Integral and Vector Calculus
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Lecture – 37
Divergence & Curl

Hello students. So, in the previous class, we introduced the concepts of gradient of a scalar function, we also talked about its properties and we also introduced a worked out one or two examples. And in today's lecture, we will actually continue working on the examples, and then I will try to introduce the concept of divergence of a vector function. So, let us work out one or two more examples before we can actually go to the divergence.

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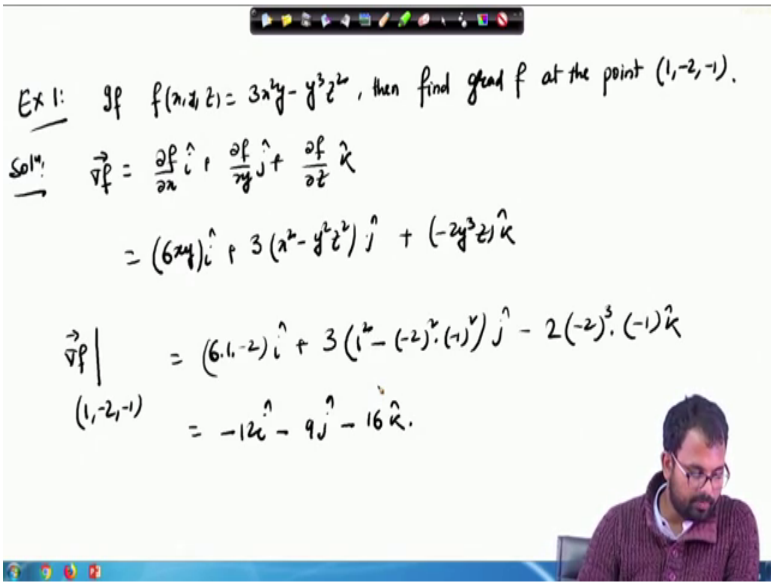

So, example let us say one for today if f x, y, z is equals to 3 x square y minus y cube z square, then find gradient of f at the point 1, minus 2, minus 1. So, here I were given a scalar function is this one, and we have to calculate gradient at a certain point. So, let us calculate or let us see how we can do that.

So, gradient of f is denoted by this symbol, and then we have del x f, del f del y j and del f del z k. So, now if I differentiate this function with respect to x, it will be 6 x y times i because the rest of the variables will be treated as a constant. Now, when we differentiate with respect to y, so it will be 3 x square minus y square z square j. And if I differentiate this with respect to z, then in that case it will be minus 2 y cube and z k. So, this is the required how to say gradient of f.

And now, if you want to calculate the gradient of f at the point 1, minus 2, minus 1. So, we just substitute these values here, so it will be 6 times 1 times minus 2 times i plus 3 x square is 1 square minus y square is minus 2 square times minus 1 square times j and this one is minus 2 y cube. So, minus 2 cube times z is minus 1 times k. So, we just substituted the value of x, y and z in the gradient of f.

So, gradient of f is basically how to say is directed outward to that to a given surface. So, if you have a surface f, f x, y, z will actually signify a surface; and the gradient of f is actually to direct it towards the normal. So, it is actually perpendicular to the surface. And at every point you can be able to calculate this gradient of f. So, on this surface f x, y, z gradient of f is directed towards the normal, it is perpendicular to the surface. So, you can be able to calculate the points where this gradient of where the gradient of f or what gradient of f is perpendicular to that surface. At a certain point this whatever we get has a calculation, so that will be actually gradient of f that is perpendicular at the point 1, minus 2, minus 1 to the surface f x, y, z.

So, if I do the calculation then we will ultimately obtain minus 12 i minus 9 j minus 16 k. So, there is the required gradient of f at a certain point minus 1, minus 2, 1, minus 2, minus 1.

