Integral and Vector Calculus

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Lecture – 34
Successive Differentiation

Hello students. So, in the previous class, we started with the concepts of vector calculus, where we introduced the how to say idea of limit continuity and differentiability of a vector function, like we already know those properties for a scalar function. We also prove very basic result which is sort of like a generalization from a result on scalar function, which is basically that if you have a constant vector then its differentiation would be 0 and if you have a differentiation of a vector as 0 then the vector must be constant.

So, this is also true for the scalar functions. So, we tried to generalize that result for the vector functions just to give you an idea that a lot of properties from the scalar functions differentiation of scalar functions, would be true for the differentiation of vector functions as well and today, I am going to list some of these properties and I will try to prove at least one or two results, and I will leave the rest of the results up to you, because they basically involve some properties from vector algebra and difference differentiation of a scalar function. So, I am pretty sure you can be able to solve the rest of the properties.

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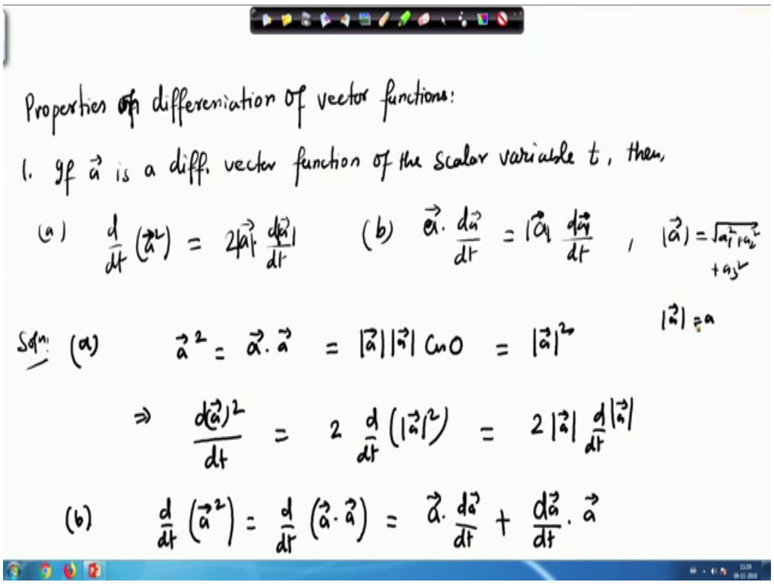

So, let us start with our first result, I am going write as our next topic as properties of, properties of differentiability, properties on differentiation. Let us call it as properties on differentiation of vector functions. So, the first property is vector functions. So, the first property is for today actually. So, is a differentiable function.

So, if a is a differentiable vector, actually differentiate vector function, differentiable vector function of the scalar variable t, of the scalar variable t then the first result is d d t of a square equals to 2 a dot product with d a d t and the second result is a dot d a d t is equal to is equal to a and d a and d t.

So, this one is actually, this here is actually the mod of d a d t and this one here is also mod of d a d t. So, here mod is basically the absolute value so; that means, a square root of square root of, if it has three components then the square root of first component. So, I can write it as a 1 square plus a 2 square plus a 3 square.

So, that is what we actually mean by mod of a. So, let us try to prove the first result. So, the first result. So, we have here a square is basically, the vector here. So, a square is equals to a dot a. So, that what, that is what we mean by the square of the vector.

So, a square is equals to a dot a which is basically mod of a times mod of a and then cos of 0. So, cos 0 is basically 0 so; that means, this is mod of a whole square. So, we can call it as a scalar quantity. So, this is basically a scalar quantity and if we do the differentiation then this will be basically 2 d d t of a scalar quantity, because mod will actually convert this whole thing into a scalar quantity and this will be 2 times mod of a times d d t of mod of a.

So, that is the first result and similarly, we can prove the second result as well. So, for the second result, we will have, so; for the second result we will have a d d t of so, for the second result we have d d t of a square equals to d d t of, we will have a dot a and from the product of, dot product of differentiation. This will be a dot d a d t plus d a d t dot a. Now, we know that differentiation of vector a square is equals to this here. So, we can use the result from section, from say from part a here. So, the left hand side of this one will be equal to this thing.

