Integral and Vector Calculus
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Lecture – 36
Gradient of a Function

Hello students, so up until last class we looked into differentiation of a vector and also integration of a vector. So, it is basically some are motivated from the integration of scalar function ah. So, we saw that if we have an integral of type let us say t running from a to b if we have a vector.

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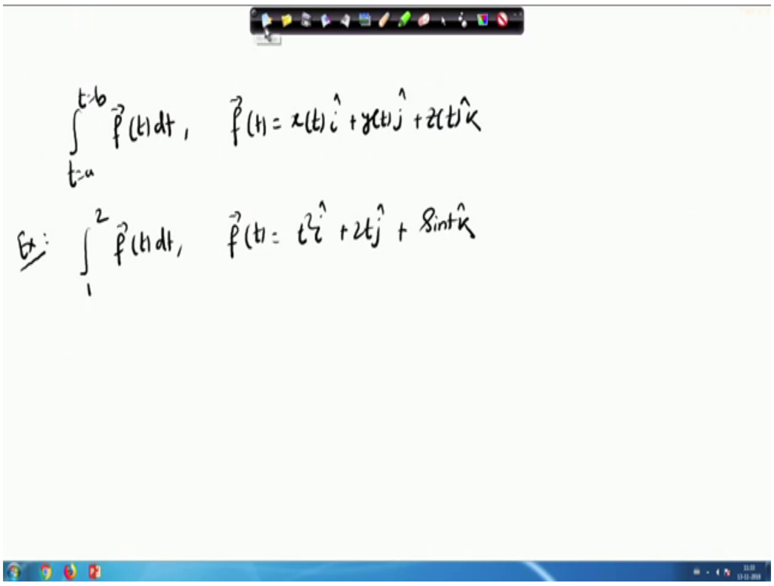

Let us say f t dt where the function f t is given which it can be x t i plus y t j plus z t k, then that case we basically substitute this in this vector f t as this one here in this integral and we integrate component wise. So for example, if we have an integral of type let us say 1 to 2 f t dt where our f t is where our f t is t square i plus 2 t j plus sin t k. So, then in that case we just integrate we substitute this a ft here in this integral and then we integrate component wise with respect to t and that would give us the integral of this vector function.

Now this is called as integration of a vector, but in couple of chapters later we will come to come to the topic of vector integration which is a slightly different. So, we will also learn about those things. Now today I am going to introduce another important topic which is quite relevant from vector calculus point of view.

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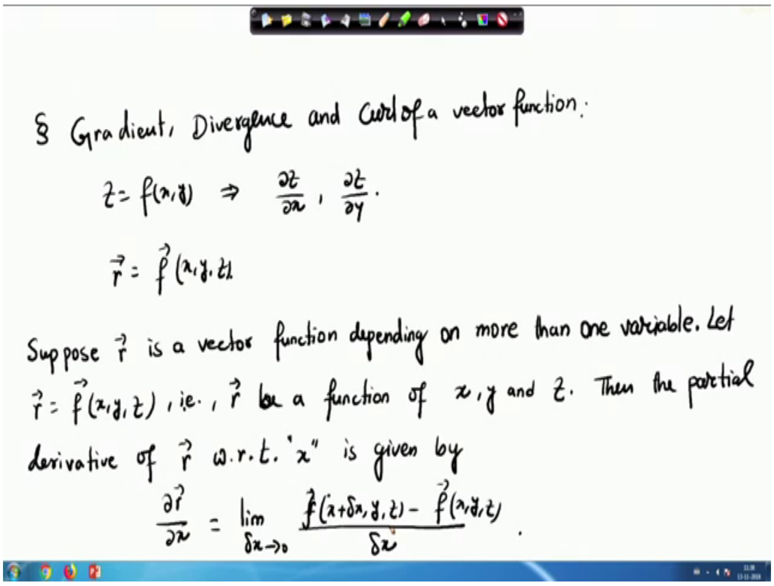

So, today's topic is Gradient Divergence and Curl of a vector function. So, by the way gradient is taken on a scalar function and the divergence and curl will be taken or is taken on a vector function. So, it is not like gradient of a vector function it is always like gradient of a scalar function and the divergence and curl of a vector function. Now to give you formal definition it goes like this. First of all what do we mean by how to say partial derivative of a vector function.

So, in case of scalar functions we saw that if we have a function of 2 variables then in that case we can talk about. So, let us say if we have a function of 2 variables then in this case in scalar case we can be able to talk about del z del x and del z del y. So, similarly in case of vector functions suppose our vector function r is equals to vector f x y z, then in that case here I can be able to talk about the partial derivative of this vector function. So, how do we how do you basically define the partial derivative of a vector function.