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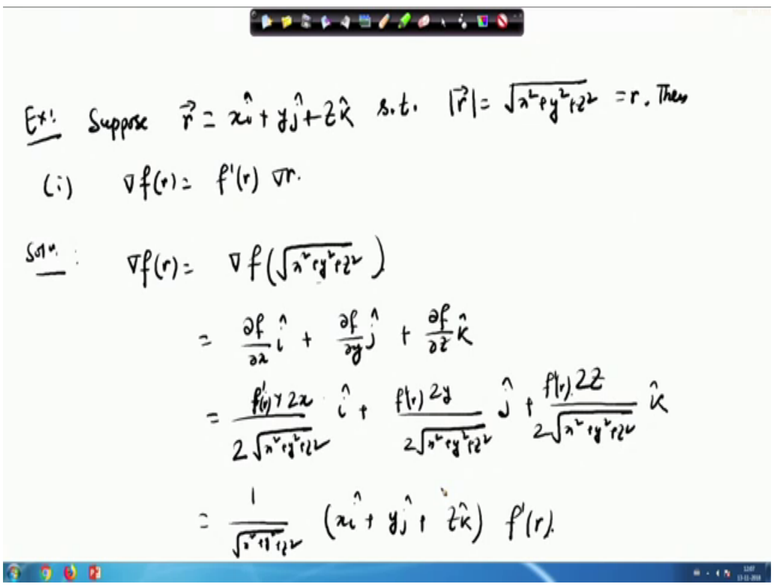

Similarly we can calculate let us say if we have suppose r equals to x i y j plus z k such that mod of r equals to x square plus y square plus z square all right. And we can simply call it as r, then the first thing that we have to show gradient of f r equals to f dash r times del r.

So, let us see how we can prove that all right. So, in order to show this result, we can write gradient of f r, now r is the, r is this how to say this here. So, r is x square plus y square plus z square. So, I can write r f of gradient of x square plus y square plus z square all right. And this is basically del f del x i del f del y j and then del f del z k.

So, now, when I am doing partial derivative with respect to x with respect to x, it will be 1 by 2 square root of x square plus y square plus z square and then 2 x times i, this here will be 2 y divided by two square root of x square plus y square plus z square times j and this will be 2 z times divided by 2 times x square plus y square plus z square k.

So, here I can be able to write 1 by x square plus y square plus z square, and I have x i plus of course, there is a there is one f dash x as well. So, there is one f dash f dash f dash r as well if this one is f dash r x square plus y square ok. And so here I have del f del x del f del y. So, this can be written as f dash x of course, f dash r, and then this.

And similarly f dash r, then this and f dash r, then this because this one would remain f dash of that and then we differentiate r. So, r will be a 1 by 2 x square and then we will have 1 by x square plus y square plus z square. And this one will be 2 x square or we can write it as y j plus z k times f dash r all right. So, now, we can now this one is basically my r. And of course, here I will have one del f del x. So, here I will have a del f del x.

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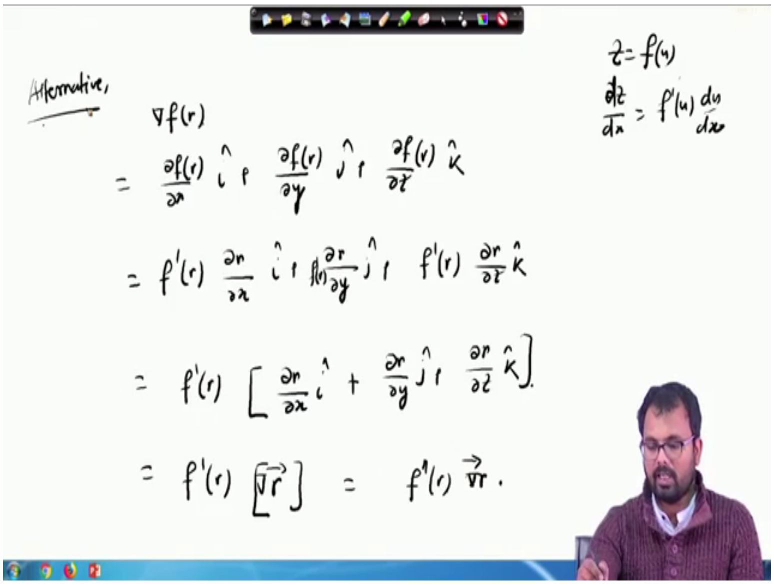

Or instead of going into this complicated calculation what we can do is we can write simply a gradient of f r. So, gradient of f r is basically del f del x r times i del f. So, we have we have to calculate the gradient of we have to calculate gradient of this here right. So, this is what we needed to calculate. So, del f del y r j and del f del z r k. So, this will be rather must simpler. So, when we differentiate, we will differentiate with respect to r. So, f dash r and then it will be del r del x. So, it is like product also.

