Integral and Vector Calculus
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Lecture - 35
Integration of Vector Function

Hello students. So, in the previous class we started doing some examples on differentiation of vector functions with respect to a scalar variable. In our case it is T and we tried to say solve some problems based on simple differentiation rule and using some simple vector algebra results. Today I will try to solve one more example which is somehow a little bit less a more interesting in a way that it will give you more idea that how to differentiate a vector function and then, we will move to our next topic.

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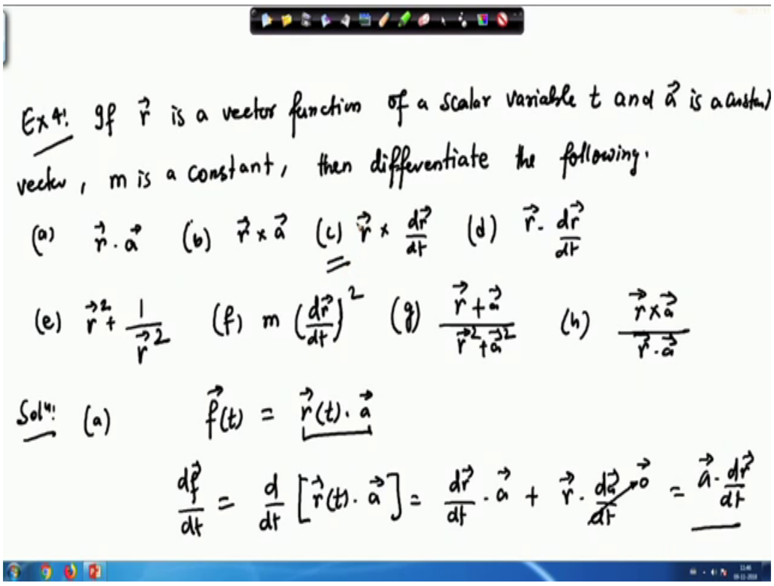

So, let us state our problem. Example, I believe it is 4 in this section, but I am not sure about it. So, you may have to follow the whole I am how to say tutorial. So, if r is a vector function, the vector function of a scalar variable, a scalar variable t and a is a constant vector as a constant vector m is a constant, then differentiate the following. So, obviously we will differentiate with respect to t. So, what are we differentiating basically?

So, we are differentiating r dot a. So, of course I would not solve all of these 8 sub problems, but I will consider at least solving 3 of them. So, let us start with a solution. So, let us start with a problem a.

So, the problem a is I am going to call it as f t. So, let us call this function as f t where f t is equals to r t because r is a vector function of scalar variable t dot product with a, where a is a constant vector. So, if I am differentiating this function, this is like our whole vector function. See if I am differentiating this vector function with respect to t.

So, this will be d dt of r t dot a and this can be written as d r dt dot product with a plus r dot da dt. Now, it is given that a is a constant vector. So, obviously it is a derivative with respect to t would be 0. So, this will be 0 alright and dot product of r with zero vector would again be a zero vector and if we add zero vector here, then it would remain unchanged.

So, we can write it as a dot d r dt and since dot product is commutative I just exchange a where a dot d r dt dot a with a dot d r dt. So, this is the required dot product, that is the required derivative also of this function r dot a, alright. Now, next let me consider let me consider problem c.

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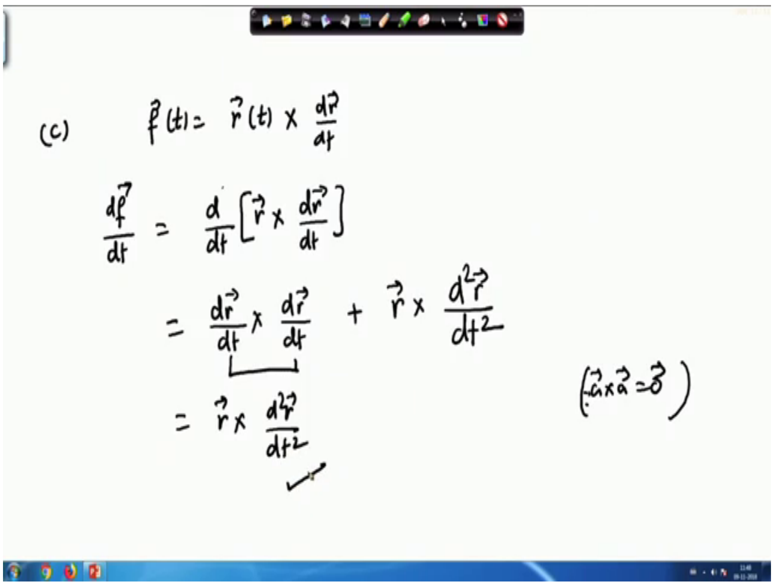

