take this to common. So, let us not write 2 here we can take 2 common times i minus of x square plus 2 z k. So, this is our required curl of f. So, I can write it as implication sign.

So, curl of f and our problem 3 is curl of curl of f right so; that means, we basically have nabla cross product with nabla cross f. So; that means, this is nothing, but nabla cross 2 of 2 times x plus z times i minus x square plus 2 z k. So, here we have to identify our f 1, f 2 and f 3 and then put everything then put everything back in this formula, and we would obtain our required result all right. So, I am leaving that task up to the students.

So, doing this is not complicated so, you can we can identify the 3 components and just follow the previous formula and that will give you the required result all right.

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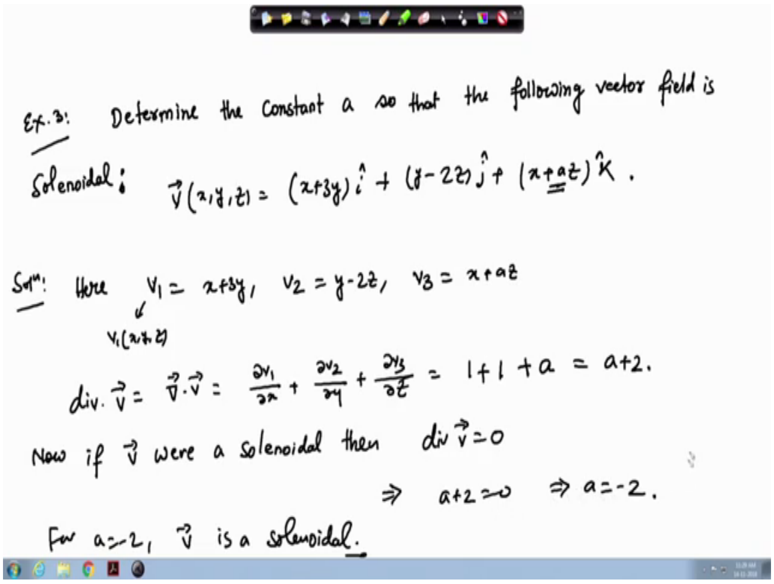


Now we will consider some a slightly different types of examples. So, let us start with them. So, here this says that determine determine the constant a so that the following vector field is solenoidal solenoidal. So, what is that? So, V xyz equals to the given vector as x plus 3 y times i plus y minus 2 z times j and x plus az times k.

So, we have to determine this a here so, that this vector field will be a solenoidal all right. So, this is our given vector field. Now we know that from the definition in the previous class that a vector field is solenoidal when divergence of V is equals to 0 so; that means, we have to make. So, we assume that the vector field is solenoidal and we make divergence of V equals to 0 and from there we have to find out that for what value of a that divergence of V equals to 0 actually occurs all right.

So, we basically solve for a equals to some constant and that will be the required how to say the answer or the required value of a for which this vector field is solenoidal. So, let us start with calculating the divergence of V. So, here we can again identify our 3 components V 1 is x plus 3 y V 2 x y z is y minus 2 z and V 3 x y z is x plus az. So, each component is a function of x y z. The correct way to write is actually V 1 of x y z all right ; however, you are allowed to write V 1 V 1 only as well, but the correct way to write is V 1 xyz because it is actually a function of x y z each component basically.

Now we calculate the divergence of V. So, divergence of V is gradient sorry nabla dot V and this will be del V 1 del x del V 2 del y plus del V 3 del z. So, del V 1 del x would be

1, then del V to del y is again 1 and del V 3 del z is a so; that means, we have a plus 2. Now if we were a solenoidal solenoidal, then divergence of V must be equals to 0 because that that is a definition and now if divergence of V is equals to 0; that means, a plus 2 must be equal to 0. So, from here we will obtain a equals 2 minus 2. So, for a equals to minus 2 V is a solenoidal field is a solenoidal all right.

So, we start with the so, after calculating the versions of we will start with the assumption that we have a solenoidal field and then in that case divergence of V must be 0 and from there we calculate a equals to minus 2 all right. So, this is our required value for a all right. Now let us move let us move on to our next example.