So, let me give you a formal definition, so suppose definition is suppose r is a vector function is a vector function depending on more than one variable depending on more than one variable more than one variable and scalar variable of course and let r equals to f x y set that is that is r and b a function of x y and z. So, basically r is a vector function of 3 scalar variables x y and z. So, you can imagine a vector function of type 3 x square z times i plus y squared z times j and plus z square x times k. So, that can be one possible vector function.

So, any vector function of the scalar variable r can be of that particular function ok. Now then the partial derivative then the partial derivative the partial derivative of r with respect to x is given by how do we define, we define it as del r del x is equals to limit delta x goes to 0 r of or f r is equal to f.

So, f x plus delta x comma y comma set minus f x y set divided by delta x. So, this is the this is how we define the partial derivative of r which is basically a vector function of 3 scalar variables x y z and similarly we can define the partial derivative of r with respect to y, then we add a small element delta y like we usually define for the scalar function.

So, for the scalar functions also we define the derivative in the similar fashion and when we define the derivative with respect to z we write del r del set and then limit delta z goes to 0 and then we small increment in the variable set. So, this is how we define the partial derivative of r with respect to x y and z. If we want to define the partial we want to define the successive derivative.

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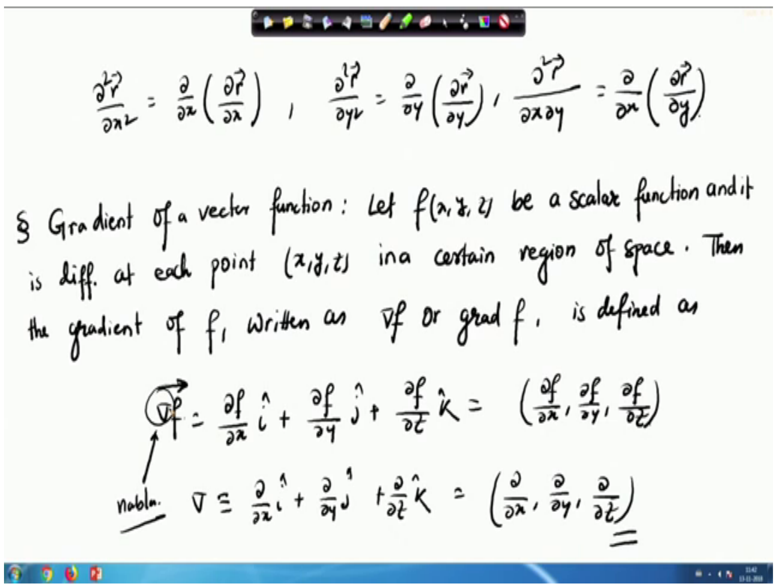

So, I mean successive partial derivative; that means, del square r by del r is del square r by del x square then we simply write del x of del r del x del square r by del y square is equals to del del y of del r del y, where r is a vector or if we are defining the mixed derivative so del x del y.

So, this will be basically del del x of partial derivative of del del r del del r del y, so this is how we define the mixed derivative all right. Now that we have introduced the concepts of partial derivative of a vector function we can actually introduce an operator and that operator is called as Gradient that operator is called as Gradient of a vector function.

So, how do we define it so let f x y z be a scalar function be a scalar function and it is differentiable at each point x y z in a certain region of space certain region of space. Then the gradient of f written as or simply grad f is defined as gradient of f is equals to del f del x times i and del f del y times j and del f del z times k some people also write it as del f del x comma del f del y comma del f del z.

So, instead of writing i j and k we can write it as a triplet and here this symbol this symbol here is called as nabla. So, usually in vector calculus nabla is nothing but so nabla is nothing but del del x times i del del y times j and del del said times k. So, this is our nebula and some people also use the notation of triplet, so del del x del del y and del del set. So, this is the notation for nabla. So, usually when we say that gradient of f then we are actually of charging this operator.

So, this is this operator is called as nabla and when you are charging on a gradient function on a scalar function then it will be called as a gradient of a scalar function and obviously it is a vector quantity.

So, gradient of a scalar function is a is a vector quantity and when of course, in this case you must have all the partial derivatives to be existing for that scalar function and we can also note that gradient of a scalar gradient of a scalar field defines a vector field, that means gradient of a scalar point function defines a vector field actually.

So, the if f is a scalar point function if f is a scalar point function which we defined earlier, then gradient of f is basically a vector point function or if f defines a scalar field then gradient of f defines a vector field alright.