So, problem c is let me call it again as a function f t which is equals to r t cross product with dr dt. So, obviously this is also function of t. So, if I differentiate then this will be df dt equals to d r dt or I can write it as d dt of r cross product with dr dt and now, if I differentiate then this will be dr dt cross product with dr dt and plus r d square r by dt square. Now, these two are same vectors. So, we know that a across a, we know that a cross a is a 0 vector that means cross product of a vector with itself is equal to 0 because they are considered as to be collinear or a parallel. So, in a way a if two vectors are parallel, then the cross product is 0.

So, a cross a it is considered as to be a parallel be considered as to be parallel and therefore, the cross product is 0. So, the cross product of first term is 0 and the second term will remain as it is, alright. So, this is how we differentiate the term c or the problem c. Now, let us consider problem e.

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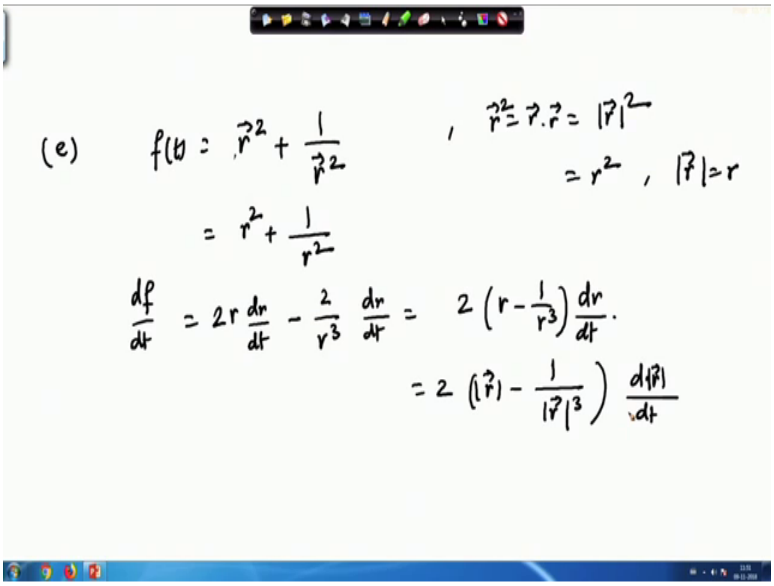

So, I am just skipping one problem, mainly because then the next problem is simpler in a way. Now, r here f t before I call this function as vector or scalar let me just write the right hand side. So, the right hand side is r square. So, let me write it as r square plus 1 by r square, right.

So, from vector calculus we know that if r is equals if we have r as a vector, then r square is basically r dot r and that is basically mod of r whole square. So, instead of writing r square I can let mod of r square. Now, mod of r is a scalar function. So, I can simply write as a square as r square. So, this mod of r is equal to a scalar r. So, remember this scalar r and this vector r, there are two different things because this is a scalar function of t whereas, this is a vector function of t. It is just that mod of r. That means, the magnitude or absolute value in a way or magnitude let us call it magnitude of r is equals to a scalar r where r scalar r is also is also a function of t, but it is no longer a vector. So, there is a big difference between a vector r and scalar r although I am using the same notation. So, that means instead of writing r square I can replace it with scalar r.

