**Integral and Vector Calculus**
**Prof. Hari Shankar Mahato**
**Department of Mathematics**
**Indian Institute of Technology, Kharagpur**

**Lecture – 33**
**Limit, Continuity, Differentiability**

Hello students. So, in the previous class, we started with the vector calculus part of our syllabus. And we introduced the concepts of vector function, and limit of a vector function, how do we define the limit using epsilon delta definition, and some results which are basically motivated from the function of scalar from the function of scalar variables.

So, like I was saying in the previous class that like the function of scalar like the scalar functions, sometimes I am confusing these two terms basically during the pronunciation, so do not get confused. So, during the scalar functions, we introduce the concepts of a continuity, differentiability, so in case of vector functions as well, we have those concepts valid. So, let me introduce the concept of continuity, and then we will go to the definition of differentiability.

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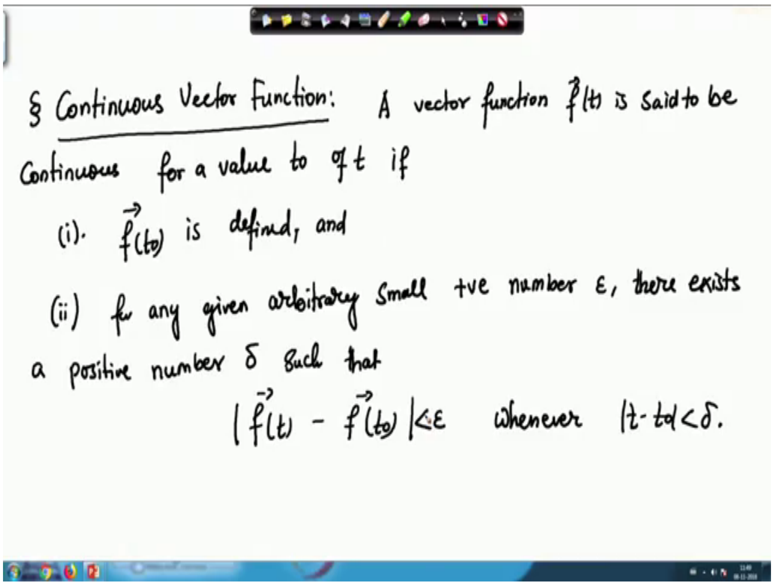


So, what do we mean by continuous function, so continuous vector function continuous vector function all right. So, a vector function f is said to be continuous a vector function f is said to be continuous for a value or at a point for a value t 0 or at a point t 0 of t if f t 0 is defined.

And for any given arbitrary small arbitrary small positive number epsilon there corresponds or there exists there exists a positive number delta such that our f t minus f t 0 is less than epsilon, whenever t minus t 0 is less than delta.

And if this is true, then the function f is said to be continuous at the point t equals to t 0. Here in case of limit we were only interested what is the value, when t is going to t 0. So, we were not interested the value of the vector function at the particular point t equals to t 0.

But, in case of continuum like scalar functions, we are interested in the value of the function at the point t is equals to t 0 as well. Because, if the function f t minus f t 0, if the absolute value is less than epsilon, only then we can say that the function f t is continuous at the point t equals to t 0. So, in this case we actually want to know, what is the value of that vector function at the point t equals to t 0. So, it is pretty much motivated from the continuity definition of scalar functions, and it is not that different actually.

However, here when we when we evaluate this mod, we have to consider the individual component, then take the absolute value, and try to obtain this how to say criteria alright.

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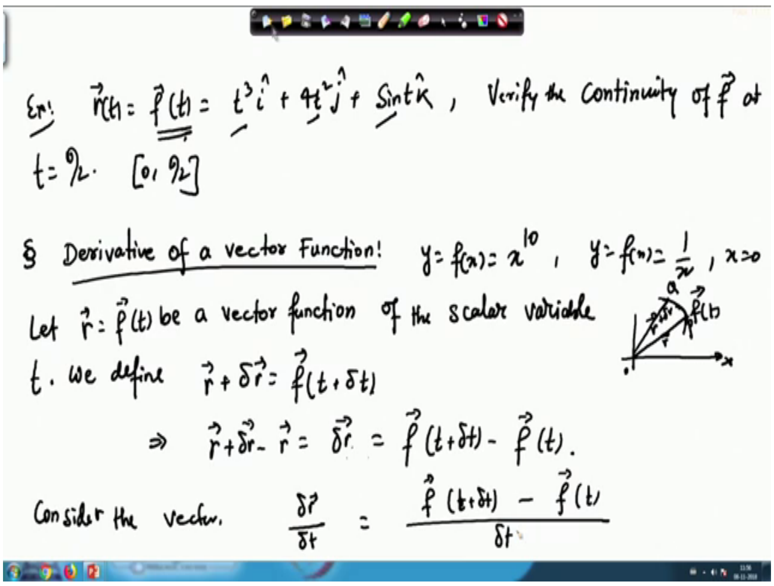

