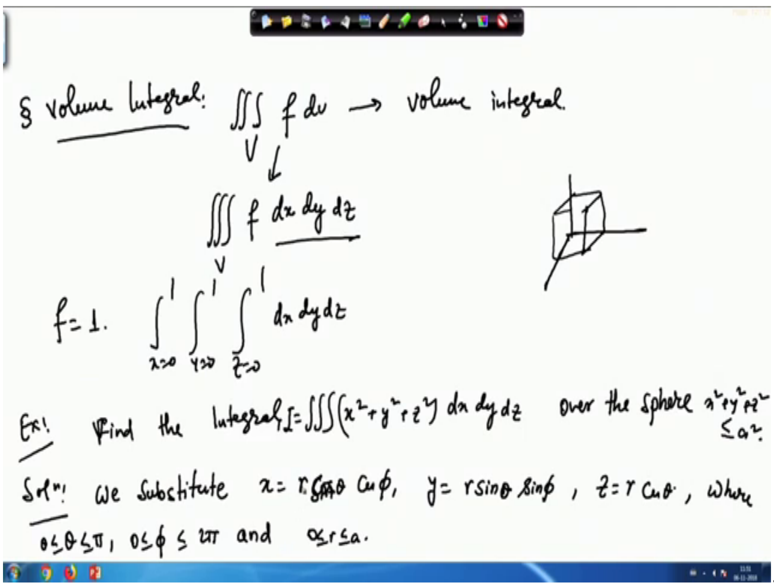
Integral and Vector Calculus
Prof. Hari Shankar Mahato
Department of Mathematics
Indian Institutes of Technology, Kharagpur

Lecture – 31
Volume Integral, Gauss Divergence Theorem

Hello students. So, in the last class we our class before the last class as well we introduced the concepts of surface integral and we also tried to cover as many good examples as possible; where we saw that how you calculate a surface area which is common between two surfaces or where two surfaces are cutting each other at some place and also, the one surface is lying in between two planes. So, we also worked out examples like that.

Now in today's class we will cover our next topic which is basically Volume Integral. So, we started with line integral, then surface integral, area between two curves and today we will start with a volume integral.

(Refer Slide Time: 01:13)



So, what do we mean by volume Integral? So, let us start with. So, volume integrals are basically integral of these types. So, triple integral let us say and I write the volume or the region where we are doing the integration as v and sum f dv. So, this is how we do not our volume integral volume integral. Now, in order to calculate the volume in t in order to calculate these type of integral we I will show that how you can do it. So, first of all we can write this one as integral, triple integral or one can also prefer to write just one integral and then, v and it depending upon how you are writing this dv. So, some people prefer to write it as dx dy dz, some prefer some people write simply dv. So, if you write 3 variables and if you just write one integral, that means that it is a volume integral.

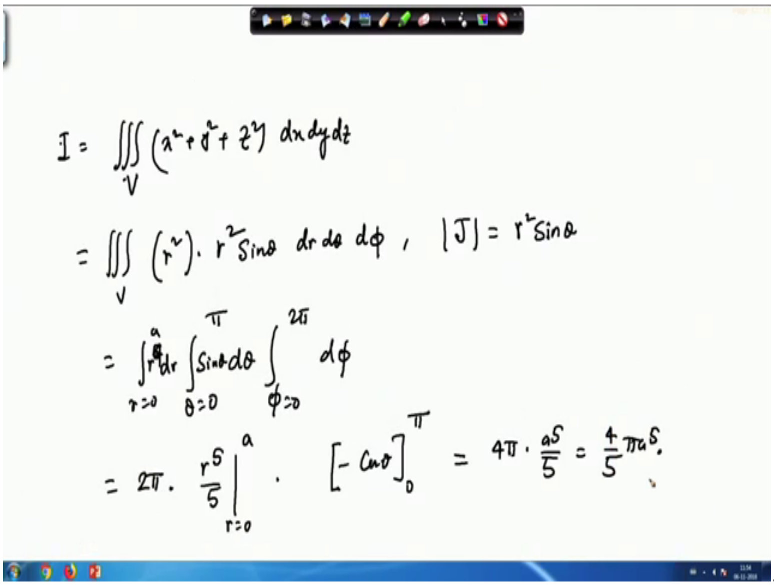
So, basically any triple integral is like a volume integral where you are calculating actually the volume enclosed by that particular surface or even if nothing is given, then basically you are calculating the volume within that range let us say. So, for example if we if our f is equal to let us say 1 and if our volume has limits x equals to x running from 0 to 1 y running from 0 to 1 z running from 0 to 1 dx dy dz, then we are basically calculating the volume of this surface here. So, where we can draw it as something like this x 0 to 1 and then, y is 0 to 1 and then, z 0 to 1, so, basically this cuboid in a way. So, that is one way we mean by what we mean by volume integral.