So, let me write it as scalar r square 1 by r square. Now this is a scalar function of t. So, instead of writing vector it is basically a scalar function. So, writing f t makes sense. Now, I am differentiating with respect to t. So, this will be 2 r times dr dt and this one is minus 2 by r cube dr dt. So, if I take two common this will be r minus 1 by r cube dr dt. Now, this r is basically mod of r, I can write it as mod of r minus 1 by mod of r cube and then, this is basically mod of r dr dt. So, this is the required result, alright.

Now, which problem should be solved? Let me choose; so, this is simple; this is not complicated. So, m is constant. So, m will come out of the differentiation and then, this will be to dr dt times d square r by dt square. So, this is simple. I can consider probably g. So, let us solve the problem g.

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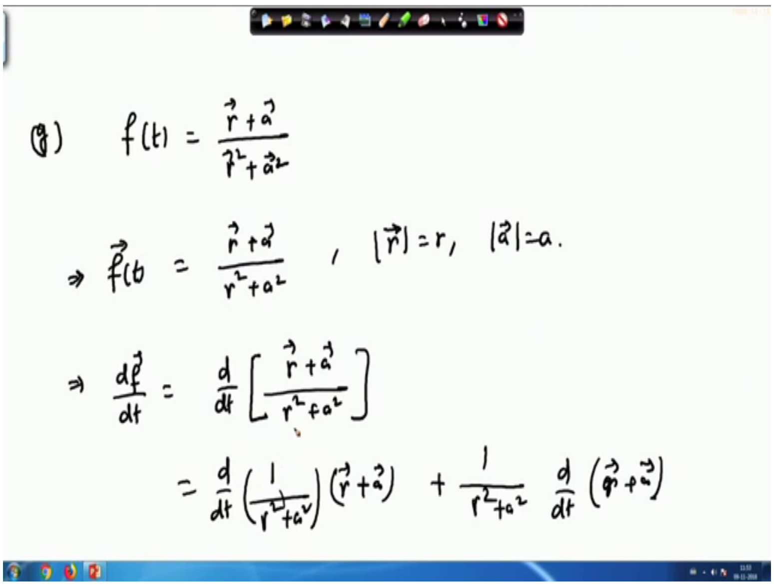

So, the problem g is let me call it as ft first and then, we will see whether it is the scalar function or a vector function. So, r square a square. So, again we have r is square vector r square. So, from the previous problem we can write it as just scalar r square and this one is vector a square which we can write it as simply a square, but a is a constant vector, alright.

So, this can be written as vector r plus vector a r square plus a square where r mod of r equals to a small r and mod of a is equals to small a. Both are scalar, alright and since r is a vector is a vector I can now write f t as a vector, alright. Now, we can differentiate with respect to t, alright. So, the differentiation with respect to t would go something like this. So, it would not be simple. It will be slightly lengthy to do the calculation. So, let us write r plus a and then, r square plus a square and now here we will apply the product rule. So, the product rule would state that we will have d dt of 1 by r square plus a square r plus a and we will have 1 by r square plus a square d dt of r plus a. So, now when we differentiate this one, it will be.

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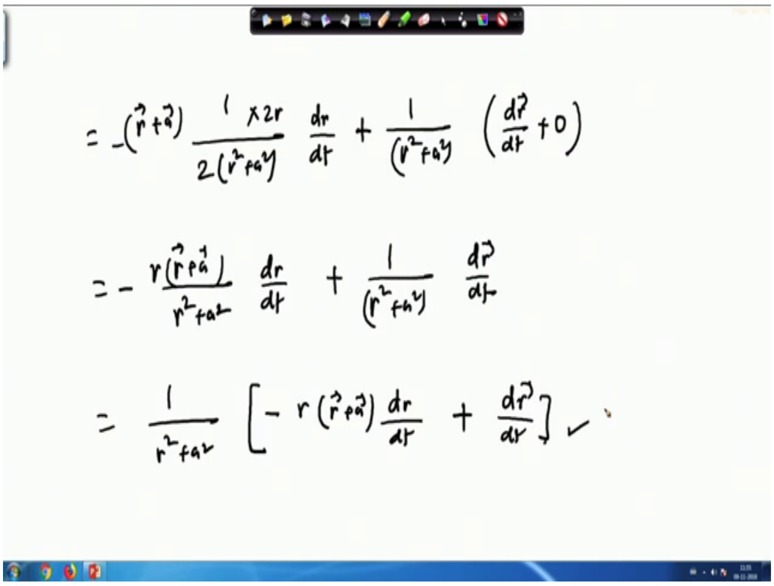