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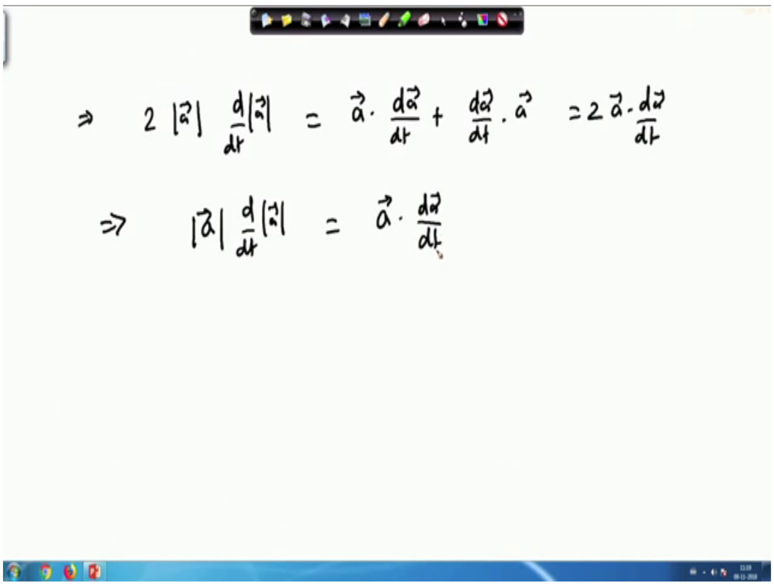

So, we can write it as 2 times vector mod of a d d t mod of a is equals to a dot d a d t plus d a d t dot a. Now, dot product is commutative. So, this can be written as 2 times a dot d a d t. So, 2 will get cancelled and therefore, we will have mod of d d t mod of a is equals to a dot d a d t. So, mod of a times d d t of mod of a is equals to a dot d a d t.

So, this is what we needed to prove in the second result. You can also call it, let us say mod of a is equals to a scalar function small a, without any vector notation. So, this a is basically scalar. So, instead of writing mod of a, we can instead of writing mod of a, we can replace mod of a by a. So, then in that case this one will be d d t of vector a square equals to 2 times a dot a times d a d t. So, this dot is usually the multiplication dot. So, this is not a dot product. So, a times d a d t. So, like scalar dot and a dot vector a dot d a d t is equals to a times. This is small a times d a d t, this is a scalar function.

So, it is better to denote the modulus or the magnitude of this vector by a scalar a. So, that there would not be any confusion; however, these properties would still remain true for this vector function a, all right.

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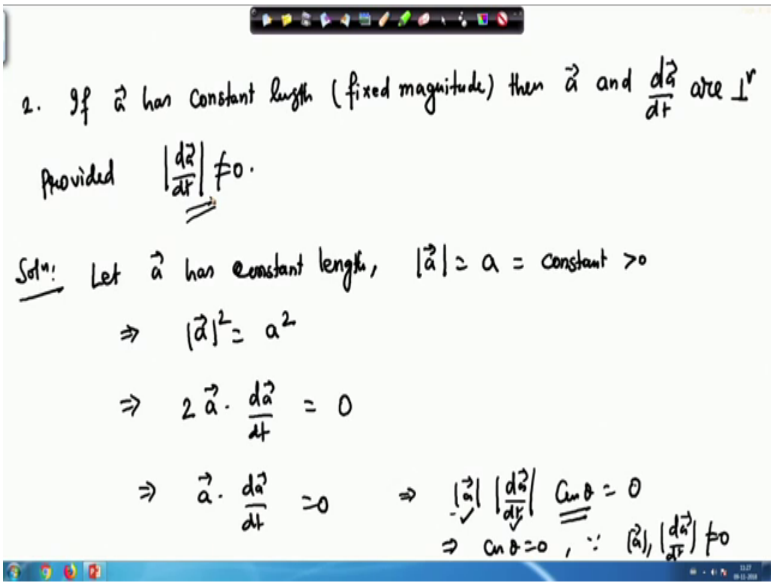

Now, the next result is or property 2 let us say is, if a has constant length, let us say constant length or fixedmagnitude, constant length means; it has fixed magnitude. Then vector a and vector d a d t are perpendicular.