So, let us consider an actual how to say example where we calculate the volume of a surface. So, first example find or calculate find the integral; find the integral. Let us say I am writing x square plus y square plus z square dx dy dz over the sphere; over the sphere x square plus y square plus z square is less or equal to a square, all right. So, basically we have to calculate this integral here over this surface. So, basically a volume integral of this integrand over this surface. So, of course everything is then how to say nice figure form, we have x square plus y square plus z square less or equal to a square.

So, we can think of it as let us say our circle situation where we had x square plus y square equals to a square and then, in that case we used to substitute x equals to r cos theta y cos to r sin theta and then, we transform the whole integral into an integral with r n theta using Jacobian and other basically change of variables. So, here in this case also we will do some substitution. So, we substitute x equals to r cos theta cos phi because we are in function of 3 variable. We have to have a spherical polar coordinate system and the spherical polar coordinate system involves the r theta and phi, all right.

So, substitute or we substitute x equals to r cos theta cos phi y equals to r sin theta sin phi. Let us take it as sin theta. So, r sin theta cos phi y equals to r sin theta sin phi and z equals to r cos theta. So, where theta would vary from 0 to pi phi would vary from 0 to 2 pi and the r would vary from 0 to a. So, we can see all those things from our spherical polar coordinate system.

(Refer Slide Time: 06:13)



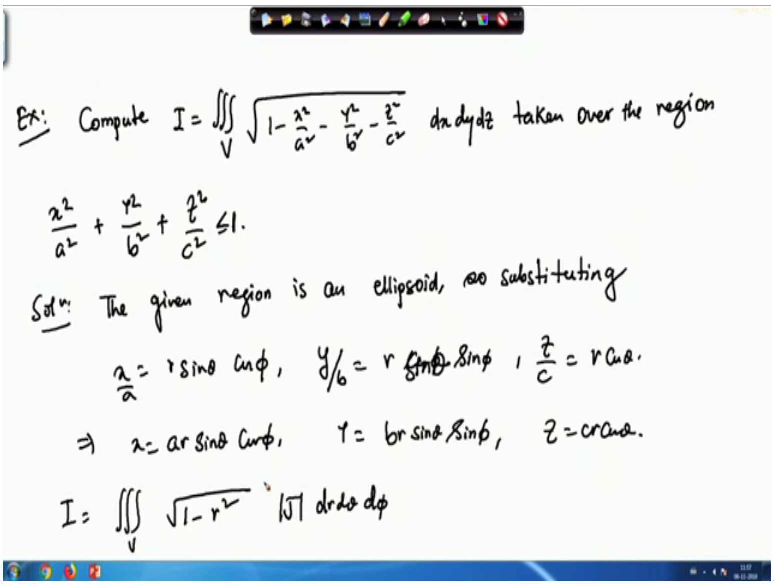
Now, we will start with our integral I. So, our integral I is basically integral over the sphere let us say which is given. So, I simply write as v and then, we have x square plus y square plus z square and here we have dx dy dz and if I change the variables, then in that case it will be a volume integral again, but in terms of r theta and phi and this one will be r square. So, first term will give a, so, if we do the square, then we will take sin squared theta common and then, it will be cos square phi plus sin square phi that will be 1 and then, again r square common.

So, it will be sin square theta plus cos square theta and then, that will again be 1. So, it will be basically r square and if I substitute if I transform dx dy dz into another variable involving r theta and phi, then this will be r square sin theta dr d theta d phi. So, our Jacobian determinant is basically r square sin theta, all right. So, this will become integral r running from 0 to a theta running from 0 to pi and phi running from 0 to 2 pi dr. This is our d theta and we can write our r to the power 5 here and sorry r to the power 4 here and then, sin theta here and then d phi here. So, phi is totally independent of r and theta.