So, we will have r plus a and then, it will be minus of 1 by 2 r square plus a square and then, when we differentiate this one, then it will be 2 r dr dt and plus r square plus a square then, we will have d r vector d t and the derivative of constant vector would be 0. So, this will be minus of vector r plus vector a divided by r square plus a square and here we will have times I can write it as scalar r dr dt of scalar r and then, this one will be 1 by r square a square d r vector dt. So, that means I can take 1 by r square plus a square common and this one will be minus of r vector r plus vector a d r dt and this one will be plus vector dr dt.

So, there is a big difference between dr dt and the vector dr dt, alright. So, we should not be; we should not be confused with them. So, this is the required, notice the result in a way in a way. So, we can leave it up to there. So, it is not a matter problem, all right. So, now we have now we have this result here. Let me go back to the problems which we had earlier. This one can also be treated in the in the similar fashion like we treated the problem g.

So, you see I am just to have a look at these results and you will be able to see that in case of vector functions you first have to make sure that the function is differentiable and if it is a differentiable function, then all the rules for the differentiation of scalar functions hold true in case of vector functions as well and all you have to do is to use some basic results from vector algebra and differentiation of scalar functions to prove these results. So, it really did not involve any kind of complicated calculations. I basically just do some simple how to say results from vector algebra and I am hoping that you also be able to solve these problems.

So, we will look, we will move to our next topic. So, the next topic in agenda is to introduce the concept of vector integration. So, limit, so first of all vector function, limit, continuity, differentiability, successive differentiation, dot product rule, cross product rule, addition, subtraction, multiplication by scalar function and all that. So, all these properties what we have learned basically for scalar function are holding true for vector functions as well.

So, we have seen limit continuity differentiability for vector functions. Now, from scalar function we also have a concept motivated which is called integration. So, integration of scalar function similarly is integration of vector function.

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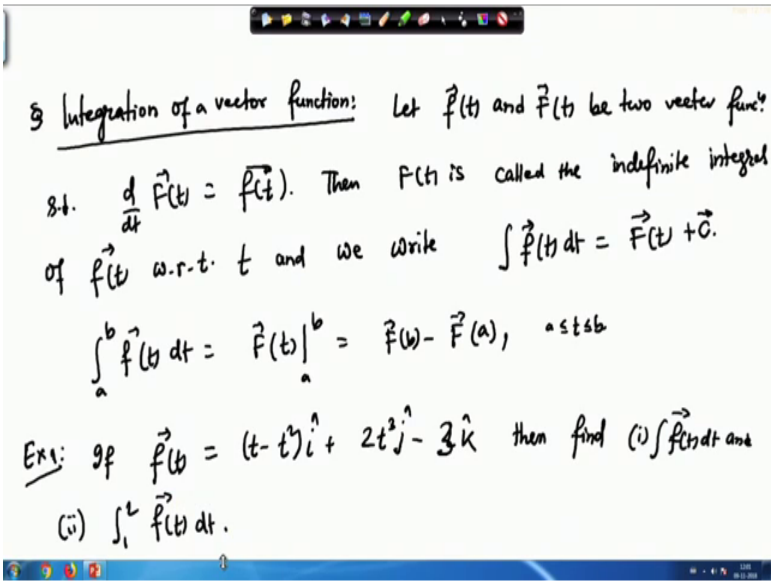

So, let us define the integration of vector function, integration of vector function; integration of a vector function. So, let small f t is a vector and of course, if we adding the definite vector integration in a way.

So, as sorry indefinite vector integration, then the integration of a vector would always result in how to say a vector function and if we are doing the definite vector integration, then the integration of a vector function would result into a vector. So, it is not like integrating a vector function definite or indefinite integral would result in a constant scalar or something. So, it will always end up being a vector, whether it is a vector function or vector constant vector. That would depend what type of integration are we doing, alright.