So, this is the perpendicular notation are, perpendicular provided, perpendicular to each other provided, the absolute or the magnitude value of d a d t is not equals to 0. So, of course, it can the condition is sufficient as necessary as well as sufficient. So, let us start with the fact that the a is constant. So, let a has constant length. So, if a has constant length then what does that mean? Absolute value of a is equal to let us say some a where a is a positive real number including 0.

So, positive set of, where a is a non negative real number all right. So, and that is basically a constant. Now, from here, we will have vector a whole square equals to a square all right and if I differentiate both sides with respect to t. So, this will be 2 times this will be 2 times a dot d a d t, we just saw that right. In the previous result, in the previous result and we just saw that, if we differentiate, this is basically, this mod of a square and then from there we will basically obtain d a d t right.

So, if I do that, then in that case the right hand side since, it is constant it will be 0 and therefore, from here we will have a dot d a d t is equal to 0 and this means that these two vectors are perpendicular to one another, because from here we will get mod of a times mod of d a d t and from here what we will get? We will get mod of a times mod of d a d t times cos of theta equals to 0. Now, a has constant length. So, mod of a cannot be 0, because this is positive basically. So, mod of a cannot be 0 and we have assumed that this d a d t mod of d a d t is not 0.

So, this is not 0, this is not 0. So, the only possibility is cos theta equals to 0. So, from here we will get cos theta equals to 0. Since mod of a and mod of d a d t is not equals to 0. So, from here cos theta equals to 0, we basically get that theta equals to pi by 2; that means, these two vectors, these two vectors are perpendicular to one another. So, therefore, the condition is condition is necessary and we can also prove that the condition is sufficient.

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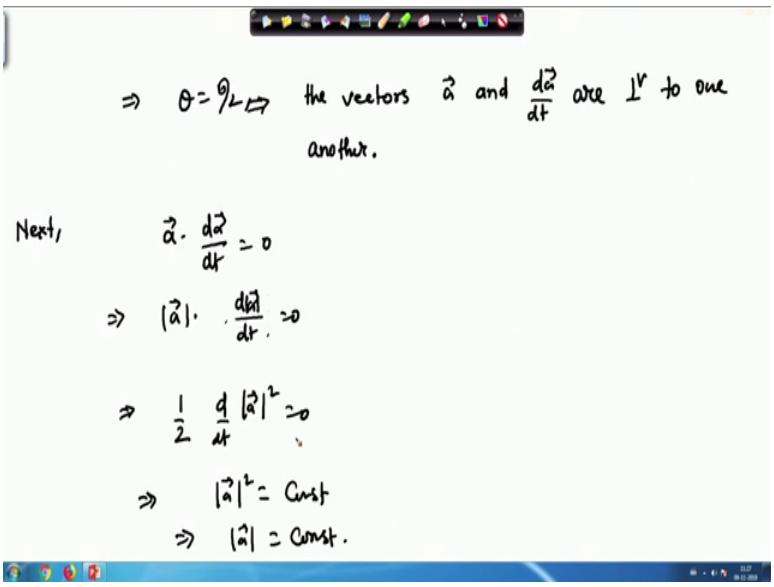

So, let us go to the next slide. So, from here what we basically get is, theta equals to pi by 2. So; that means, the vectors. So, this implies that the vectors a dot a, the vector a and the vector d a d t are perpendicular to one another. Next, we have is other way around. So, let us assume that d a d t they are perpendicular. So, this is equals to 0 and from here we will get d a d t equals to, we will get from here.

This will be equal to if I go to the previous slide. So, this will be equal to a dot vector a dot d a d t. So, vector a dot d a d t equals to basically, we will obtain a dot d a d t. So, I will obtain this as vector a dot d a d t.

So, from the previous result I can use this here and sorry, this here all right and this means that I can be able to write this as 1 by 2 d d t of mod of a whole square equal to 0 and from here we will basically obtain mod of a square equals to some constant and if mod of a square equals to some constant are basically constant.