And for example, we can have let us say a function a vector function of type, I am writing r or r t equals to f t equals to let us say t cube i plus 4 t square j plus sine t k, then verify the continuity verify the continuity of f at t equals to let us say pi by 2.

So, of course the function is continuous at t equals to pi by 2, because at f pi by 2 at t equals to pi by 2, this basically of pi by 2 will get a value, then we will do that epsilon delta definition, and we can be able to show that the value can be made arbitrary small. And here of course you can have an interval instead of having this point, because this is an algebraic function, this is an algebraic function, this is a trigonometrical function. So, you can even choose the interval as 0 to pi by 2, and this vector value this vector function is continuous throughout this interval as well. So, it is a really nice behaving vector function. And therefore, it is continuous even in that interval also.

So, now what we will do is we will define the differentiability of a vector function. So, let us define differentiability or derivative let us write derivative of a vector function alright. So, like function of a scalar functions, we can also differentiate the vector functions as well, but in which context do we mean.

So, for example a function like y is equals to f x equals to let us say x to the power 10 is 10 times differentiable, so it is a very nicely behaving function, basically a polynomial function. And you can be able to differentiate at least 10 times or if because the exponent is 10. However, if we had a function like, let us say y is equals to f x equals to 1 by x. And suppose you want to check its differentiability at x equals to 0, then of course it is not a differentiable function at x equals to 0. And mainly because it is also not continuous at x equals to 0.

So, similarly we can have similar kind of situations, in case of vector functions as well. So, what how or what way we can define the differentiability of a vector function is the question here. So, in case of scalar functions, you remember we used to take x plus delta x, and the value of f at x plus delta x minus f x is defined as the as divided by delta x is defined as the derivative, so here also we will do the similar thing.

So, the definition would be so suppose if so suppose if we have a curve like this, so this is basically our f of t. So, this is the vector function, and that is the origin, and this is my vector r, and this is the point p. Now, this curve can go like this, and then this will be point q all right the point p and q. And at the point p we have the point, we have the vector r. And at the point q, we have the vector r plus delta all right. So, this is how we define this.

And let us now put everything in words. So, let r is equals to f t be a vector function of the scalar variable t. And we define r plus delta r, so the point q basically as f of t plus delta t, so that means, from here if I write r plus delta r minus r, then this will be basically f of then this will be basically this will be basically delta r equals to f of t plus delta t minus f of t, so that is this small how to say movement from p to q. So, this small arc is our delta r, and the delta r is defined in this way.

Now, if we consider this vector so consider the vector delta r by delta t, then this will be f of t plus delta t minus f of t by delta t.

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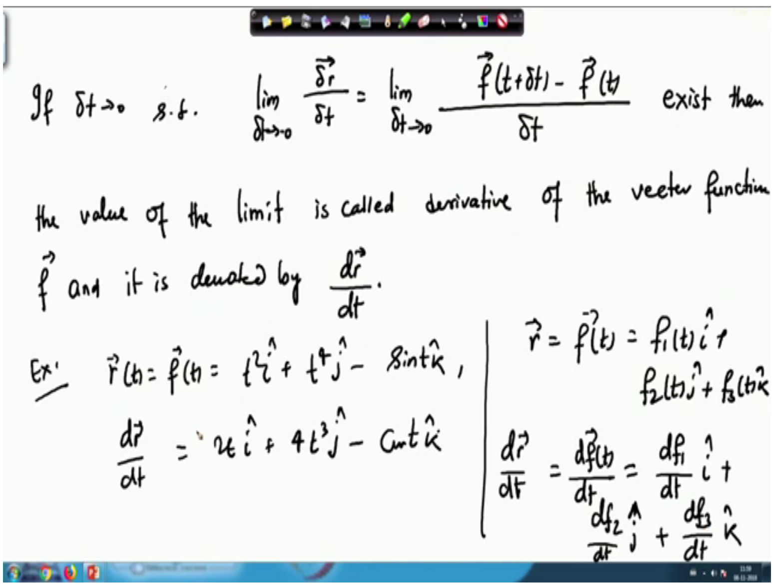