So, let f t and capital F t be two vector functions; be two vector functions, such that d dt of capital F t is equals to small f t small f t, then capital F t is called the indefinite integral of small f t with respect to t and we write integral of vector small f t dt equals to capital F t plus some constant c, where c is a constant vector, alright and yes. So, this is how we write the vector integration for two vector functions and you can see that it is pretty much motivated or sort of like a generalization of integration of scalar function.

So, there we also had integration of f x d x is equals to some capital F x plus a constant c. So, here c is a constant vector. So, let us consider a few examples and of course, if we have a definite integral if we have definite integral let us say a to b f t dt, then obviously we write integral ft from a to b as so equals to capital F t from a to b. So, this will be f b minus f a, alright. So, where a less are equal to t less or equal to b. So, if we are integrating with respect to t between the range a to b, then in that case this will be our required definite integral. So, let us solve few examples. So, our first example if f t the given vector function equals to t minus t squared times i plus 2 t cube j minus 3 k. So, then find first integral f t dt and second integral 1 to 2 ft dt, alright.

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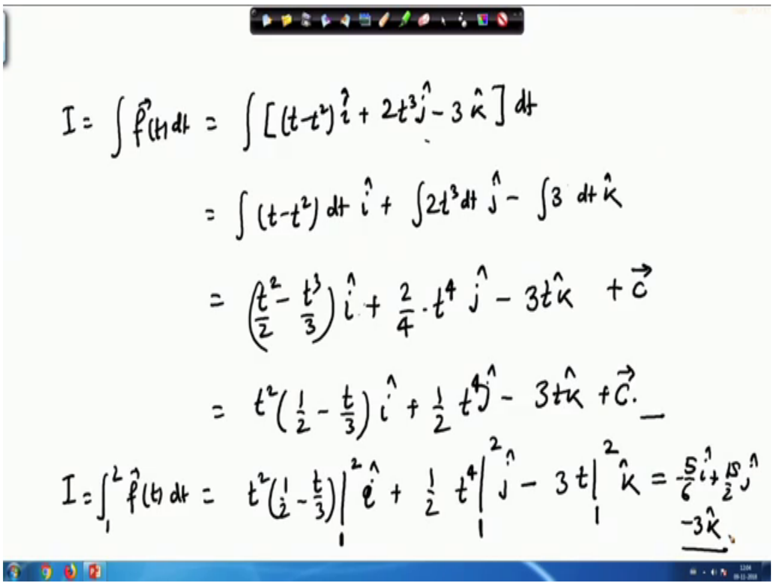

So, let us solve the integral the definite integral first. So, f t dt now we can write f t as t minus t square plus 2 t cube j minus 3 k times dt and when we are doing the integration, so the integral like the differentiation that you; like the differentiation where the differentiation went to the each component in case of integral as well the integral will go to each component.

So, the integral will be will be done on the first component. So, t minus t square dt i plus 2 t cube dt j minus 3 t dt k and then, when we integrate this will result in as t square by 2 minus t cube by 3 dt time simply i not dt. So, just i and this one will be 2 by 4 t to the power 4 times j minus 3. So, this is 3 t plus a constant vector c. So, this is or I can simplify it a little bit. So, t is square common 1 by 2 minus t by 3 times i plus 1 by 2 t to the power 4 j minus 3 t k plus a constant c. So, this is the required integral for the given vector function f t and as you can see it is basically doing the integration of three scalar functions. So, you just send your integral to three different components the component of i, j and k and do the scale and integration, alright.

So, this is our required integral and now we can calculate our definite integral. So, 1 to 2 f t dt. So, if I calculate 1 to 2 f t dt, then this will be t square 1 by 2 minus t by 3 evaluated at 1 2 and 2, then I then 1 by 2 t to the power 4 t at 1 t at 2 times j minus 3 t evaluated at 1 2, then k and then c would not appear because then it will be c at 2 minus c at 1.

