Integral and Vector Calculus
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Lecture – 32
Vector Calculus

Hello students. So, in the previous class we looked into the concepts of volume integral and before that class we worked the examples on surface integral. For the time being we go to the Vector Calculus part because that is also very important and at the end I will try to cover the center of gravity chapter where we have some formulas, from integral calculus that will help us calculate center of gravity and moment of inertia.

But for the moment, I will start with the vector calculus section which is also very interesting and you can be able to relate the things which you have learnt in your function scalar functions like limits continuity and differentiability, integration of scalar functions which we were doing up until last class.

So, in the vector calculus as well these things are there and we will see how we actually define the vector function and its limits continuity differentiability things like that. So, today we are going to today we are going to start with this chapter here; so, this here.

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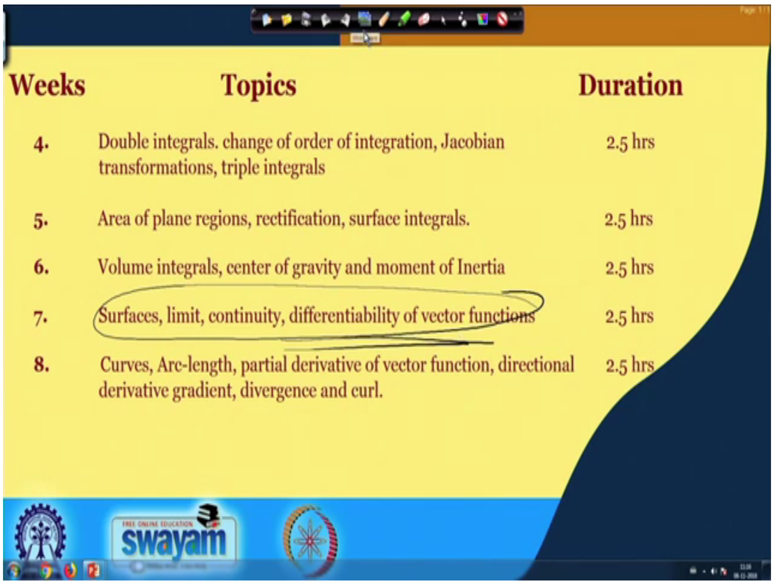

So, scalar sorry so, limit continuity differentiability of vector functions. So, this is what we are going to start with all right.

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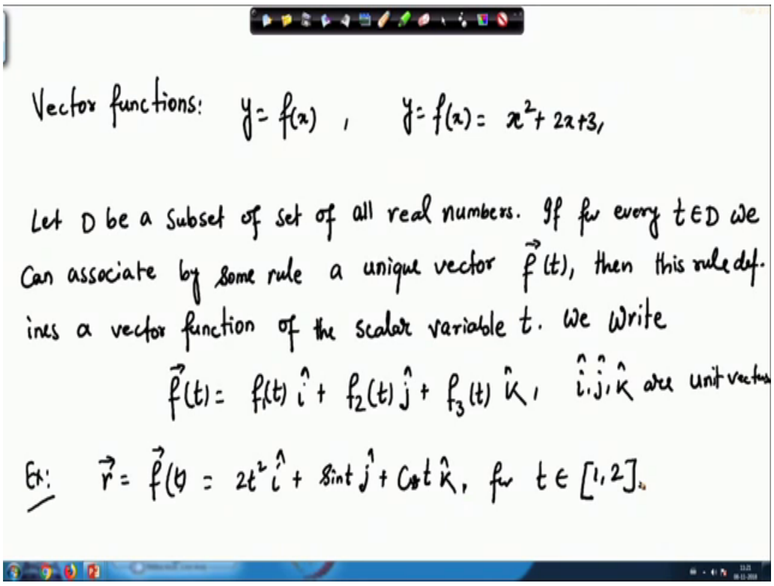

So, let me start a new page and I am going to increase the thickness a little bit yes. So, today's topic is vector functions. So, what do we mean by vector functions?

So, usually the function of scalar variable or scalar functions are written as y is equals to f x where x is the independent variable and y is the dependent variable. Possible examples could be y is equals to f x equals to let us say x square plus 2 x plus 3 or y f x equals to sin x or exponential x any such functions which we have learnt basically. So, those are the scalar functions; function of one variable.