And if I make delta t goes to 0 so if delta t goes to 0, then limit delta t goes to 0 delta r by delta t equals to limit delta t goes to 0 f of t plus delta t minus f of t divided by delta t exist. So, or if delta t goes to 0, so instead of then we can write such that so such that this exist, then the value then the then the value of the limit, then the value of the limit is called the derivative of the vector function of the vector function f, and it is denoted by d r d t.

So, when t goes to delta t goes to 0, then this right hand side basically this right hand side is basically, if the limit exist is called as the derivative of the vector function f, and it is denoted by dr dt all right. So, for example, if we have a function let us say r equals to t square i plus t to the power 4 j minus sine t k, then it is derivative dr dt will be 2 t i plus 4 t to the power 3 j minus sine t is cos t ok. So, we basically differentiate every component.

So, when it comes to differentiating a vector function when it comes to differentiating a vector function, which is given by r is equals to f t equals to f 1 t i plus f 2 t j plus f 3 t k. Then in that case the derivative is dr dt given by df dt, and then this one is df 1 by dt times i df 2 by dt times j plus df 3 by dt times k. So, these are all functions of t.

So, this is how we differentiate a vector valid function. So, we differentiate every component, and if any one of the component does not exist at a given point, then in that case the function the vector function is not differentiable. So, suppose we are evaluating the differentiability at t equals to 0, and we have instead of t to the power 4, we have 1 by t. Then in that case of course, the second component is not differentiable, and therefore the overall function vector function would not be differentiable.

So, if we want to talk about the differentiability of the vector function, we have to talk about the differentiability of each individual component or each individual scalar function involved in that vector function, so that is how we define the differentiability of a vector function all right.

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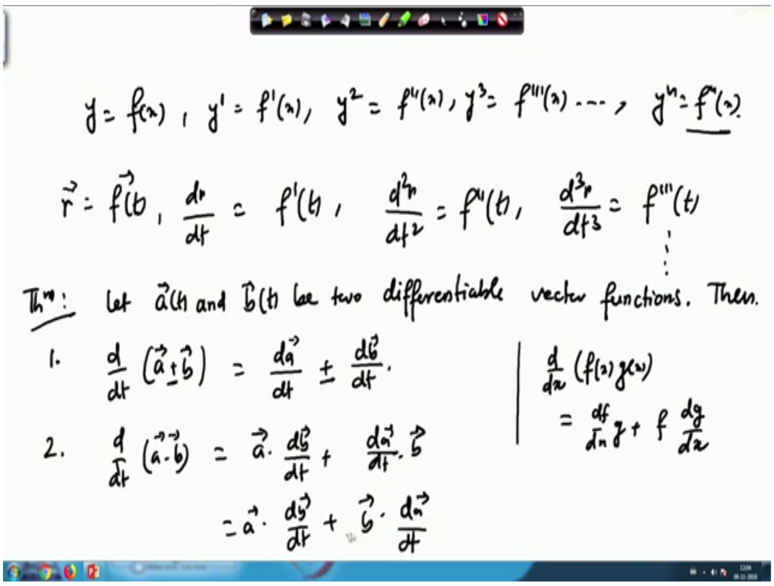

Now, like function of one variable or scalar functions, we had y dy dx let us say y 1 equals to f dash x, then we had y 2 which is d square by dx square equals to f double dash x, then we had y 3 which is basically d 3 y by dx 3 is equals to f triple dash x and dot dot and so on, so y n equals to f n x all right.

So, similarly we have such rule for the vector function as well. So, if we have r equals to let us say f t, then in that case we will have dr dt equals to f dash t, which is basically df dt. Similarly, d square r by dt square equals to f double dash t. Similarly, d 3 r by d 3 is d 3 dt 3 is equal to f triple dash 3 t and dot dot so on.

So, you can do successive derivative for the vector functions as well. So, you can differentiate a vector functions as many times as you want, if it is differentiable that many times all right. So, this is how we define the derivative of a vector function or successive derivative. So, this is basically called successive derivative.