So, we can integrate and put the value phi equals to 2 pi right away our second term is r to the power let us say 4. So, this will become r to the power 5 by 5, r is evaluated at 0, r is evaluated at 1 and sin theta will be minus of cos theta evaluated at 0 and pi. So, cos pi is minus 1 and minus minus plus and minus minus plus cos 0, so, again plus; so, basically 2. So, this is 4 pi a to the power 5 by 5. So, that means 4 by 5 pi a to the power 5. So, this is our required volume integral, our triple integral. So, here in this case basically the given equation was a sphere, the given equation of the wall of the surface was just sphere.

So, it is fairly easy to come up with this substitution and then, just substitute the substitute for the variable x and x y and z and use some Jacobian that will give us the required answer as this one, all right. We can consider another example. So, another example could be let me find an interesting example which is yes.

(Refer Slide Time: 09:23)



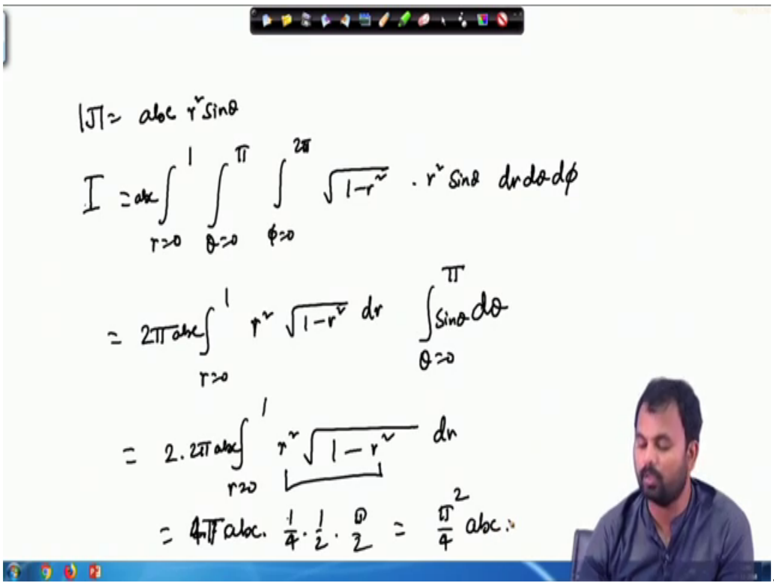
So, now another example is compute I equals two triple integral or volume integral let us write it in this way 1 minus x square by a square minus y square by b square minus z square by c square dx dy dz taken over the region all right. So, in this case we have the ellipsoid basically. So, previously we had a sphere and in this case we have that ellipsoid. So, what we will have is the given region is an ellipsoid.

So, substituting x equals to x by equals to r sin theta cos phi y equals y by b equals to r cos theta r sin theta sin phi sin theta sin phi and z by c equals to r cos theta. So, basically if I substitute if we can see it as a circle as a sphere as well. So, instead of considering x square by a square if I consider x by a as a variable let us say, then in that case x by a is a variable u as the y by b is a variable v and z by c is a variable w, then it is basically u square plus v square plus w square is less or equal to 1. So, then again it says sphere.

So, that is what we are doing. We are taking the substitution for x by a not x and similarly for y by b and z by c. So, now, I can write it as x equals to a r sin theta cos phi y equals to b r sin theta sin phi and z equals to c r cos theta, right. So, now I will do the change a variable. So, here we will have the volume integral v and if I do the change a variable, then I mean this whole thing will reduce to basically 1 minus r square because we will be having this a cos square plus sin square formula and everything will keep getting reduced to 1 and dx dy dz.

So, here we will have our Jacobian determinant and then dr d theta d phi. So, the Jacobian determinant can be given by abc r square sin theta; this we can calculate easily.

(Refer Slide Time: 12:37)



So, Jacobian determinant can be calculated easily and this is abc r square sin theta. So, I will substitute all these values in our volume integral, right. So, let us substitute all these values in our volume integral I am calling it as I and the integral r running from 0 to 1 because then in that case our circle is of unit radius. Sorry sphere is of unit radius and then our theta will be 0 to pi and phi will be 0 to 2 pi; this will be 1 minus r square times r square sin theta dr d theta d phi. Now, we can have this a separate.