Now how do we define the function of; how to say it is a vector functions actually? So, I mean we know that vector quantity they have directions as well as the magnitude. So, how do you define functions related to that?

So, we can give a formal definition for a vector function. So, the formal definition goes like this. So, let D be a subset be a subset of set of all real numbers set of all real numbers and if for every t in D, we can associate or we associate we can associate by some rule by some rule. So, what is this rule? I will define, but for the time being let us write it as by some rule a unique vector say f t, then this rule defines a vector function all right; of the scalar variable t. And here f is basically a vector quantity or a vector function for the variable t.

So, f can be called as basically a vector function. So, one possible ah; so, we represent this. So, since a vector quantity has three directions. So, i j and k we can be able to with the help of i j and k we can be able to write a vector quantity or a vector basically. So, here this vector function f t can also be represented with respect to those unit vectors i j and k. So, we represent or we write f t equals to f 1 t i f 2 t j f 3 t k where i j k are unit vectors. So, basically along x axis y axis and z axis so, they are perpendicular to each other and they are the unit vectors.

So, here f 1 t is a function of t f 2 t is a scalar function of t f 3 t is a scalar function of t. So, all of them are basically scalar function of t and while writing with the help of i j and k they constitute a vector function. So, one possible example could be let us say r equals to f t. So, we write basically the vector function by r and r is a function of t and we write r equals to f t equals to let us say I could write 2 t square i plus sin t j plus cos t cos t ok. So, this is one such example for let us say t is an interval 1 to 2.

So, this could be our a vector function. So, like this we can define several types of vector functions and they are how to say how we represent them using i j and k their domain well; for example, where the variable t belongs to things like that all right.

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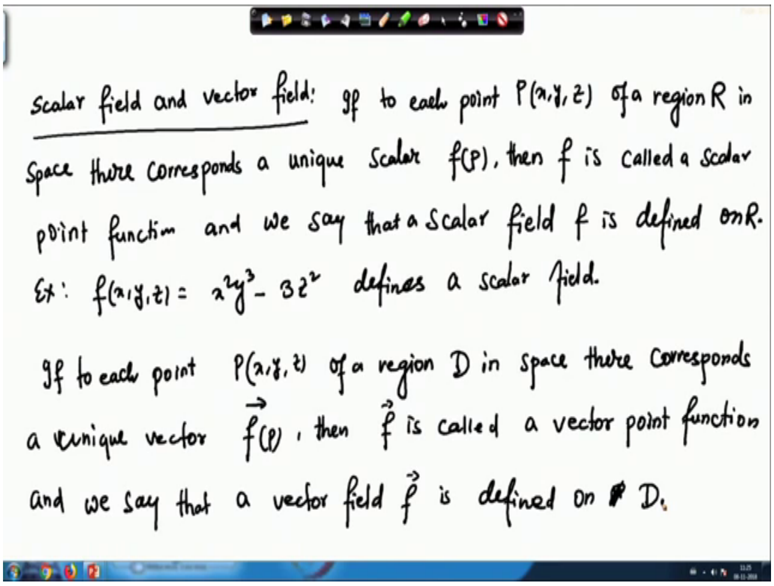

So, now what do we mean by let us say scalar field and vector field. So, scalar field and vector field scalar field and vector field what do we mean by them? So, scalar field is basically; so, if to each point P x y z of a region r let us say of a region R in space, there corresponds a unique scalar f of P, then f is called a scalar point function and we say that a scalar, we say that a scalar field f is defined on R right on R or in R basically on R is fine yes.

So, if for every point P in that region R; if we can correspond a unique scalar f P. So, then in that case that is unique scalar is called as the, is called as the scalar point function and we say that a scalar field f is defined on R. So, similarly and so for example, we have f x y z. So, for example, if we have xyz equals to let us say x square y cube minus 3 z square. So, this defines a scalar field. So, this defines a scalar field all right.

So, next we define the vector field. So, if to each point if to each point P x y z of a region D let us say, in space there corresponds a vector for a unique vector actually a unique vector f P, then f is called a vector point function a vector point function. And we say that a vector field f is defined on R sorry on D. So, I defined on D. So, that is how we define a vector field.