So, from here we can write mod of a is equals to some square root of some constant. So, this will be again a constant, so that means, if they are perpendicular to one another then in that case mod of a would also be constant. So, therefore, the condition is also sufficient so; that means, if the dot product, so, if the dot product is 0 then in that case we can also say that the vector is of constant length.

So, of course, this condition is necessary as well as sufficient, provided this is true all right. So, this were the two basic results that we proved here we can also try to prove some more results, but I think they are all analogous and motivated from the vector algebra or some differentiation of the scalar functions.

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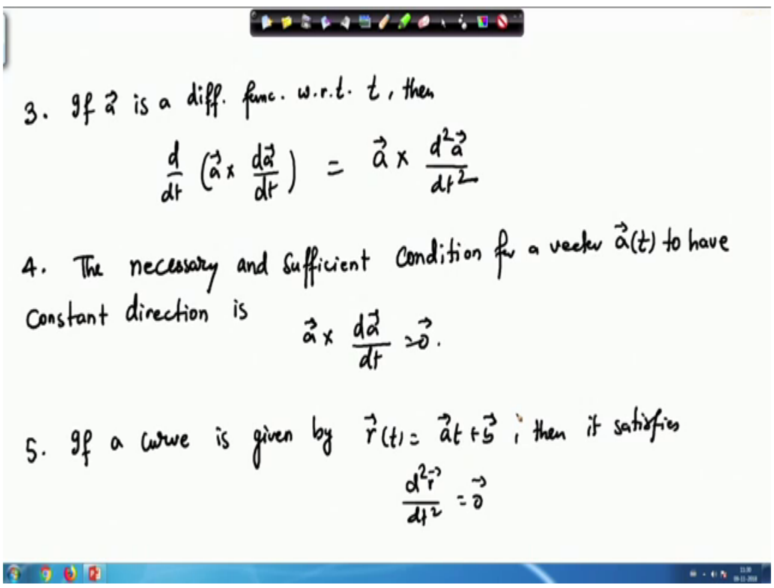

So, we will skip those, but I am going to write those properties. So, if a is a differentiable function with respect to t, if a is a differentiable function with respect to say t and then we will have d d t of vector ath cross product with d a d t is equals to a cross product with d square a by d d t of vector a times d a d t is equals to vector a times d square a by d t square mainly, because when you take d a d t here then the 2 vectors will become parallel.

So, this is obviously, true thus a fourth property is the necessary and sufficient condition. It says that the necessary, the fourth property is the necessary and sufficient condition for a vector a t to have constant direction is a t times d a d t equals to 0 vector. So, this is also nice property for a vector to have a constant direction.

So, that was for the constant length, this one is for the constant direction and this is very how to say obvious to prove. So, you just have to use some derivative result for a cross a and from there you can be able to obtain this. So, it is really not complicated I can also list some more properties for example, if you, so, if a curve is given by let us say r t equals to a constant plus b constant then it satisfies d square r by d t square equals to a 0 vector.

So, this is also not complicated to show. So, we just differentiate with respect to t. So, derivative of a constant vector would be 0 and then you differentiate again with respect to t, then you will have constant vector a that derivative of that constant vector would again be 0 and then you will be, you will be basically obtain this d square r by d t square equals to 0.

So, like that if you go through some vector, vector calculus books which I have listed in the references, you come across a lot of results which have basically motivated from vector algebra and differentiation of scalar functions. So, I leave those results up to you for practice and hopefully I will try to include some examples in your assignment sheets as well all right.

So, next we will practice some problems from vector cal differentiation of vectors. So, these were some results. Now, we will practice some problems, before we move on to the next topic.