And like continuity or limit rule, we also have rules for some difference product of vector functions, and they are different derivative. So, theorem-2 you may call. So, let a and a t and b t be two differentiable vector functions.

Then so the first rule is d dt of a plus b is equals to da dt plus dv dt, so that means differentiation of the sum is equal to sum of the differentiation all right so. Secondly, if you have subtraction, so differentiation of this of the difference is equal to the difference of the derivative. So, derivative of the difference is equal to difference of the derivative.

Third is the dot product. So, derivative of the dot product is equals to a dot db dt plus da dt dot b, it is like differentiation of the product of two functions. So, in case of differentiation of product of two function, which was something like d dx of let us say f x times g x. So, it was something like df dx times g plus f times dg dx. So, we used to differentiate one function leaving another. And then differentiation of second function leaving the first function. So, it is the same rule here.

And as since we have a dot product dot product is commutative. So, we can play with this a little bit, and we can arrange this dot product, so because of the commutativity of the dot product. We can write b dot da dt, so that is the dot product.

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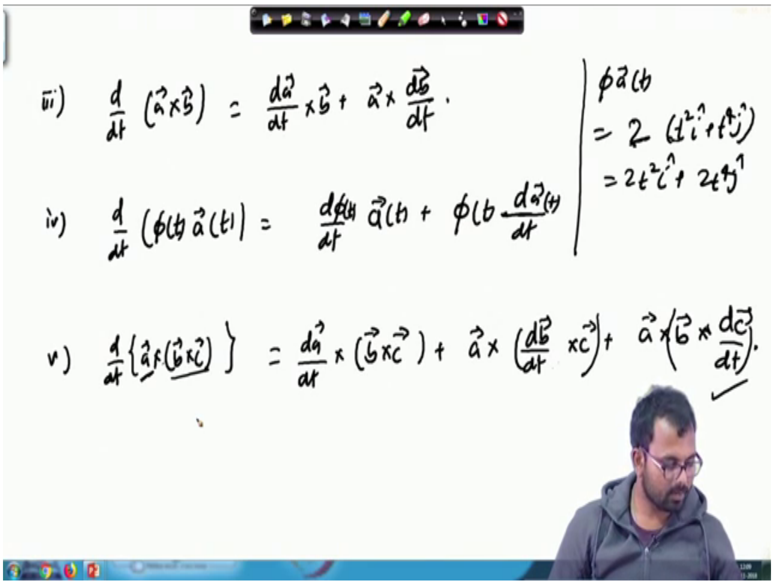


And if you have let us say the cross product d dt of a cross b, then it is defined as d a cross da dt cross b plus a cross db dt. So, this is the cross product rule. And we have to remember that here we cannot change the order, because if we write b dot b cross da dt, then we have to put a minus sign. So, a cross b is equals to minus of b cross a, so that is that is that is what we have to take care of here.

Now, similarly if we have let us say a scalar function which is which is multiplied by vector function, so I have phi t which is also a differentiable scalar function. And then I have then I have a vector function say a t. So, then in that case it can be written as d phi dt times a t plus phi t times da dt all right. So, it is just that there would not be any kind of dot or cross here, but it is just that the function phi is multiplied in usual multiplication sense.

So, when I say usual multiplication sense, it means that if we have a phi times a t, so let us say our function a t is t is square i plus t to the power 4 j. And my phi function is a my phi function is something say a constant basically say 2, then in that case it will be a usual multiplication of these two scalar quantities. So, it is 2 t square i plus 2 t to the power 4 j, so it will not be like a dot product. So, dot product will usually give you a real number actually, but in this case you will still get a vector function all right.

Now, if you have let us say instead of instead of a scalar function and a vector function, we can also have da dt times say vector a vector b and sorry a a cross let me write this formula properly. So, we can also have something like a cross b cross c, I cannot have a dot b dot c, because then b dot c is a scalar function, and that is then it is the same definition here all right.

So, we are not writing that formula. So, here in this case, it will be da dt cross product with b cross c plus a cross db dt cross c and then a cross b cross dc dt all right. So, again we have to take care of the order. So, here b cross c, and then I have to make sure that this the cross product is here, and then we have a cross product here. So, this is how we will do the cross product for derivative of the cross product of three vector functions.