If we have z equals to f of u. So, when we differentiate. So, del dz dx is equals to we know that we can write f dash u times d u dx. So, we can write it as d u dx times. So, here we can write dd u times du dx and then differentiation of this is f dash u and then d u dx. So, similar thing we are doing here times i and then del r f dash r then del r del y times j and then f dash r del r del z times k. So, this one is rather much simpler. So, I can take f dash r common and then this one will be del r del x times i and del r del y times j and then del r del z times k.

So, this will be basically del r del x del r del y del r del z. So, this is our vector gradient of r all right. So, here r is basically a that a scalar function. So, this is this is how we prove this identity.

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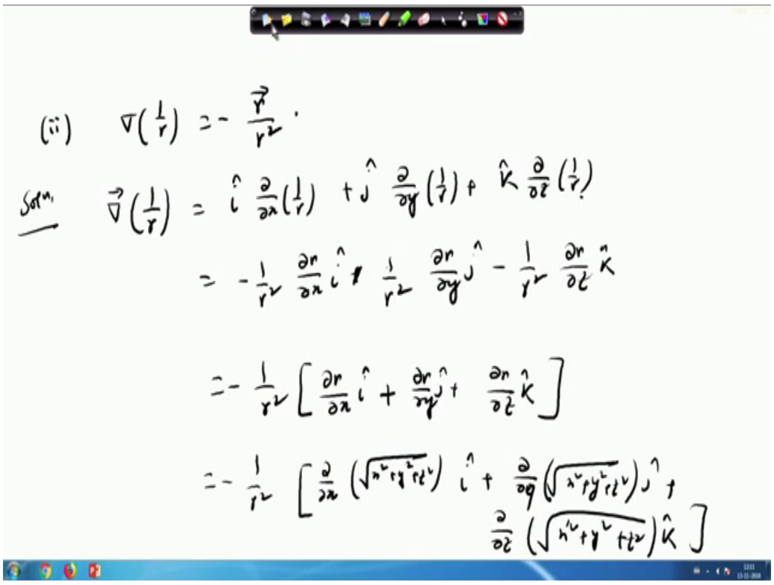

And the second result will be let us go to the second result. So, let us consider this consider we have something like gradient of 1 by r. So, what will be the result? So, it will be minus of gradient a minus of vector r divided by vector r mod of vector r whole square. So, this is scalar function r all right. Now, so if I write gradient of 1 by small r where r is a scalar function.

So, this can be written as i times del del x of 1 by r plus j times del del y of 1 by r plus k times del del set of 1 by r. So, this will be minus 1 by r square del r del x times i plus minus basically 1 by r square del r del y times j and minus 1 by r square del r del z times k. So, if I take minus 1 by r square common, it will be del r del x all right, and then it will be del r del x, this will be del r del y and this will be del r del z k j. Now, r is basically now this r is basically this r is basically our x square plus y square plus z square.

So, I can be able to write it as del del x of x square root of x square plus y square plus z square times i, x is del del x of square root of x square plus y square plus z square times j and del del z of so here del del y del del z of x square plus y square plus z square times k all right.