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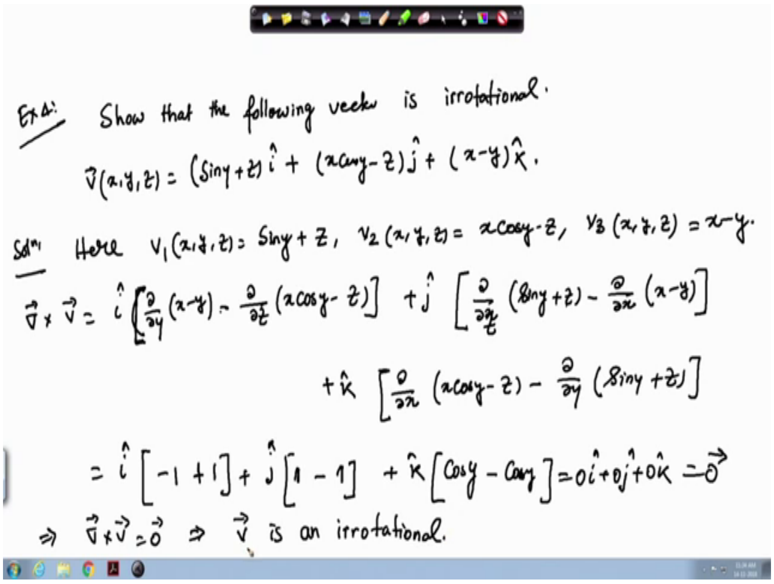


So, our next example says that show that the following vector so that the following vector is irrotational. So, our given vector is V x y z. So, this is our given vector function all right sin y plus z times i plus x cos y minus of z times j plus x minus y times k. So, we have a given vector function or a vector field and we have to show that this vector function is actually irrotational. So, from the definition of irrotational which we learnt in the previous lecture, our vector function is irrotational when that curl of that vector function is 0.

So, here we have to show that curl of V is equals to 0 and then it will be a irrotational field. So, let us first calculate or identify the 3 components and then we calculate curl of f all right. So, here our V 1 our V 1 xyz is sin y plus z then V 2 x y z is x cos y minus z and

V 3 x y z is x minus y all right. Now we calculate the curl of v. So, curl of V is nabla cross V and from the formula, we have let us go back to the formula. So, it says del V 3 by del y minus del V 2 by del z. So, we can come back to the form of del V 3 by del y so, del del y of V 3, V 3 is x minus y minus del del z of V 2. So, del del z of V 2 is x cos y minus of z. We can use the third bracket the bigger bracket actually times j and then we have del del x of then we have del del z sorry del del z of del del z of f 1.

So, f 1 is sin y plus z minus del del x of f 3; f 3 is x minus y plus k times del del x of f 2; f 2 is x cos y minus z minus del del y of f 1 ; f 1 is sin y plus z is not it all right. So, now, when we calculate so, del del y of x is 0 because x will be treated as a constant and del del y of y will be minus 1 minus del del z of x cos y would be 0 because there is no z here and minus of minus; so, this will be plus delta z of z is again 1.

Now j times del del z of sin y plus z. So, sin y is constant when we are differentiating with respect to z and partial derivative of z with respect to z would be one minus del del x of x is 1 and del del x of y is 0 and k times del del x of x cos y will be cos y and cos y and del del x of z would be 0 minus del del y of sin y will be cos y and del del y of that would be 0.

So, basically we have 0 0 0 so; that means, we have 0 times i plus 0 times j plus 0 times k. So, ultimately we obtain a 0 vector. So, from here, we have curl of V curl of V equals to 0 vector and this implies that V is an l irrotational vector field. So, if the curl of V equals to 0, then the vector field is said to be irrotational and we proved that the given vector function V is actually a irrotational.

So, all you have to do is remember these formulas that if the versions of V is 0, then it is solenoidal and if curl of V is 0, then it is a irrotational and you just have to verify or if you were asked to calculate the value of a certain parameter for which it is either divergence or a either solenoidal or irrotational, then you basically do what we did in the previous example all right.

Now, let us just consider a few more examples; some remarks are also to be addressed.

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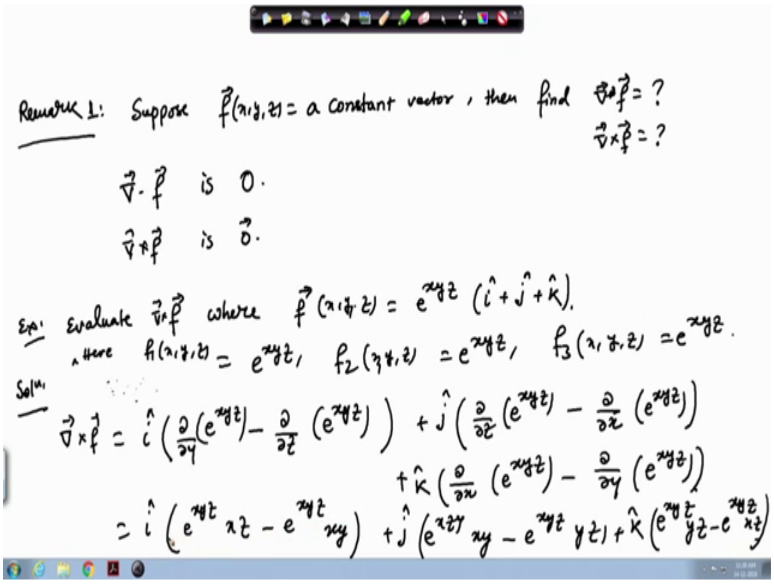