So, we keep r integrals at one place, theta integrals in one place and phi integrals are one place. So, since there are no variables involving phi in the integrands, it will be simply 2 pi because then it is fine and phi evaluated at 0 and 2 pi. So, just 2 pi and this will be r running from 0 to 1 r square times 1 minus r square dr and this will be theta running from 0 to pi d theta sin theta here and then, this will be cos theta minus of cos theta and just like previous example my cos pi and minus cos ultimately this will reduce to two times 2 pi r running from 0 to 1 r square 1 minus r square dr.

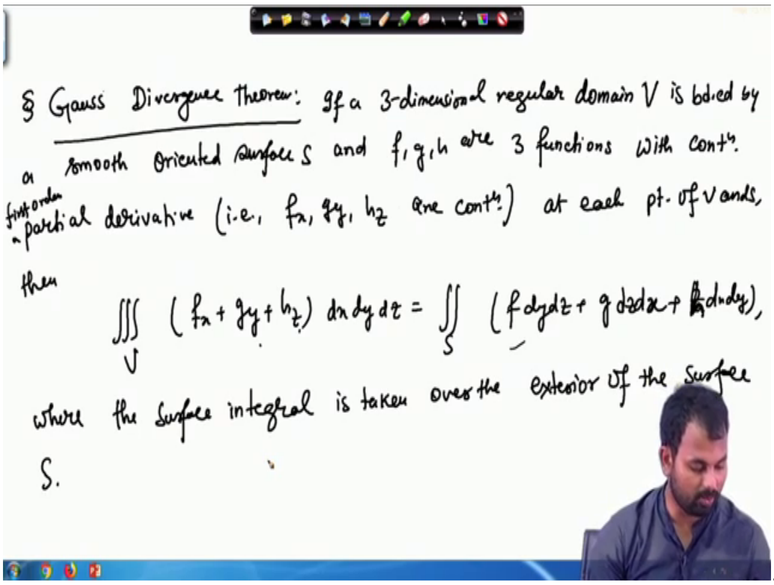
So, now calculating this integral is fairly simple. So, we just have to use some kind of substitution. So, we substitute r is equals to sin squared t and then, just transform this limits and ultimately we will be able to obtain, so, its not very complicated. So, 4 pi times where is our abc. So, this is abc, right. So, let us let write our abc, so, abc. So, 4 pi abc times we will have 1 by 4 times 1 by 2 times pi by 2. So, ultimately pi by 4 square abc. So, this is the required volume integral of this integrand here taken over the region given by this ellipsoid.

So, this is how we basically calculate the volume integral. So, volume integral is not that complicated in a way because if someone has worked out a line integral and area between the curve or surface integral, then volume integral is basically very how to say small generalization. It is not too complicated that is maybe small is not the right word, but its not very complicated once you are perfect with the area between the curves or area between different surfaces.

Now in volume integral, there is a very important theorem. Of course one can view it more from the vector calculus point of view, but it is we can also see this theorem from integral calculus point of view and that is called a Gauss Divergence theorem.

So, Gauss Divergence theorem gives us a tool to transform any surface integral into a volume integral. So, if you are given a surface integral which may appear to you a little bit complicated, then instead of solving the surface integral if it is in certain particular form if it is in a certain form, then you can transform the surface integral into a volume integral. So, how we can do that let us see in our statement.

(Refer Slide Time: 16:15)