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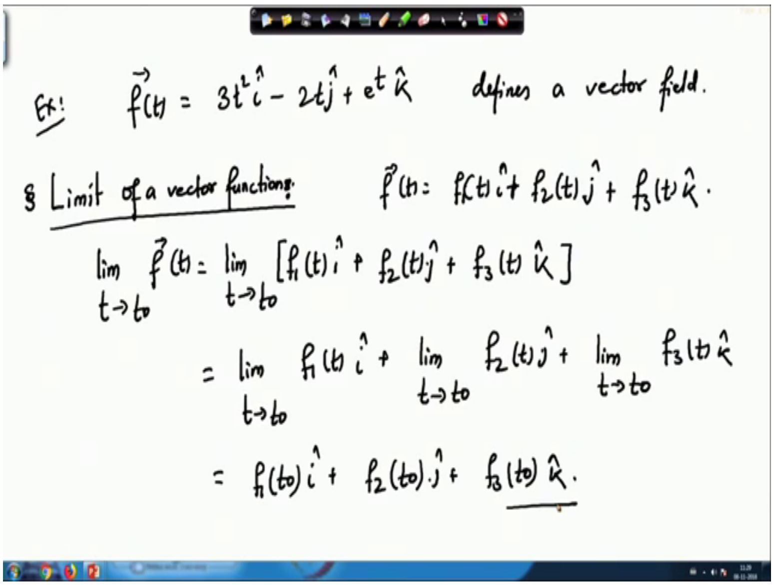

And a possible example could be let us say f t equals to 3 t square i minus 2 t j plus e to the power t k. So, this defines a vector field on the given domain actually. So, we can have any domain and then this will actually defining a vector field all right. So, that is how we define the vector field.

Next we have is so, we know in the in the similar fashion we can define all sorts of vector functions. So, defining the vector function and having its domain of definition is really not a very complicated thing. So, those things are very can be can be done very easily.

Now the next topic we know that from a function of scalar variable is scalar functions actually after the function we have we when we study usually the limit our first scalar function. So, here also we will basically look into the limit of a vector function. So, how do we define it what do we mean by it and we will work out some examples. So, let us start limit of a vector function of a vector function not functions. So, usually we know that the limit f t the function f t is given by f 1 t i plus f 2 t and j plus f 3 t right.

So, when we pass the limit on the left hand side Do, let us we have limit t goes to t 0; that means, when t is going to some real number t 0 and I am passing the limit on the left hand side. So, what would happen to the right hand side? How do we define the limits on this f 1 t i f 2 tj f 3 tk? So, we the idea to define the limit is basically to define the limit in every component. So, we define the limit in every component all right. So, here we are defining the limits. Now if I define the limits then this should be looking like; so, this is the required way to define the limit.

So, we passed the limit on individual components. So, suppose after making t goes to t 0 our f 1 t will be f 1 at t 0 times i f 2 t at t 0 times j and f 3 t at t 0 times ok. So, this is how we define the limits on individual how to say components.

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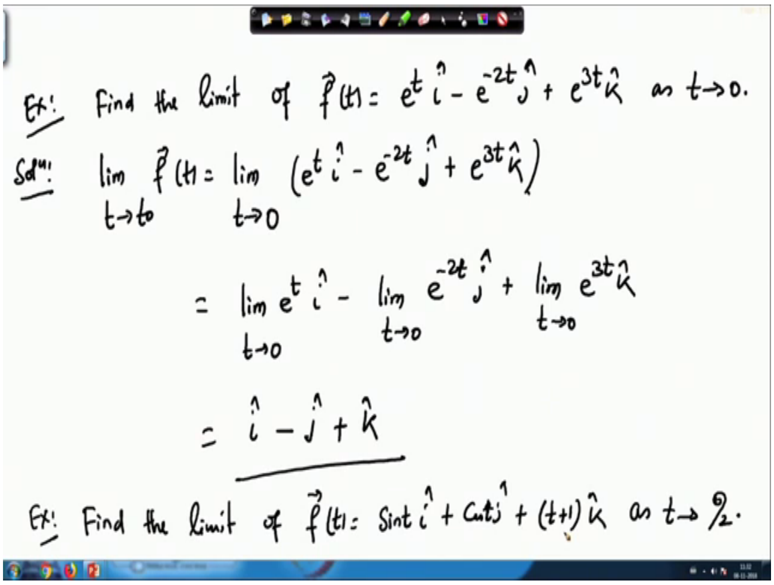

So, we can look into an example. So, find the limit of let us say f t equals to e to the power t i minus e to the power minus 2 t j plus e to the power 3 tk as t goes to 0 all right. So, solution would be we will start like limit t goes to t 0.