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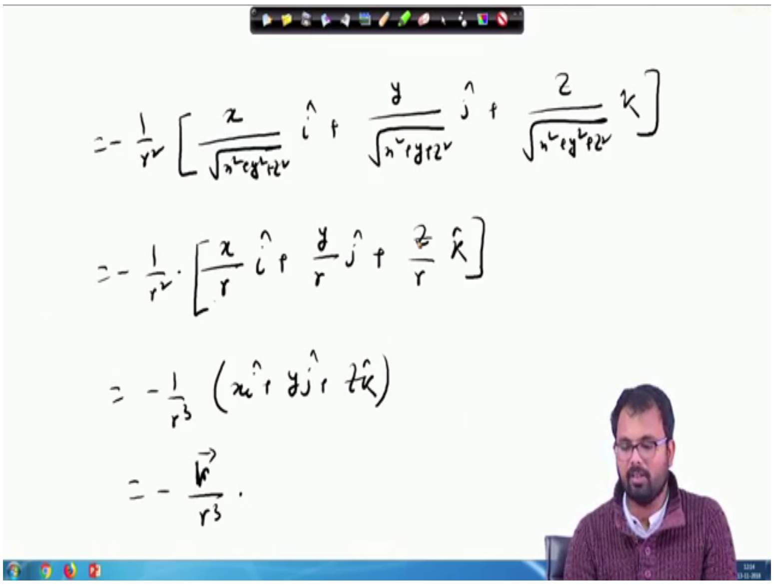

So, now, when we differentiate this whole thing it will be converted into minus 1 by r square, we can be able to write x by x square plus y square plus z square times i. Similarly, we can be able to write y by x square, we are just doing the partial derivative, we are not doing anything else. So, just doing some partial derivative we will be able to obtain x square plus y square plus z square times k. Now, this is our mod of r. So, I can be able to write one by r square mod of r or simply small r. So, let me write just small r. So, this is our small r x i, this is a small r and this is small r k.

So, I can take 1 by r common, and then this will be minus 1 by r cube and then this is x i plus y j plus z k. And this is nothing but minus vector r divided by r cube. So, I think here it was r cube all right. So, this one's, this one has to be r cube. So, here we can see that by doing just some simple differential how to say a partial derivative, just calculating some partial derivative, we can be able to obtain the results.

And gradient divergence curl is nothing there is nothing but just doing some partial derivatives it might get a slightly complicated at some places, but it is just that at the end of the day we are doing some partial derivatives to calculate these del f del x del f del y and del f del del z the three components of the given scalar function or a vector function. And with the help of which we can be able to calculate the gradient divergence of curl of a vector function.

So, sometimes the calculation might get a little bit tedious, but you know what I mean actually. So, here for example, in this case it is not that we cannot be able to obtain, this is just that it was leading to a complicated situation. So, I decided that we can be able to do it in a much simpler way than doing it in a complicated way. So, then we considered this.

So, this is an alternative method. So, this is an alternative method. And with this method, we can be able to solve the problem one in a rather simpler way than going into a complicated way all right. And this is our example 2. So, there can be some other forms to show that instead of this a gradient of 1 by r, we may be having something like I do not know let me let me give you an another example.

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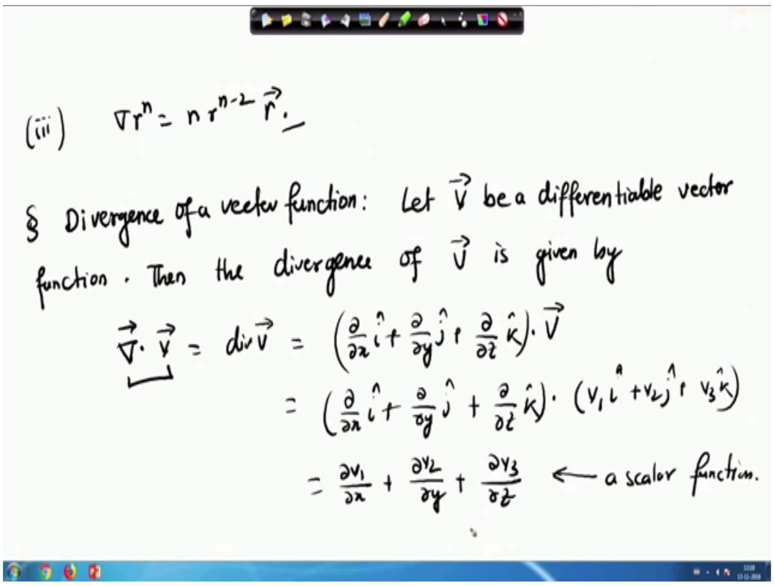

So, third example could be calculate gradient of r to the power n n r to the power n minus 2 vector r. So, here you just have to do some partial derivative play with the exponent of n, there is nothing much complicated here. It will just have to be really good at calculating these partial derivatives and that will be pretty much it. So, I am leaving this problem up to the students; I am pretty sure you can be able to solve it.