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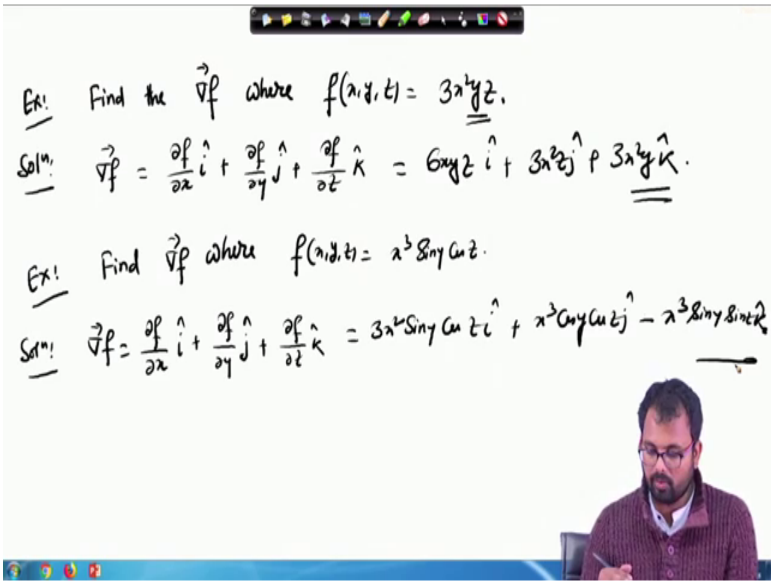

Now, let us calculate so find example. So, find the gradient of f where f is obviously a scalar function. So, where f x y z is equals to 3 x square y z, so that is our given function and we need to find gradient of f here. So, let us do that gradient of f is equals to del f del x i del f del y j and del f del said k. So now, if I do del f del x for this function then it will be 6 y x y z because we are differentiating partially with respect to x only.

So, y and z will be treated as constant when we are differentiating with respect to x only times i plus when we are differentiating with respect to y, then x and z will be treated as constants. So, this will be 3 x squared set times j and when we are differentiating with respect to z then x and y will be treated as constants and then it will be 3 x square.

So, this is the required gradient of the function f and that is how we calculate. So, based on this you can calculate the gradient of any scalar function provided it has a partial it has a partial derivatives that exist ah. So, we can we can consider an another example; let us say to find a gradient of f where f x y z yes 3 or let us not have 3 I can consider x cubed sin y cos z.

So here obviously, this is a differentiable function at least with respect to x y and z individually. So, then we can write gradient of f is equals to del f del x times i del f del y times j and del f del z times k.

So now, when we differentiate f with respect to x only then it will be 3 x square sin y cos z i plus when we differentiate with respect to y, so it will be x cube sin y is cos y cos z j and when we differentiate with respect to z then it will be minus of sine z. So, I can have here instead of plus I can have minus, so this will be minus x cube sin y and then sin z, so that is how we basically calculate gradient of a scalar function alright.

Now there are certain results that that are I mean how to say valid for 2 functions basically. So, suppose if you have us if you have 2 addition of 2 functions, then if you charge the gradient on that sum of 2 functions what would happen to the sum. So, like we did for the differentiation or a limit and continuity for the sum of 2 functions or different our difference of 2 functions and things like that. So, I would call them as properties.

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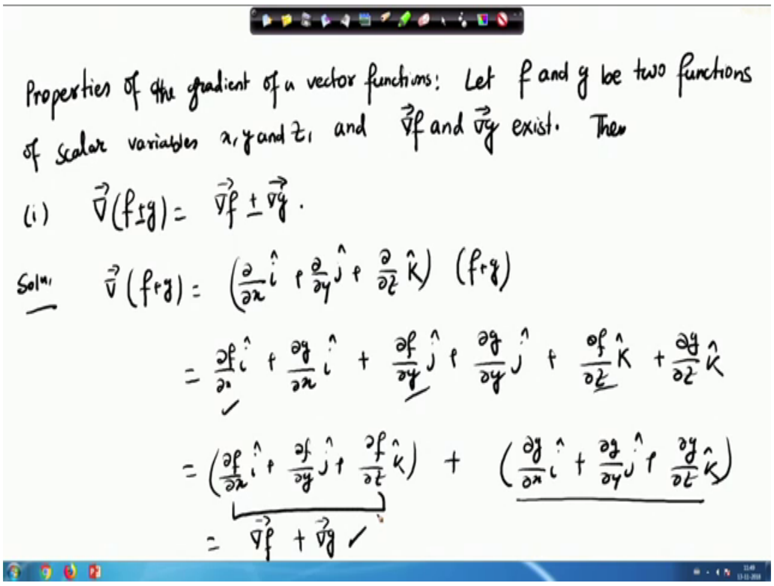

So, properties of the gradient of the gradient of a vector function, so what are the properties. So, statement is let f and g be 2 functions of scalar variable x y and z and gradient of f and the gradient of g exists. So, they are defined in a way then exists actually.