So, ultimately value of constant at 2 and value of constant and 1 would again remain constant and then, constant minus constant would be 0. So, c would no longer appear and if you solve this whole thing, then you will end up getting minus 5 by 6 i plus 15 by 2 j minus 3 k. So, this is the required answer of this integral of f t dt between the range 1 to 2. So, this is how we solve the problems from my integral in the concept of integral calculus and let us solve one more example.

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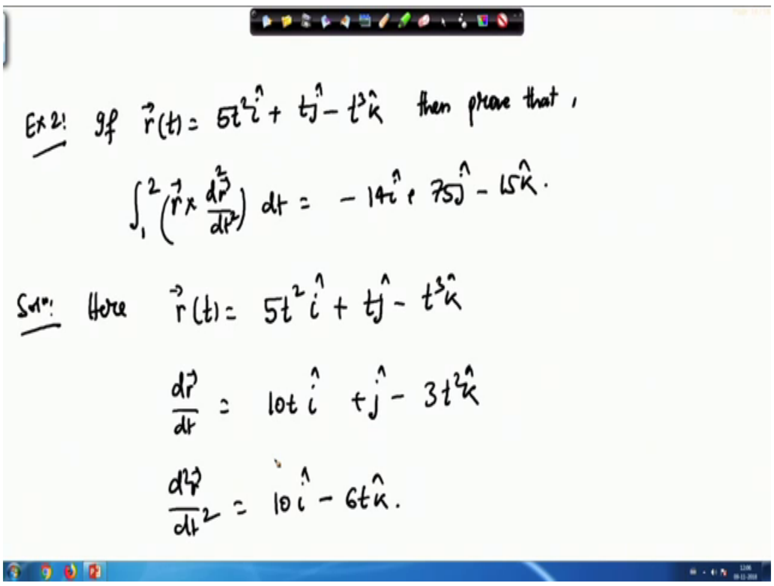

So, this one is see you might be getting the idea that as we did the differentiation of vector function, the differentiation of the integral of vector function is pretty much following the similar rules. It is just that here we are doing the integration that is it is nothing special. So, let us consider our next problem.

So, if r t equals to 5 t squared times i plus tj minus t cube k, then prove that integral from 1 to 2 r cross dr dt or r cross d square by dt squared dt equals to minus 14 i plus 75 j minus 15. So, the solution would go like this. Here r t equals to 5 t square i plus t j minus t cube k. So, we have to calculate d square r by dt square. So, first we will calculate d r dt. So, this will be 10 t i plus j minus 3 t square k, then we have to calculate d square r by dt square. So, this will be 10 i plus 0. So, we will not write that component minus 60 k, right and then, we will calculate the cross product of r and d square by dt square. So, cross product of r and d square by dt square. So, how do we calculate? So, in order to calculate that we will simply write r d square r by dt square.

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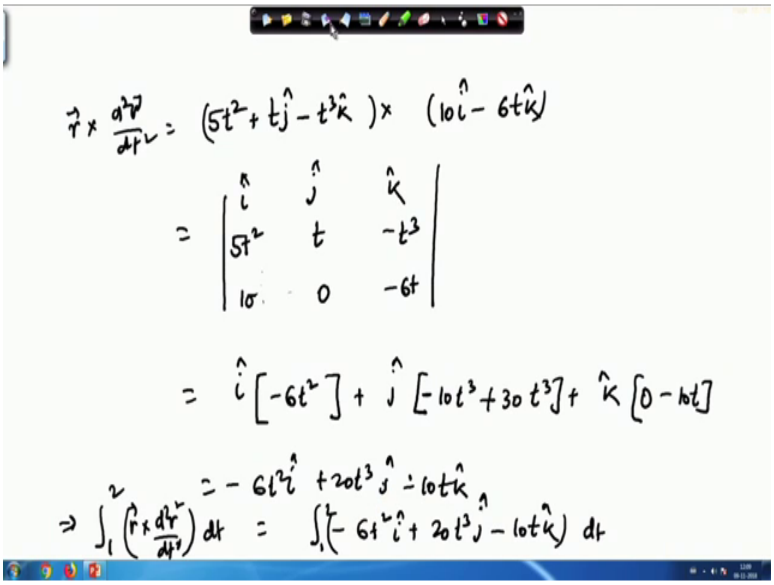

So, our given vector r is 5 t square plus tj minus t cube by t cube k times. What did we have? 10 i minus 6 to 10 i minus 6 t k right minus 60 k. So, calculating this is this one is not difficult. So, i j k I will write 5 t square t j. So, t only minus of eq this one is 10 i minus sorry 10 i and then here it is 0 and then, it is minus of 60. So, ultimately this if you simplified this whole thing this whole thing, then we probably end up getting an expression.