And we will move on to our next problem or yes. So, we will move on to our next how to say topic, and this is basically how to say curl divergence sorry. So, the next topic is divergence of a vector function. So, divergence of a vector function. Now, how do we define the divergence of a vector function, let me get the how to say formal definition.

So, the divergence of a vector function is let V be a differentiable vector function vector function, then the divergence of V is given by we write it with the same nabla symbol, so and then dot product with V. So, here since we have V as a vector function when we are charging this operator nabla which is itself a vector function vector quantity, we have to have some kind of how to say operation here. So, either it is a dot product or either it is a cross product because that is how we how to say multiply or come or charge one vector on another. So, either it has to be a dot product or it has to be a cross product all right.

So, divergence deals with the dot product. And when we learned about curl then in that case it will be a cross product all right. So, the divergence of f sometimes some people write it as div div V naught F here we have v. So, divergence of V is div V and this is also denoted like this. And it is basically del del x of i a del del x i del del y j and del del z k times V.

So, V since it is a vector quantity or vector function, it also must have three components let me call them as V 1 i V 2 j and V 3 k. So, now, we have dot product of two vectors and dot product will yield del V 1 del x del V 2 del y and del V 3 del z. So, ultimately we are getting a scalar function. So, this is a scalar function so that means, gradient of a scalar function is a vector quantity, whereas the divergence of a vector function is a scalar quantity. So, here we see that the versions of V is actually a scalar function all right. And whatever we get after calculating that is a different thing.

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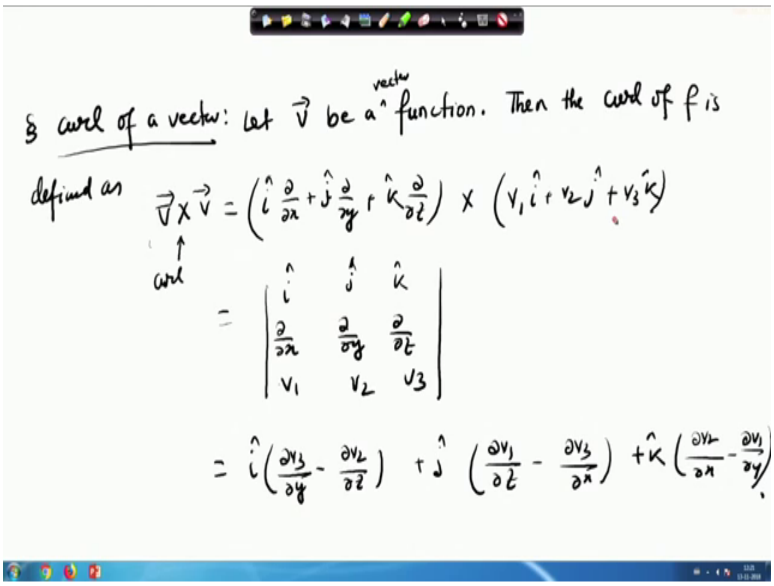

Now we can also define the curl of a vector, the curl of a vector. So, as I was saying that when we are charging nabla on a vector quantity, so let us call it as let V be a vector function be a vector function, then the curl of f is defined as. So, here in this case also the new operator nabla will be charged on the vector V.

But since nabla is a vector quantity charging on a vector function, there has to be some kind of operator or operation acting in between. Now, the first operation which is basically dot product is resolved for the divergence. So, the second operation that is left is basically this cross product. And this cross product this cross product of gradient and V is called as the curl. And this is nothing, but i del f del x del del x plus j del del y and k del del z cross product with f one sorry v 1, so we can write it as v 1 i v 2 j and v 3 k.

So, from here we can calculate the cross product as i j and k then we have del del x del del y del del z then v 1 v 2 v 3. So, if we break this course or this determinant, then this will be del v 3 del y minus del v 2 del z plus j times del v 1 del z minus del v 3 del x plus k times del v 2 del x minus del v 1 del y. So, this is the required curl of a vector function V whose components are v 1, v 2, v 3.

