Integral and Vector Calculus
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Lecture – 30
Surface Integral (Contd.)

Hello students. So, in the last class we were looking into the Concepts of Surface Integral and we also started with one two examples. Since, it is one of the important topics in integral calculus, we will practice maybe a couple of more examples on surface integral and then we move to the volume integral.

Of course, surface integral has two aspects basically. So, one can look to them from the integral calculus point of view and also from the vector calculus point of view. So, we will also work out a few surface integrals when we look into vector calculus part, but right now we are more analyzing it from the integral calculus point of view.

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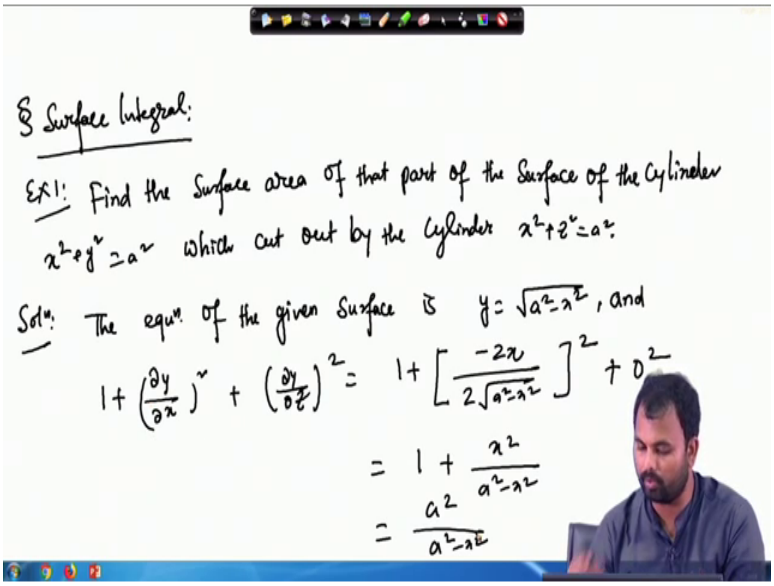

So, let us start with our first example for today. So, our first example states that. So, example 1, so find the surface area, so find the surface area of that part of the of that part of the surface, that part of the surface of the cylinder of the cylinder x square plus y square equals to a square which is cut out by the cylinder x square plus z square equals to a square.

So, basically we have to find out the surface area of the first cylinder of the first cylinder cut off by the second cylinder. So, the first cylinder has the base in x y plane and the second cylinder has the base in x z plane. So, of course, one cylinder is like this, another one is like this, so they will intersect at the they will intersect somewhere and we have to find out the area which is basically how to say intersected or cut off by these two cylinders.

So, this is our given cylinder; so, the given the given surface is basically this one here. So, let us write the equation of the given surface is y equals to square root of a square minus x square. So, I take x square on the other side and we can calculate. So, the surface is basically this the base line in x y plane. So, basically we will we will do del y del x and del y del z square. So, del y del x will be square root of 2 square root of a square minus x square, this one is minus of 2 x plus 0 square. So, this will be 1 plus x square by a square minus x square. So, this will ultimately end a square by a square minus x square. So, we will get this value equals to this thing.

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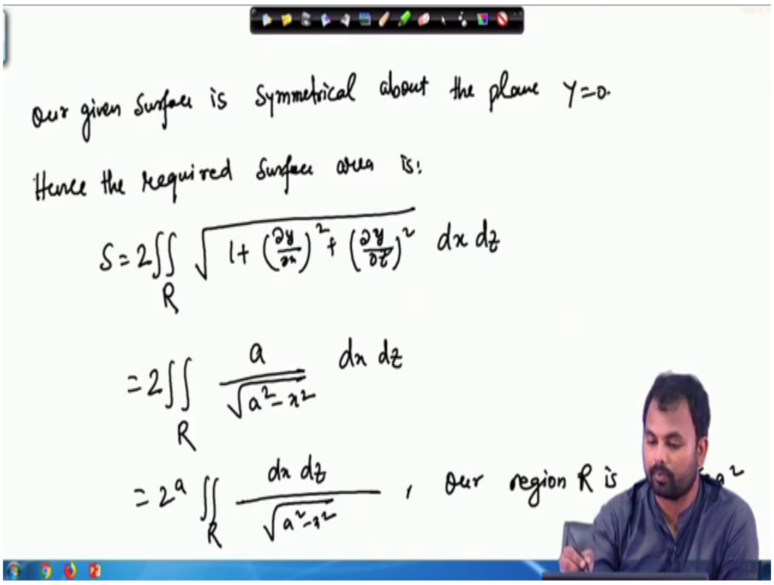

Now, we see that the given surface, our given surface is symmetrical is symmetrical about why about the plane y equals to 0. So, of course, it is symmetrical because we have x square plus y equals to y square equals to a square. So, it does not matter whatever value we put for x, we will get always how to say the same value for y and that means, that the cylinder is lying on the both half of the plane y equals to 0. So, it is symmetrical on the both sides of the plane y equals to 0 that means, along x axis, along x z actually. So, along the plane basically y equals to 0.