So, that t 0 is actually 0 ft. So, let me write limit t goes to 0; f t is basically e to the power t i minus e to the power minus 2 t j plus e to the power 3 t k. So, now, we will pass the limits in individual functions. So, e to the power t i minus limit t goes to 0 e to the power minus 2 t j plus limit t goes to 0 e to the power 3 t right. And now we will pass the limit on individual functions. So, let us pass the limiting the first function. So, when t goes to 0 e to the power 0 will go to 1.

So, i when again t goes to 0 e to the power minus 2 t will go to 1. So, this is e to the power minus 2 t as t goes to 0 is 1 and then here also it will be 1. So, ultimately this thing so, that will be the limit of this vector valued function or vector function not vector valued, but a vector function. So, similarly you can work out some more examples. So, I am I am just writing them, I think it is very straightforward. So, you may try to do them by your by your own. So, for example, find the limit find the limit of let us say f t equals to sin t i plus cos tj plus t plus one k as t goes to pi by 2.

So, when t goes to pi by 2 we will pass the limit. Then this will be 1, this will be 0, this will be pi by 2 plus 1. So, ultimately the answer will be i plus pi by 2 plus 1 times k. So, basically the component of j so, our component of any unit vectors if it is 0, then we do not write that vector. So, we simply avoid that vector that that unit vector basically that or that particular component and we write the remaining 2 components as a plus symbol and it is understood that the component of the missing unit vector is basically 0 and that is why you are not writing it.

So, here in this case at t is equals to pi by 2 this is becoming cos this is becoming cos pi by 2. So, cos pi by 2 is 0. So, basically we do not write the jth component. So, we just write ith and kth component all right and.

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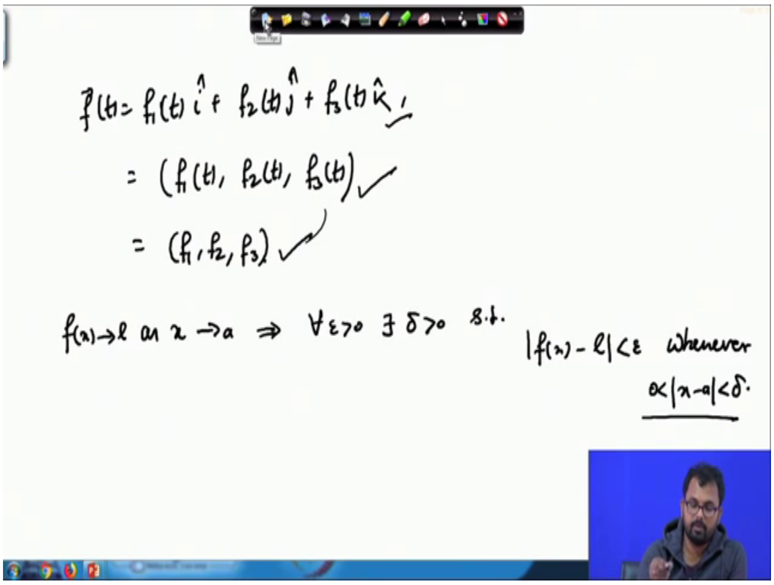

There are some other ways to denote a vector function or to denote a vector quantity. So, so far we have been writing ft equals to f 1 t i f 2 t and j and f 3 t k, but we can also write. So, there are several other ways to so, we can also write f 1 t comma f 2 t comma f 3 t. So, like a point of a point in 3 dimension or we can simply write f 1 f 2 f 3.

So, all these are just equivalent ways to write this vector a vector function in commas basically. So, it is up to you and in several books you may find the first notation, if you may find the second notation or you may find the third notation.