Then the first property is gradient of f plus g would be gradient of f plus gradient of g. So that means, a gradient of the sum is equal to sum of the gradient, similarly if we have a difference here, then gradient of the difference is equal to difference of the gradient all right. So, let us write a minus sign here I can try to prove addition and then the subtraction follows in the similar way, so let us try to show at least one result.

So, gradient of f plus g is equal to del del x times i del del y times j plus del del z times k times f plus z. So, this would be del del x del f del x timesiI plus del g del x times i del f del y times j plus del g del y times j plus del f del z times k plus del g del z times k.

Now we gather the f come how to say f terms at one place and g terms at one place, so this will be del f del x times i plus del f del y times j plus del f del z times k and then plus we will gather the terms for i. So, del g del x times i plus del g del y times j and del g del z times k.

So, now this is basically our gradient of f that is how we defined the gradient of f. So, I can write this as gradient of f plus this one is my gradient of g. So, I can write gradient of g and this is the required result. So, some of the so the gradient of the sum is equal to sum of the gradient similarly we can prove the gradient of the difference is equals to the difference of the gradient. So, the proof is not that complicated.

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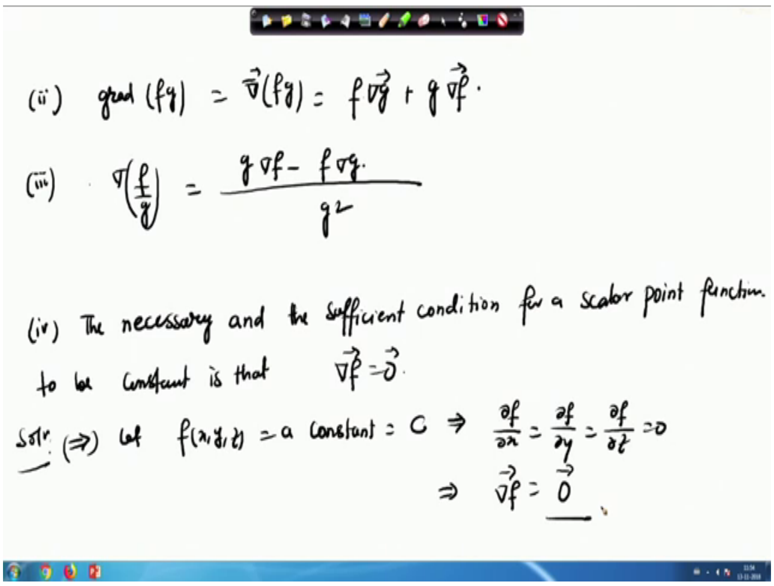

Now, we can also have a result or property that gradient or gradient of the product or we also write as by nabla symbol. So, gradient of the product equals to f times gradient of g plus g times gradient of f. So, it is the same similarly like a rule like differentiation of 2 functions.

So, the first function would remain unchanged the differentiation of the second function and then second function would remain unchanged and the differentiation of the first function. So, it follows very much the similar exactly the similar rule like the differentiation of 2 functions.

So, this proof is also very easy and I am leaving this up to the students and I am pretty sure you can be able to solve it. The third property is the third property is let f and g are to be to vector functions such that their gradient of f and gradient of g exist, then we can write gradient of the quotient.

So, the gradient of the quotient would be g square g times gradient of f minus f times gradient of g. So, this is how we define the gradient of the quotient and of course the proof is done and that complicated to say because here we have a quotient.

So, if you differentiate with respect to x y or z, then it will always be 1 by g square and then you are doing the differentiation for the rest of the function. So, it is a fairly simple to prove it might be a little bit lengthy but of course, you will get to practice a little bit more in this sense, but the proof is very straightforward so I am leaving this task up to the students.

So, they can be able to solve it I am pretty sure about it and then we have an another small result. So, the or the fourth property that the necessary the necessary and the sufficient the necessary and the sufficient condition for a scalar point function to be constant, is that gradient of f must be a 0 vector. You always have to pay very close attention that gradient is a vector function, but divergence is a scalar function which we will define in probably in our next lecture. However, again the call of a vector function is again a vector.