So, let us consider our first remark suppose f x y z equals to a constant vector all right; a constant vector, then find gradient sorry divergence divergence of f and curl of f. So, the thing is solution or the thing is if f is a constant, then in that case a constant vector then in that case when you charge this in nabla operator nabla operator has partial derivatives del del x del del y del del z. Now how it has been charged depending on whether you are charging the divergence or whether you are charging the curl, but whatever that vector function is it will always be differentiated with a with respect to xy and z.

So, if your f is a constant function I mean, it will always be differentiated with respect to xy and z and the minute you differentiate every component of that constant vector, it will lead to a 0 vector because a constant vector differentiation of it will always be 0. So, regardless whether you are calculating divergence or curl, the answer would always be 0.

In the first case it will be scalar 0 and in the second case, it would be a 0 vector because in case of curl we get a vector whose every component is actually a 0. So, it is very simple to see and I am leaving this result up to the students. So, divergence of f is actually 0 and curl of f is actually a 0 vector. So, these 2 things are quite straightforward to see and I am leaving these up to the students all right.

Next is so next we consider an example. So, next we consider an example. Evaluate curl of curl f where our f x y z, it is an little bit interesting function; it is a slightly interesting function. So, e to the power x y z i plus j plus k so, here we have to calculate the curl of f.

So, let us go to the solution. So, curl of f we know that the formula. So, here our 3 components are e xyz e xyz e xyz because they are all same. So, I can write here ah; here f 1 x y z. So, here f 1 x y z equals to e to the power x y z f 2 x y z equals to e to the power x y z and similarly f 3 x y z is equals to e to the power x y z. So, now, we calculate curl of f. So, curl of f would be i times it would be del del y of f 3.

So, del del y of f 3 is e to the power x y z minus del del z of f 2 del del z of f 2 would be e to the power x y z, then j times del del z of f 1. So, del del z of e to the power x y z minus del del x of f 3; f 3 is e to the power x y z plus k times del del x of f 2 which is e to the power x y z minus del del y of f 1 which is again e to the power x y z.

So, now, when we differentiate e to the power x y z with respect to del with respect to y so, this will yield e to the power x y z times again we differentiate this one. So, times x z minus e to the power x y z times this will be xy plus j times e to the power x y z and then this will be x y minus del del x. So, this will be e to the power xyz yz plus k times e to the power x y z. This will be yz minus e to the power x y z and this will be x z. So, now, we will take it to the power x, why is that common? So, here it will be x times e to the power.

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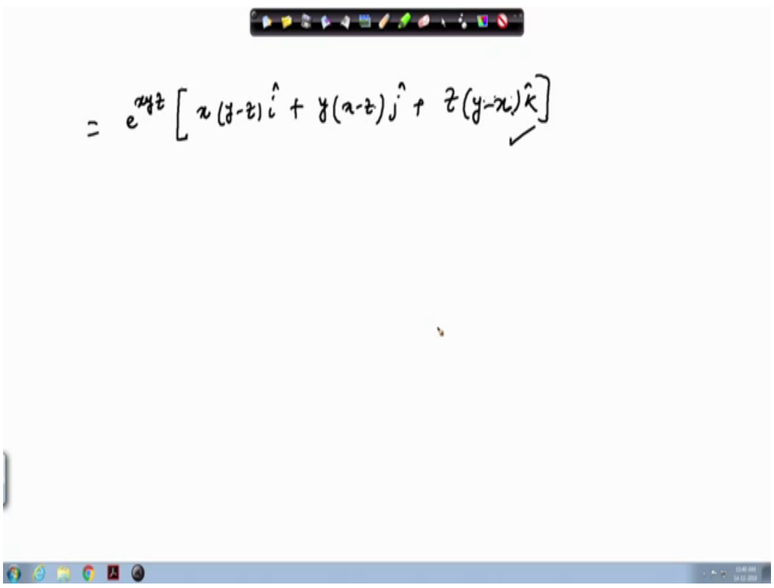

So, this will be basically e to the power x y z common and then we have x times y minus z times i plus y times z minus x times j and this will be z times x minus y times k. Did I do it correctly? y common then this will be x minus z and that will be y minus x all right.

So, this will be y minus x this will be y minus x and this will be x minus z. So, let us correct it. So, x minus z and y minus x all right; x minus z and y minus x so, this is the required curl of f for this given vector function f. Now today we try to cover as many examples as possible. We will probably continue with our examples in our next class and I look forward to it.

Thank you for attention.