There can be some other notations as well they basically mean this first line here that you have component wise addition of these scalar functions multiplied by this unit vectors i j and k respectively. And also you should not change the order. So, if you write f 1 t f 2 t f

3 t; that means, the component of i or the coefficient of i is f 1 coefficient of j is f 2 and coefficient of k is f 3.

So, do not how to say with the order of this of this notation all right. So, in a function of a scalar variable or in scalar functions, we learnt about limits, but we also learnt about their epsilon delta definition. So, epsilon delta definition is something like for scalar function is if we have f x goes going to l as x going to a, then we usually then this implies that, then we usually used to write that for every epsilon positive there exist a delta positive such that we had fx minus l is less than epsilon whenever 0 less than x minus a less than delta.

So, this was our epsilon delta definition for the function of scalar variable for when f x is going to l as x going to m. Now in case of function of a vector functions basically, we can have the similar epsilon delta definition as well.

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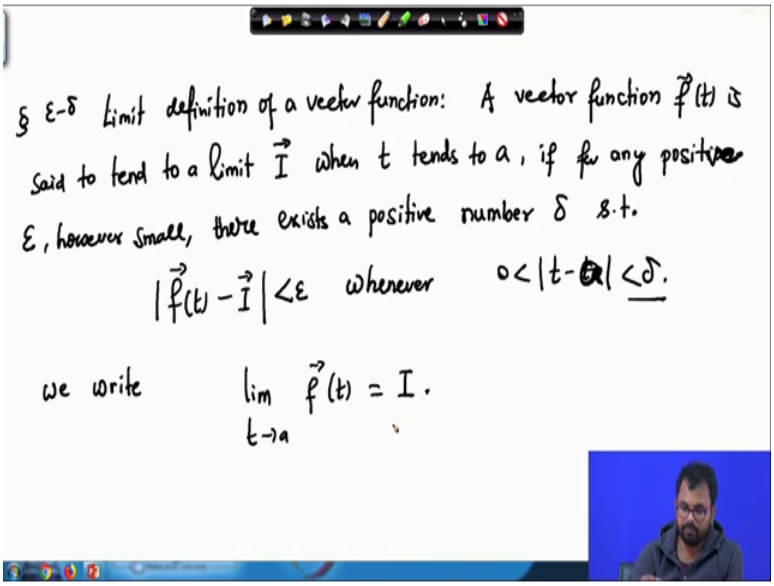

So, I can write epsilon delta definition epsilon delta limit definition epsilon delta limit definition of a vector function all right. So, the definition goes like this a vector function f t is said to tend to a limit I when t tends to a if for any positive epsilon positive epsilon. However, small or arbitrarily small; however, small there exists a delta positive or a positive number delta positive number delta such that such that our f t minus I so, Ii is also vector is less than epsilon whenever 0 less than t minus t 0 or in this case we have a.

So, t minus a is less than delta. So, this is and if this if this is true, then in that case we write simply limit t goes to a f t equals to I. So, it is not l basically it is I. So, this is basically our epsilon delta definition for the limit of a function. So, if we say that the function f has a limit let us say I, then in that case as t tends to it then in that case for every for any positive not position for every positive ok

So, for any positive epsilon; however, small there exist a positive number delta such that f t minus I is less than epsilon whenever t minus l less than delta and we basically write limit t goes to a f t equals to I. So, this is this is how we define the limit of a function using the epsilon delta definition.

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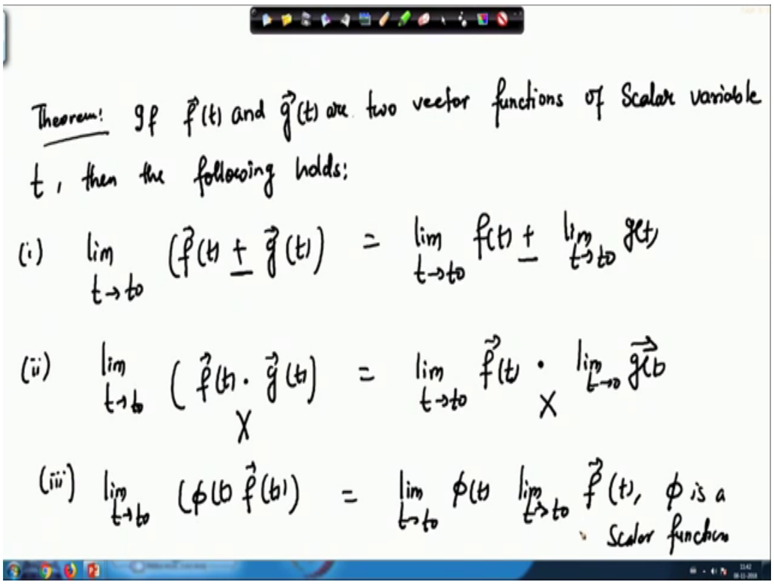