So, in case of scalar function when we are dividing the gradient, we are obtaining a scalar function a vector function. But in case of divergence, it is a scalar function. And again in case of curl, it is a vector function. So, this is how we define the curl of a vector.

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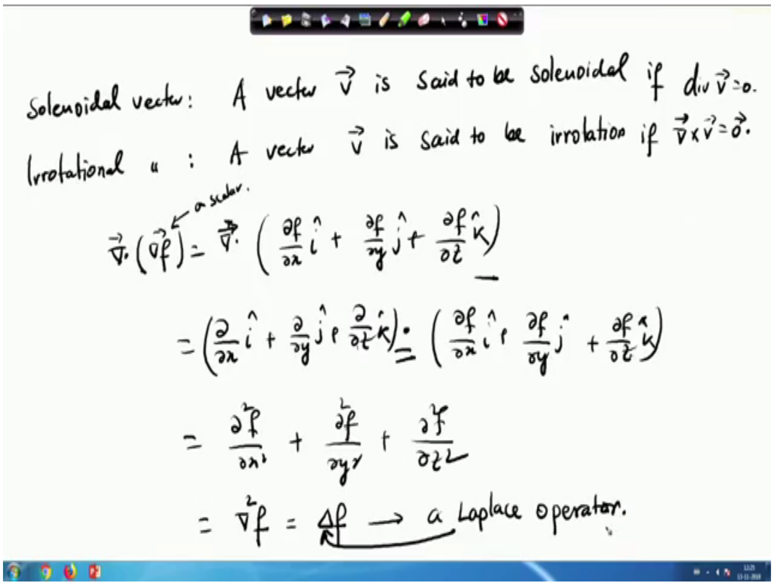

And we have a two quick definitions. So, the first definition is sol solenoidal vector solenoidal solenoidel vector. So, a vector V is said to be solenoidal if divergence of V equal to 0. And conservative vector and then we have another definition which is basically irrotational or irrotational, irrotational, irrotational vector. So, a vector V is said to be irrotational if curl of V equals to a zero vector all right. So, we have a solenoidal vector field and irrotational vector fields.

So, if V divergence of V equals to 0, then we is said to be solenoidal vector. And if curl of V equals to 0, then it will be call it will be called as irrotational and from divergence and gradient, we can also define a very important operator in vector calculus or in partial differential equations which is something of very important.

So, what happens if you charge let us say divergence on the gradient of a scalar function. So, what would this be f is a scalar function. So, here f is a scalar all right. So, what would happen let us see. So, this is gradient, and then we have del f del x i plus del f del y j plus del f del z k the first gradient would give us this that is pretty much straightforward.

Now, let us break this gradient here. So, del del x i del del y j and del del z k dot del f del x del f del y and del f del z. So, now, we have a dot product. And if we have a dot product, then the dot product would look like del square f by del x square, because this partial derivative will be charged here, this partial derivative will be charged here, and this partial derivative charged here. So, this will be del f del square f by del y square and del square f by del z square.

And this second order partial derivative of f with respect to x, y and z, can simply be written as del square f or some people write it as inverted nabla. So, sometimes it is called as delta. This and this is nothing but a Laplace operator. So, this in this delta, this delta, or this nabla whole square is called as the Laplace operator. And the Laplace equation, so when we when you have del square f equals to 0 or del square f equals to some g or whatever function you have then such equations are called as Laplace equation which is very important in diffusion reaction equations in and when they often comes as a Poisson’s equation.

So, basically it has a very how to say deep applied mathematical connection. So, this Laplace or the nabla whole square or delta is an important operator and not only in the vector calculus, but in other fields as well. And how do we obtain it we basically charged this nabla as a divergence on the nabla of f which is actually a vector function, so this one here. And this is our required Laplace operator.

So, now, that we have introduced the concepts of divergence and curl. In our next class, we will practice some more examples motivated from divergence and curl of a vector and we will also try to show when a vector field is conservative, when a vector field is solenoidal or irrotational, and then we move to our next topic which is level surfaces and directional derivative.

So, I thank you for your attention. And I will see you in the next class.