And now, that we have the symmetrical part the required surface area, hence the required surface area, so we will take the projection on x z plane, all right surface area is S equals to integral over the region R. So, R is the surface and then we have 1 plus square root of del y del x whole square plus del z del x sorry del y del z whole square dx dz. So, when we take the projection on x z plane.

Now we will have here a 2, because this the plane is symmetrical along the y. So, we have to just calculate the surface area above the x z plane and just multiplied by 2 and that will give us the whole surface area because of this symmetricity condition symmetric condition. So, we have 2 here and if I take the square root so over the region R I will have a and then in the denominator, I will have a square minus x square dx dz.

Now, the projection on the x z plane will be projection on the x z plane for this for this for this cylinder here is this one actually and therefore, we can write 2 a integral over the region R dx dz divided by a square minus x square and our region R is we will just write the equation. Region or surface whatever you would like to call. So, our surface R or region R is this one.

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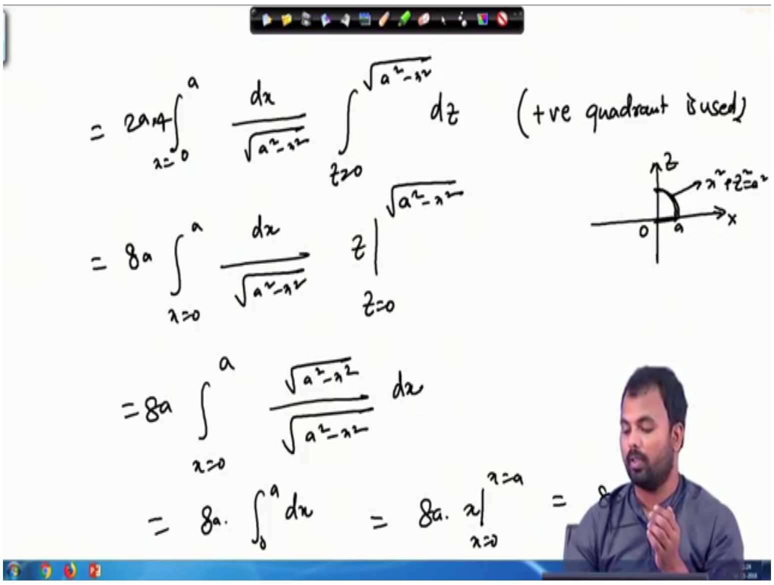

And this can be written as integral from 0 to 2 a 0 to a dx by a square minus x square and range for z will be 0 to a square minus x square dz, positive quadrant will be used or is used and then we put a 4 here basically. So, we put a 4 here. Because now the projection is taken in the x z plane, and x z plane what will happen is it is basically a circle and the surfaces since circle is in a way symmetrical along x and z axis then in that case we do not have to calculate the how to say the surface area along the whole circle we just take one of the one of the quadrants and then multiplied by 4. So, that is what we are doing.

We, instead of calculating over the whole x square plus z square equals to a square we just calculate on the first quadrant and then we multiply the whole thing by 4 and that will give you the whole surface area. So, the this is also convenient because now we are only focused in the first quadrant and in the first quadrant if we draw the curve if you draw if you draw the curve like this, so this is our first quadrant and that is the equation of this circle. Let us say, so x plane and z plane.

So, in the first quadrant our a is varying from 0 to a. So, this is our x limit and then z will vary from 0 to a square minus x square. So, z will vary from 0 to a square minus x square. So, that is what we have written as the limit of z. So, now, this will result in 8a integral x running from 0 to a dx divided by a square minus x square and then here we will have z evaluated from at 0 and evaluated at a square minus x square.

So, this will become 8a x running from 0 to a, a square minus x square in the numerator, and a square minus x square in the denominator times dx. So, both will cancel out and then it will become 8a times x dx sorry only dx and then this will be 8a x evaluated at x equals to 0 and x equals to a. So, this will be basically 8a square.

So, here we had to pay a very very close attention that that the two cylinders, I mean first of all the we have to look at the equation of the cylinder along which plane it is symmetric, and then we just have to calculate the surface area on the upper half and multiplied by 2 that will give you the area surface area of the whole cylinder.

Now, it is being intersected or it is being cut off by some other cylinder. So, depending on what type of equation that cylinder has we can use the projection in that plane and even in that plane here we have use the property that its basically a circle and then we do not have to calculate along the whole circle in a way or disk in a way. We just have to look into our first quadrant and then multiplied by 4 that is what we are doing here. And this is this is your required answer. So, this is what we are doing here and that is why required answer.

So, this is how we calculate the surface area in this problem. We will work out a similar problem like that, before we move on to our next topic.