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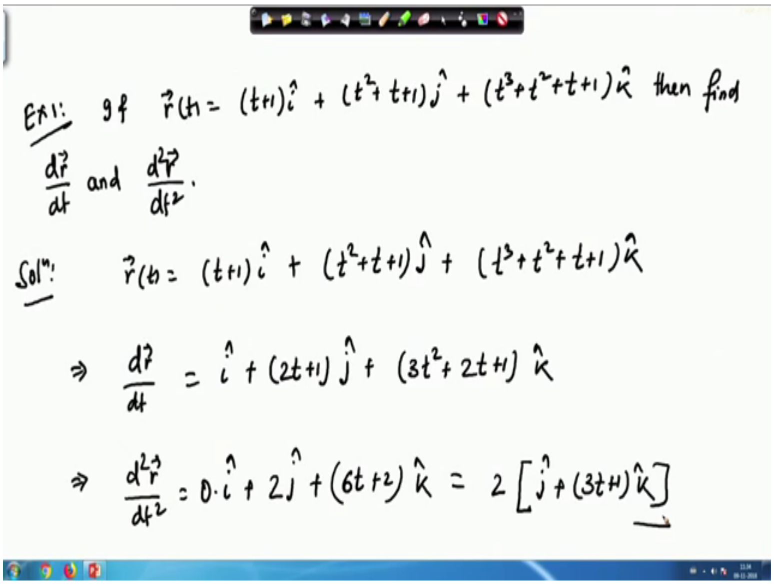

So, let us practice a few examples. So, example 1; if our t equals to say t plus 1 times i t square plus t plus 1 times j and t cube plus t square plus t plus 1 times k then find d r d t and d square r by d t square. So, now, we are practicing few examples all right. So, we have a given function vector function r t.

So, let us write the vector function r t again, which is t plus 1 times i then t square plus t plus 1 times j and then t cube plus t square plus t plus 1 times k. So, these are all the components along the x y and z directions. Now, we have to find out d r d t when we study about application of vector calculus.

We will get to know what does this d r d t mean and how similarly? What does d square r by d t square mean, but so for at the moment, we will just take to the basic calculation of d r d t and d square r by d t square. So, this one will be just a differentiation component wise with respect to t. So, if I differentiate this one, then differentiation of 1 is 0, differentiation of t would be 1.

So, this will be i similarly, this will be 2 t plus 1 and this will be 3 t square plus 2 t plus 1 times k and when i differentiate again with respect to t. So, this will be differentiation the component of i is 1. So, differentiation of 1 is 0 times i, differentiation of 2 t is 2 and 1 is 0. So, 2 j differentiation of 3 t square would be 6 t plus 2 times k. So, since 0 times i 0. So, we will not write that component i take 2 common. So, j plus 3 t plus 1 times k.

So, this is the required answer for d square r by d t square. So, that is how you basically calculate the derivative of the, given vector function.

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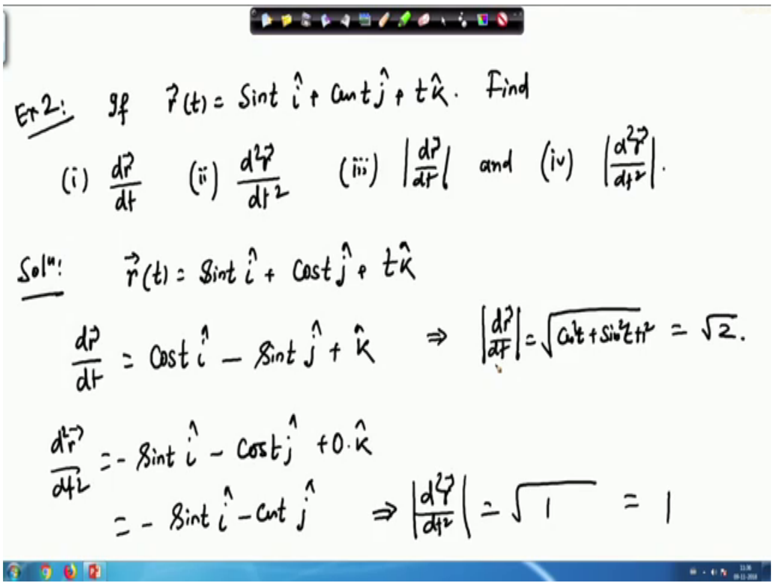

Let us practice an another example. So, suppose if r t equals to, we have sine t times i cos t times j plus t times k. Then find d r d t d square r by d t square mod of d r d t and mod of d square r by d t square. So, the solution would look like this, our given function r t is sine t then cos t times j and then t times k.