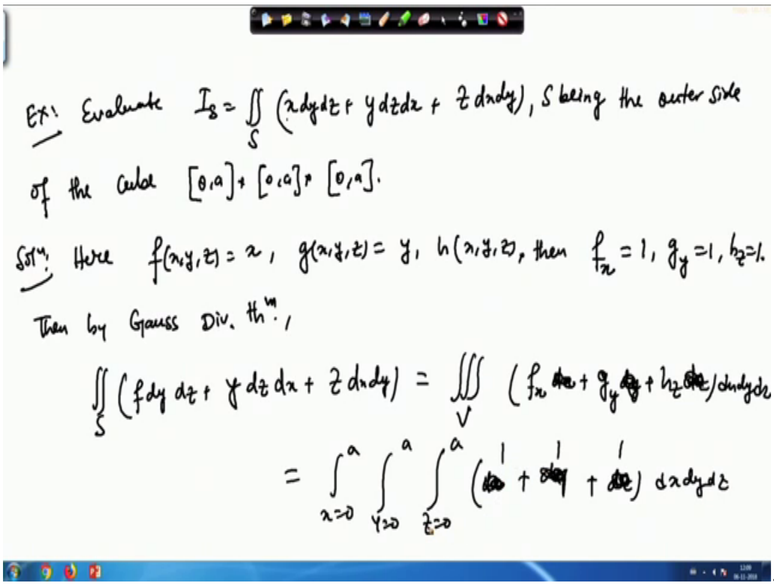
So, Gauss divergence theorem; so, the Gauss divergence theorem, so it says that if a 3 dimensional a regular domain 3 dimensional regular domain let us call it as V is bounded by a smooth orientated, it is a oriented surface lets us say S and f g h are 3 functions are 3 functions with continuous partial derivatives. That means, fx that is fx g y and hz are continuous, alright.

So, the first order with continuous maybe first order it should be written first order first order partial derivatives at each point of v and s, then our volume integral fx plus gy plus h z. It means that f is a function of whatever if it can be a function of x and y, but partial derivative of f with respect to x, partial derivative of g with respect to y and partial derivative of h with respect to z dx dy dz is equal to surface integral f dy dz g dz dx plus z dx dy where, the surface integral; where the surface integral is taken over the exterior of the surface as where the surface integral is taken over the exterior of the surfaces. That means, the normal or basically on the outer side of this surface and if that.

So, this integral basically this Gauss Divergence theorem basically gives you a tool that can help you transform a surface integral into a volume integral. So, we have to make sure that these functions fg and z fg and h naught z f g and h they are they have a continuous partial derivative and if they do then in that case you see you have a function f, but if you if they have continuous partial derivative just looking at them, you can be able to understand whether they have continuous partial derivatives or not its very simple. So, even if they involve lot of complicated algebraic expressions when you take fx, then it is actually always one derivative one powerless.

So, it could be a rather simpler expression to deal with then dealing with this surface integral, right. So, with the help of Gauss Divergence theorem we can always be able to transform the surface integral into a volume integral and most probably I mean in most of the cases it happens that this volume integral is very simple to evaluate compared to this surface integral. So, this is the only theorem from vector calculus in a way we will learn in our integral calculus part and we will work out four examples to see its application that whether actually a given surface integral can be handled using a Gauss Divergence theorem or not.

(Refer Slide Time: 21:03)



So, let us start with our first example. So, evaluate let us say I surface is equals to integral over the surface s x dy dz plus y dz dx plus z dx dy s being the outer side of the cube 0 to a times 0 to a times 0 to a. So, that is where the surface integral is taking place on the outer side. Now, of course it may not be a very complicated expression. So, here we have x y and z and we can just integrate how to say I need surfaces and then, that will give us the surface integral, but at the end of the day we have to integrate on each surfaces. Ultimately it will become a very big example or big calculation to do.

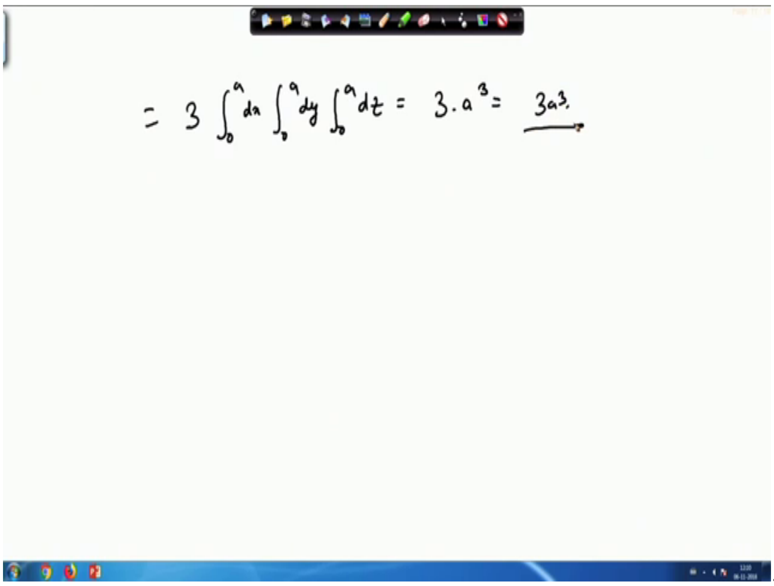
So, what we can do here, we will apply Gauss Divergence theorem because obviously x y and z are continuous functions and they also have continuous first order partial derivatives. So, these fy and f g and h have continuous partial derivatives. So, I can write here f x y z as x g x y z as y and h x y z. So, they can be function of x y z as well here in this case we are lucky that they are only functions of x y and z respectively here.