So, obviously we treat these two as one vector, and the obviously we treat these two as one vector, and this one as another vector, and based on that we can do this derivative. So, basically we differentiated da dt times b cross c, and then I used instead of b in this formula, I am using b cross c. So, then our cross product with this and d dt of this b cross c will we will we will get common, and then we do the derivative one b cross c. And that will give us the whole cross derivative of the cross product of a cross b cross c.

So, there are several other formulas of derivative of vector functions, we can look into any text book. And there you can find all sorts of formulas, and for the interested readers, you can consult any one of those vector calculus book for more formulas. I have just listed few, because we might be using them at some point in the future, and therefore it is a good collection to have.

Now, that we defined these formulas, we can also try to prove them, but it is just that the proof is very straightforward, and it is also present in those textbooks. So, we will not do that for the time being.

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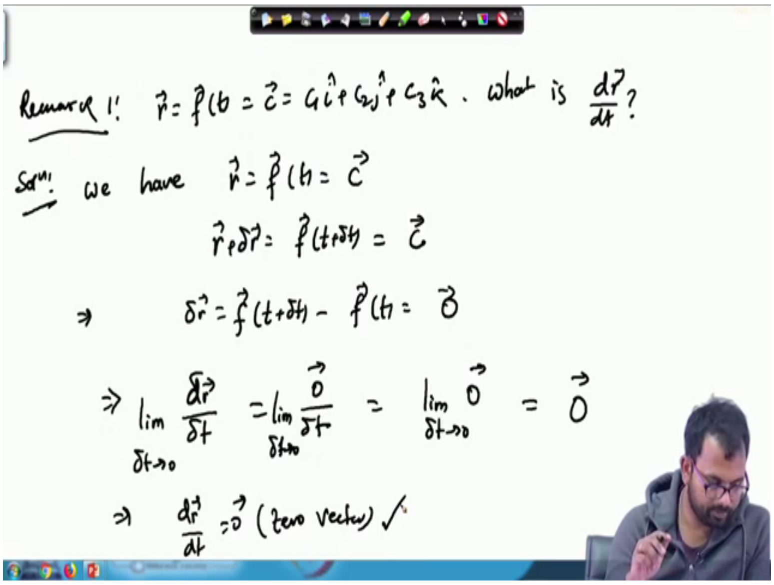

Next we will try to prove some basic result, so let us say remark-1. So, we know that derivative of a constant function is 0, so that is a very well-known result in our function of one variable. Now, what is the derivative of a constant vector so like constant function, we also have a scalar functions, we also have constant vector functions.

So, suppose we have r equals to f t equals to a constant function, so whose component are c 1 i plus c 2 j plus c 3 k, so what is dr dt? So, c 1, c 2, c 3 are all real numbers. So, they are all real numbers, and then we have to calculate the derivative of this vector function.

So, solution we have r equals to f t equals to sum c. So, from here r plus delta r would be f dash t plus delta t equals to again c. And therefore, from here we will have delta r equals to 0, so f of t plus delta t minus f of t is equal to 0.

So, from here we will get dr del r delta t equals to 0 by delta t. And now I will take limit on both sides. So, this will be limit delta t goes to 0, and then this will be limit delta t goes to 0 0. So, limit of a constant function is basically constant that particular constant, so here it is 0.

And therefore, dr dt is equals to 0 which is a 0 vector, I can put a vector sign here, so which is basically a 0 vector. So, like derivative of a scalar function else derivative of a a constant in function of in scalar functions gives us constant. In here in case of is a vector functions, the derivative of a constant vector will give us a 0 vector. So, this result a also holds true from, what we what we have learned in our scalar functions.

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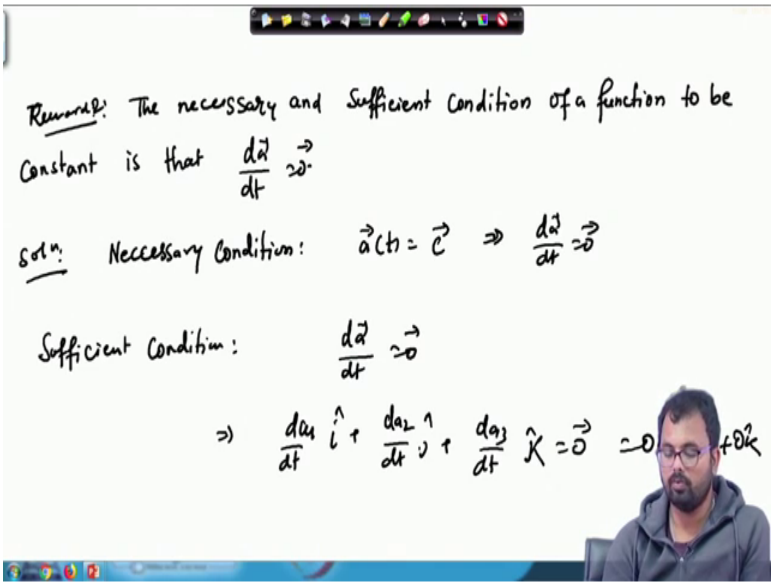