So, divergence is a scalar function divergence of a vector function is a scalar, however the gradient of a scalar function and curl of a vector function they are all vector quantities. So, you have to pay very close attention that whether it is a 0 vector or just 0 all right. So, the necessary condition is that if we have a constant function then they are gradient then the gradient would be 0. So, it is very easy so necessary condition this implication. So, it is very easy to see if f is constant. So, let f x y z is equals to a constant.

So, I call this constant as C, so obviously from here we will have del f del x del f del y and del f del z equals to 0 because the function is constant. So, partial derivative with respect to x y and z would always be 0 and therefore from here if I write gradient of f. So, gradient of f would be del f del x times I del f del y times j and del f del z times k and since all these components there is a coefficients are 0. So, 0 times I plus 0 times j plus 0 times k this will be actually a 0 vector. So, that condition is necessary and that has been established alright. Now let us go the reverse way.

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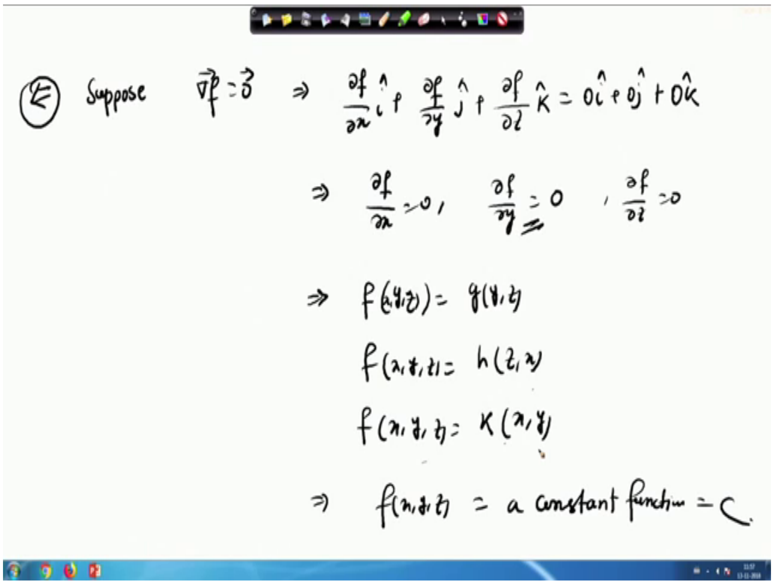

So, the reverse way suppose gradient of f is equals to a 0 vector. So, then what would happen what would happen is we can write it as del f del x times i plus del f del y times j and del f del z times k is equals to 0 vector. So, 0 vector can be written as 0 times i plus 0 times j plus 0 times k.

So now, this is sort of like an identity so the coefficients must be equal so; that means, that the coefficient of i must be equal to coefficient of i on the right hand side coefficient of j on the left hand side must be equal to coefficient of j on the right hand side and similarly for the coefficient of k. So, from here I can write del f del x equals to 0 and del f del y equals to 0 and del f del z equals to 0.

So, if I integrate on both sides of these equations, so if I integrate then f will be a function of f will be a function of y z only. So, I can write g y z, if I integrate the second equation then f x y z equals to f will be only a function of h z x unless the partial derivative cannot be 0. So, unless the function f is a function of h and z x only the partial derivative cannot be 0 and we cannot write constant because we do not know whether the whether the function f is a constant or not. So, even if the function is a function f is a function of z and x only it is partial derivative can still be 0 without the function of being a constant.

So, once you integrate you should not write constant you should always write the function of z and x all right and when we integrate f then g h let us say I have k, then integrating the third equation will give me f as a function of k x y. Now here we have a problem one function cannot be equals to 3 different functions unless the function it is elf is constant because the function cannot be just a function of y and z, it cannot be function of z and x and it cannot be function of x and y. So, the only possibility for this function f to be existing is that f x y z has to be a constant.

So that means, a constant function so that means, it must be it goes to C. So, unless this function f is constant these 3 possibilities cannot happen, so if f is equals to f is equals to how to say g y if f is a function of g y as a function of y z z x and x y. So, from first equation from the first solution we can see that f is independent of x from second equation we see that f is independent of y from third equation we see that f is independent of z.

So, this one and that means that f is independent of x y and z and since f is a function of a 3 variable so the only possibility f has is to be a constant. So, it cannot be any other function so that means if the gradient of f is 0 then the function has to be constant and if the function is constant and obviously the gradient of the function is 0.

So, this is what we needed to prove and these are some simple results which we can definitely practice and we will practice some more examples on gradient of a function and we will also introduce that we will also introduce the concept of Divergence of a vector function in our next class.

So, I thank you for your attention and I will see you in the next class.