That is what we have. Then, our partial derivative of f with respect to x will be 1, partial derivative of g with respect to y will also be 1 and partial derivative of h with respect to z is also 1, then by Gauss Divergence theorem then by gauss divergence theorem we will have surface integral f dy dz y dz dx and z dx dy is equals to, then surface integral will be converted into volume integral f dx y g y dy and h z dz.

So, now volume integral x is running from 0 to a, y is running from 0 to a and z is running from 0 to a. This will be dx because fx is 1, this will be dy because gy is 1 and this will be dz because h z is 1. So, now we will integrate how to say first with respect to. So, we will basically write here d x. So, that will go into each basically individual integral and this will then reduce to sorry so, this is not dx dy. So, fx dx and then, dx dy dz is here yes. So, that is I was wondering, so, why it is looking that way?

So, we have dx dy dz and we have 1 here, 1 here, 1 here.

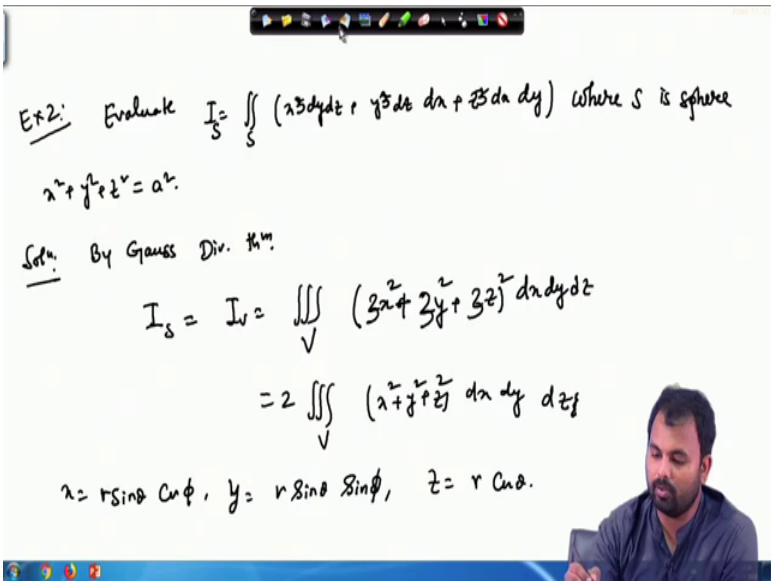
(Refer Slide Time: 24:39)



So, this will ultimately be 3 times integral from 0 to a dx 0 to a dy and 0 to a d z. So, ultimately we will obtain a square a a a. So, 3 times a, that means a cube so, 3 a cube. So, this is the required volume integral. So, here yes the formula is correct. So, here you can see we could have calculated the surface integral is just that s being the outer surface of the cube. So, we have to go on every surface one by one, one by one, so, there could be a possibility that we may have to evaluate 8 types of surface integral.

So, instead of doing that we just use Gauss Divergence theorem because our functions are nicely behaving and we can calculate the volume integral and the answer of which is as same as the answer of the surface integral. So, as you can see it is proven to be a very efficient tool in order to handle these kind of integrals. We can consider one more example.