Similarly so, like limit of scalar functions we had several properties. In case of limit of vector functions, we will also have several properties. So, I can just summarize them in a small theorem. So, if f t and g t the two vector functions or are. So, f t and g t are two vector functions of scalar variable of scalar variable say t then the following holds.

So, what are the following? First one is limit t goes to t 0 vector function f t plus vector function g t is equals to limit t goes to t 0 f t plus limit t goes to t 0 g t. So, the limit of the sum is equal to sum of the limit all right. Now similarly if you have a subtraction so, the limit of the difference is equals to difference of the limit. So, this is also true. Now if we have let us say product. So, the product of two vector function which could be scalar product or a dot product, then it will be so, if you have scalar product we simply do not have product in vector functions. We either have scalar product which is also called as dot product or the cross product.

So, if you have let us say a dot product of two vectors functions and their limit will be equal to limit t goes to t 0 f t dot product with limit t goes to t 0 g t. So that means, limit of dot limit of the dot product is equal to dot product of the limits similarly limit of the cross product is equals to cross product of the limit all right and the third and fourth property is basically limit t goes to t 0 phi t times f t is equals to limit t goes to t 0 phi t times limit t goes to t 0 f t where phi is a scalar function; phi is a scalar function.

So; that means, if you have a vector function multiplied by scalar function and if you take the limit, then basically limit of this scalar function times limit of the vector function. So, that is what we get after taking the limit.

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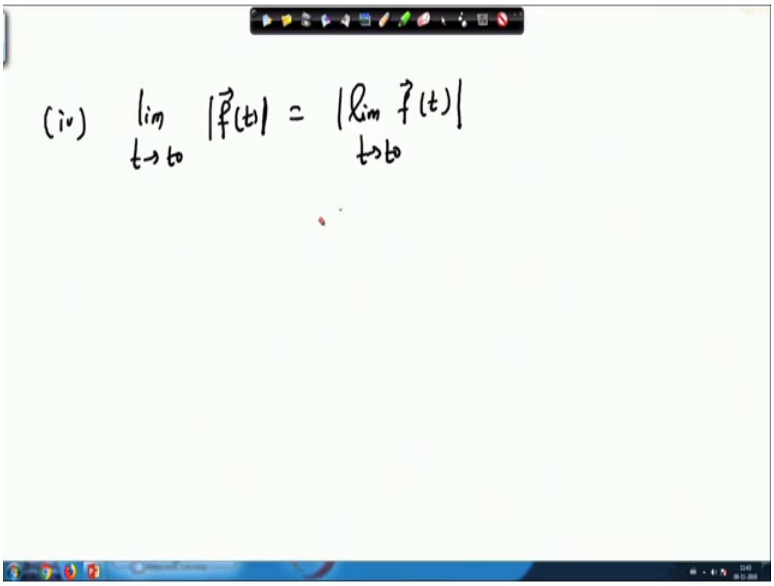

And fourth result is limit of the mode basically is equal to mode of the limit. So, limit of the absolute value is equals to absolute value of the limit. So, whatever we will get after calculation. So, these are the results which also holds for the vector functions and the proof is we will basically prove it using the epsilon delta definition and it follows on the similar arguments for the results of the is based on the results on a scalar function.

So, I am not proving these results and also they are not important from the context of the syllabus. So, these are the results which we will be using as quite often and I mean for the interested readers you can look into any vector calculus book suggested by me and there you might be able to find the proof as well, but these are the results which are valid in case of vector functions as well.

Now in the next class we will start with the continuity of a vector function and probably will also introduce the concept of differentiability of a vector function. So, I will stop here for today and I will see you in the next class.

Thank you.