Now, the next thing is the next property is the next property is we can have it again as a remark. The necessary we can say that the necessary another remark say two the necessary and sufficient condition of a function to be constant is that da dt equals to 0.

So, the necessary condition, we have proved. So, the necessary condition necessary condition we have already proved that if we have a is equals to a constant function, then from here da dt is 0, this is we have already seen, this we have already seen just like couple of minutes ago, so this is true.

Now, the condition is sufficient, so that means the sufficient condition is sufficient condition to prove the sufficient condition. We will have da dt equals to 0, now the vector function a has three components. So, let us write da dd a 1 t i plus da 2 dt times j plus da 3 time and divided by dt times k is equals to 0, now this is zero vector actually. So, this is basically our zero vector. So, this zero vector can be written as 0 i plus 0 j, where each component is 0. So, we can write each component and 0.

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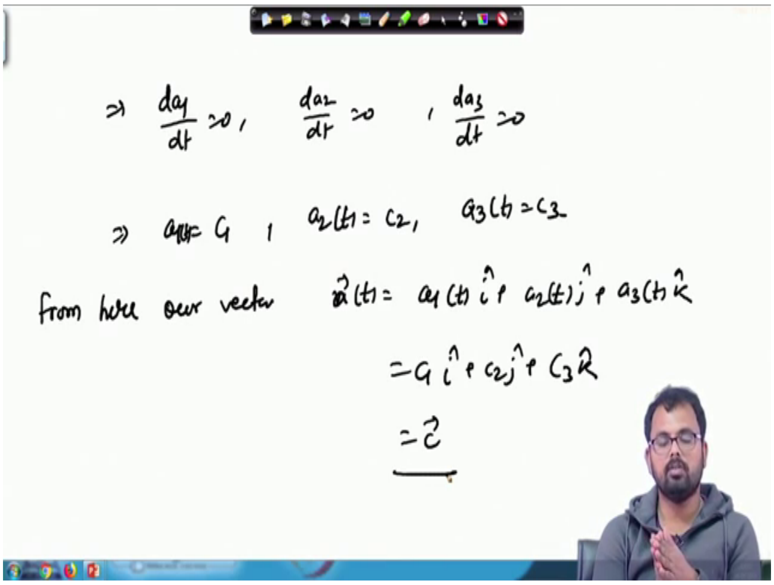

And now we equate the coefficients of i j and k. So, if we equate the coefficients of i, j, and k, then this will give da 1 d t equals to 0, da 2 dt equals to 0, and da 3 dt equals to 0. So, from here if we integrate both sides, then we will get a 1 equals to constant say c 1 a 1 t equals to constant says c 1, a 2 t equals to constant say c 2, and a 3 t equals to constant say c 3, so that means, a 1 2 a 2 2 a 1 2 a 1 t, a 2 t, and a 3 t are basically constant.

So, from here our vector r from here our vector a t is a 1 t times i, a 2 t times j, and a 3 t times k, which is basically c 1 i c 2 j and c 3 k. Since, c 1, c 2, c 3 are all constant, this is basically a constant vector. So, the condition is sufficient as well.

So, you see like a function scalar functions, if the function is given to be constant, then it is derivative is 0. In case of vector function as well, if you have a constant vector, then it is derivative would also be 0. And it is necessary and sufficient condition, so that means if the derivative is 0, then the function vector function has to be a constant function. And if the vector function is a constant function, then its derivative has to be 0.

So, we learned this basic result in vector calculus. And in our next class, we will try to work out some examples on derivative of a vector function, and then we move to our next topic. So, I will stop here for today.

And I thank you for your attention, and I look forward to you next class.