So, we will probably end up, so we will probably we will end up getting an expression something like i times minus of 60 is square. So, plus 0 that will be 0 plus a times minus of. So, there is no 10 i here. So, here we will get some minus of 10 t cube minus plus 30 t cube and then, k times 0 and minus 10 t. So, ultimately we will obtain minus of 60 square and this will be 20 t cube times j plus minus, sorry 10 tk and now we will do the integration.

So, now we will do the integration from 1 to 2, right. We were asked to evaluate the integral from 1 to 2. So, 1 to 2 r cross d square r by dt square dt. So, this will be integral from 1 to 2 minus of 60 square i plus 20 t cube j minus 10 t k dt and if we do the integration, then we will ultimately obtain.

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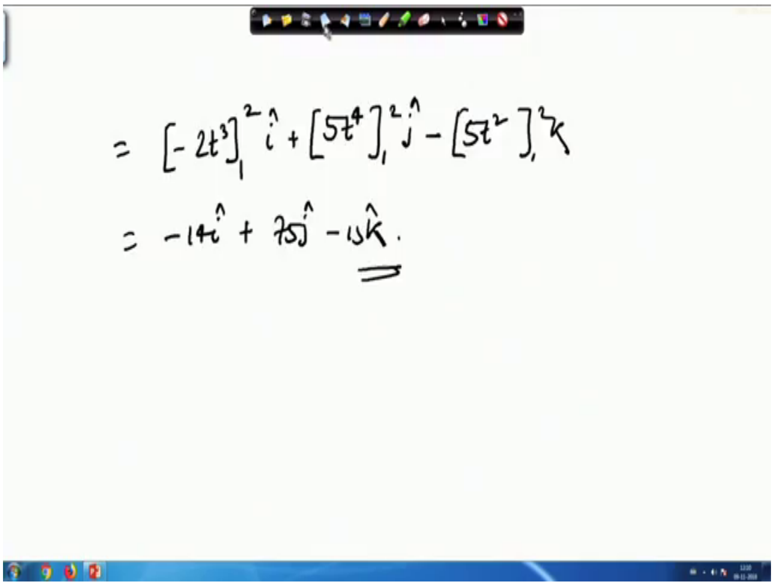

This as minus of 2 t cube from 1 to 2 times i plus 5 t to the power 4 1 to 2 j minus 5 t square 1 2 2 k, alright. So, integrating individual components and now we will substitute the value evaluate and then, you will probably get 16 and then plus 2. So, here we will have 14 i and then this one will be 75 j and this one will be minus 15 k.

So, this is what we needed to prove. So, you remember we had to differentiate an expression like that in the differentiation part. So, similarly we had in this problem we have to integrate this r cross d square divided by dt square and this one is a definite integral problem. So, we integrated between 1 to 2 and once we got our required integral, we just substituted the value of t at t equals to 2 and t equals to 1 and this is the required answer which we needed to needed to find.

So, integration of vector function is more or less similar to integration of scalar functions and like differentiation of vector functions, we follow the similar results of integration of scalar functions for the integration of vector functions. So, it is all of like generalizing the concepts of scalar functions to the vector functions and we see that all of them holds true in this case. It is just that in case of vector calculus, you are just doing the differentiation or integration component wise.

So, we are not doing just for one function, we are doing it for the every functions involved in each component basically. I will try to solve maybe 1 or 2 more examples based on integration of vector functions and afterwards we will move to our next topic which is basically Divergence Gradient and Car.

So, we will stop here for today and I thank you for attention. I will see you in the next class.