(Refer Slide Time: 25:45)

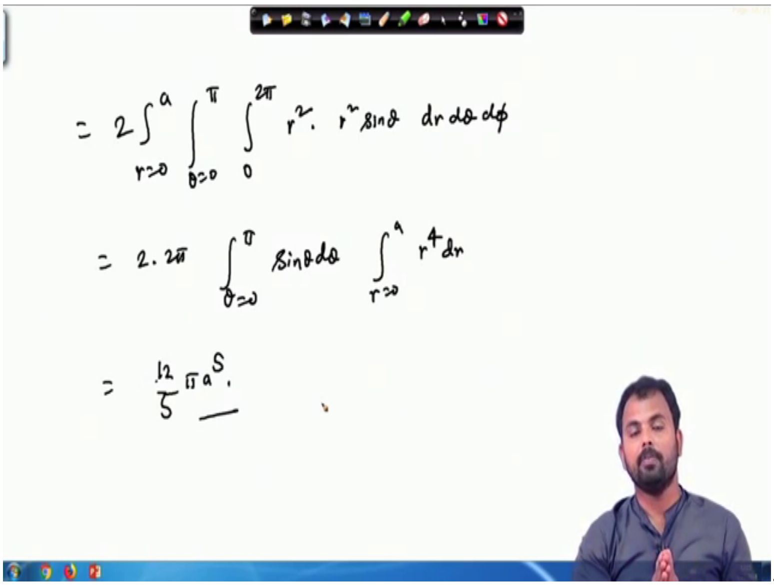


And let us consider another example. So, evaluate I equals two surface integral x square dy dz y square dz dx and z square dx dy, where S is the sphere. Our surface is basically as sphere x square plus y square plus z square is equals to a square.

So, obviously here we can do some substitution, but before that if we want to evaluate the surface integral over the surface, it will certainly be little quite complicated. So, instead of doing that what we can do is we can use Gauss Divergence theorem. So, by Gauss Divergence theorem obviously we can see that this is our f, this is our g and this is our h. So, since they are all algebraic or quadratic function, they will have continuous first order partial derivatives and if we use the Gauss Divergence theorem, then this I surface will be reduced to I v an I v is basically surface integral over the volume v fx. So, that means 2 x n plus fy.

So, that gy that means 2 y and then hz that means 2 z and then, here we will have dx dy dz. So, basically two common integral volume integral x plus y plus z dx dy d z, all right and now we basically use that substitution x equals to r sin theta cos phi y equals to r sin theta sin phi and z equals to r cos theta. So, then this integral will reduce to then this integral will reduce to two volume integrals.

(Refer Slide Time: 28:03)



So, this will reduce to r running from 0 to a, theta running from 0 to pi and phi running from 0 to 2 pi. This thing will reduce to r sin theta cos theta or we can modify this example a little bit, so that just to make things a little bit easier. So, this will be 3 3 3 x square y square z square; x square y square z square. So, then in that case we will have r square sin square theta plus times cos square phi plus sin square phi. So, ultimately it will reduce to a very nice simple formula.

So, we can do that and then, this will be 0 to 2 pi, then this is r square times the value of the Jacobian will be r square sin theta and we will have dr d theta d phi. So, now we can put d phi here, so, this will be 2 pi and we can have theta variable at one place, theta integral at one place and we will have r integral at one place.

So, ultimately we can evaluate this whole thing and it will reduce to 12 by 5 pi a to the power 5. So, here you can see just looking at the integral. So, x cube y cube and z cube just for the simplicity we have considered x cube y cube and z cube. So, just looking at the integral we can realize that it could be a little bit complicated to do the surface integral.

However, if we use the volume integral Gauss Divergence theorem, then this whole thing will reduce to a very simple integral where we just use some method of substitution; where we just use some method of substitution or change of variables to obtain a rather simpler integral and this is very easy to calculate.

So, there could be a possibility that this surface integral may have lead to a complicated calculation, but just by the help of Gauss Divergence theorem we finish the problem in like 5 or 6 steps. So, the Gauss Divergence theorem is proven to be a very important tool in integral calculus and we will also see in vector calculus that how with the help of Gauss Divergence theorem we can reduce a surface integral to a volume integral more from the vector calculus point of view.

So, I will stop here for today and in the next class, we will continue with our next topic and I will try to include some examples based on this Gauss Divergence and you know in your assignments sheet and I look forward to your next class.

Thank you.