Now, d r d t, what is our d r d t? So, d r d t would be differentiation of sine t is cos t i differentiation of cos t is minus of sine t j and differentiation of t k would be just k. So, from here our mod of d r d t would be square root of cos square t plus sine square t plus 1 square. So, square of cos square t plus sine square t would be 1 and then 1 square is 1. So, this is basically, square root of 2 and again we will calculate d square r by d t square from here.

So, this will be minus of sine t i then minus of cos t j and differentiation of 1 is 0. So, 0 times k. So, basically minus of sine t i minus of cos t j and from here we can calculate our d square r by d t square as square root of minus of sine square t plus square root of minus of cos square t.

So, ultimately 1. So, that is the required answer and it cannot be plus minus 1 or plus minus 2, because we are talking about the, the magnitude and the magnitude cannot be negative. So, that is why we are always taking the positive value. So, this is the required solution to this problem all right.

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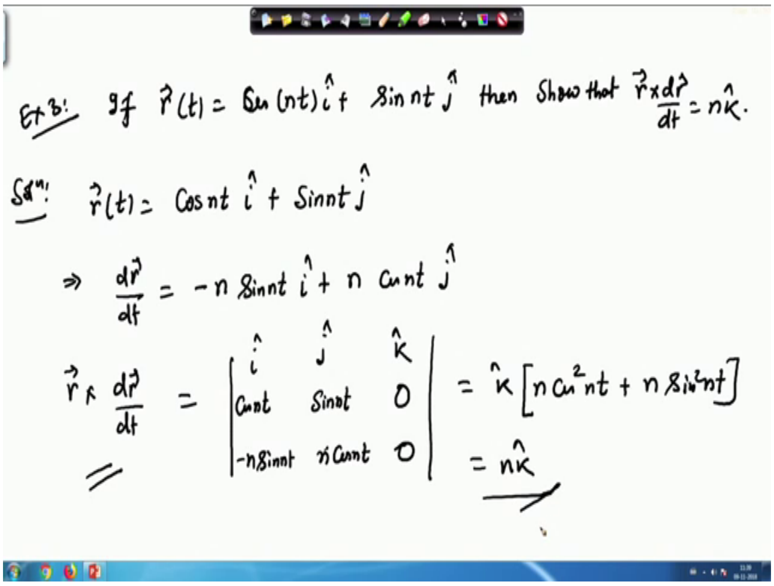

An another example of this type could be that you might be asked to evaluate something like this. So, suppose if r is equals to, if r t equals to cos n t sorry, cos n t i plus sine n t and j, then show that r cross d r d t equals to n times k all right or you might be asked to, so that r dot d r d t is equals to something else.

So, here we will, we have r t equals to cos of n t i plus sine of n t j. So, from here we can calculate our d r d t, because we are needed to take the cross product with of r and d r d t. So, here d r d t would be minus of n sine n t i and this one is n cos n t j. Now, we will calculate the cross product r cross d r d t. So, this will be i j k.

We know that how we calculate the cross product. So, we take the determinant and then here we will have component of r. So, cos n t sine n t and then component of k is 0 and here minus of n sine n t n cos n t times here, the component of d r d t or the kth component is again 0. So, if you break this determinant then you will find that both the both the components i and j are 0, because their coefficients are 0.

So, i and j would not occur and similarly and simply we will have k times n cos square n t plus n sine square n t. So, if I take n common then this will be a cos square n t plus sine square n t. So, that is one and therefore, this will be n times k cap and this is what we needed to prove.

So, that you might come across examples, which are of this type that you are given a vector function and then you will be asked to evaluate r cross d square r by d t square or r dot d square r by d t square or you will be given an expression d square r by d t square plus r cross d d r d r d t plus r dot d r d t and then you will be given an expression or you will be asked to find a certain expression.

So, just remember that all you have to do some simple differentiation with respect to t of the vector function r and then just calculate those expressions. So, substitute the values, simplify them and that will give you the required answer. So, this type of examples would not be that difficult and I am pretty sure, you can be able to solve all of these by doing some simple differentiation.

So, we may try to solve maybe, one or two examples more little bit different than these in our next class and I am hoping that you would find them interesting. So, thank you for your attention for today and I look forward to your next class.