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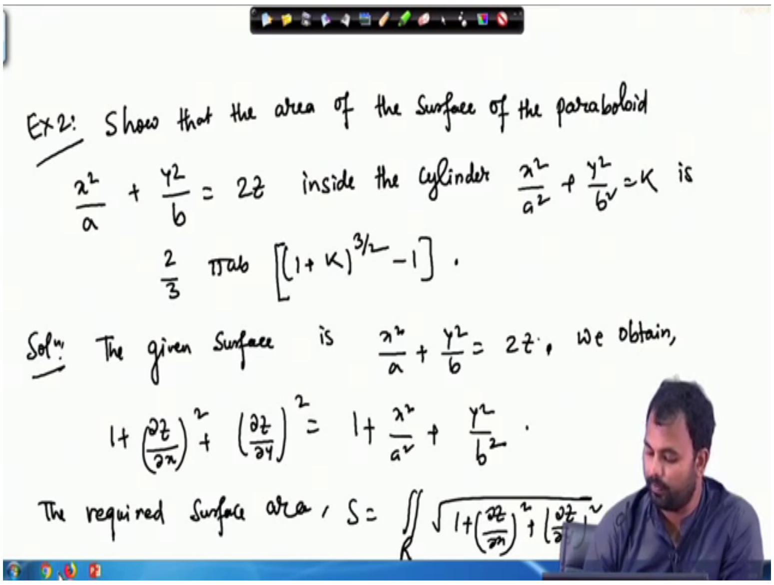

So, let us start with our next example. So, our next example is, so our next example is show that the area of the surface of the paraboloid x square by a plus y square by b equals to 2 z inside the cylinder, x square by a square plus y square by b square equals to k, k is 2 by 3 pi ab, 1 plus k whole to the power 3 by 2 minus 1.

So, instead of having two regular cylinders we have one of them as paraboloid and it is being how to say in a way it is cut off or it is inside that cylinder given by this equation. So, here the equation of the cylinder the base basically lies in x y plane and based on that first of all the given surface, we can write the given surface is x square by a plus y square by b equals to 2 z.

So, from here we obtain the given surface is this, full stop. We obtain del z del x, so let us write the whole thing. So, del z del x del z del y. Now, from here if I calculate del z del x and del z del y then this will be 1 plus 2 x by a and then 1 2 will get cancelled. So, basically x square by a square and then this will be y square by b square, right, yes. So, this is what we obtain here.

Now, the required surface area, so the required surface area since our ellipse lies in x y plane we take the projection on x y plane. So, the required surface area S can be written as, so R denotes our surface this one and we can write 1 plus del z del x plus del z del y dx dy, all right.

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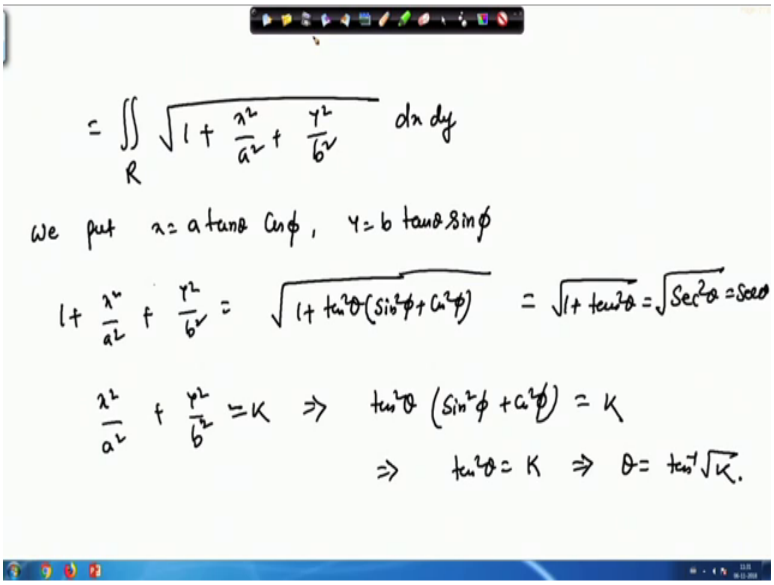

So, now let us substitute the values here. So, this will be over the region R, 1 plus x square by a square plus y square by v square dx dy, all right.

So, now, what we can do in first of all instead of how to say calculating this integral in x and y variable, what we will do we will do some change of variables here. And, if we do the change of variables basically doing some method of substitution we can be able to reduce this whole thing into a rather simpler integral. So, how we can do that? We can substitute, so we put x equals to a tan theta cos phi and y equals to b tan theta sin phi.

Since, we have integral of two variable we have to assume the substitution in two variables as well and from here I can calculate I can calculate our dx dy. So, those things can be calculated easily. Now, what we can do? We will basically see that 1 plus x square by a square plus y square by b square is equals to 1 plus tan square theta will be common we will have sin square phi plus cos square phi and this will reduce to the square root of 1 plus tan square theta. So, this is basically sec square theta so that means, sec theta, all right. So, this is our sec theta.

And x square by a square and our surface and our surface x square by a square plus y square by b square equals to k will transform to tan square theta would get common and then we will have sin square phi plus cos square phi. So, this is our given surface, due to this change of variable the surface will get also changed in that variable and therefore, this will be tan square theta equals to k. And from here we will obtain theta equals to tan inverse square root of k, all right. So, we have got the value of theta.

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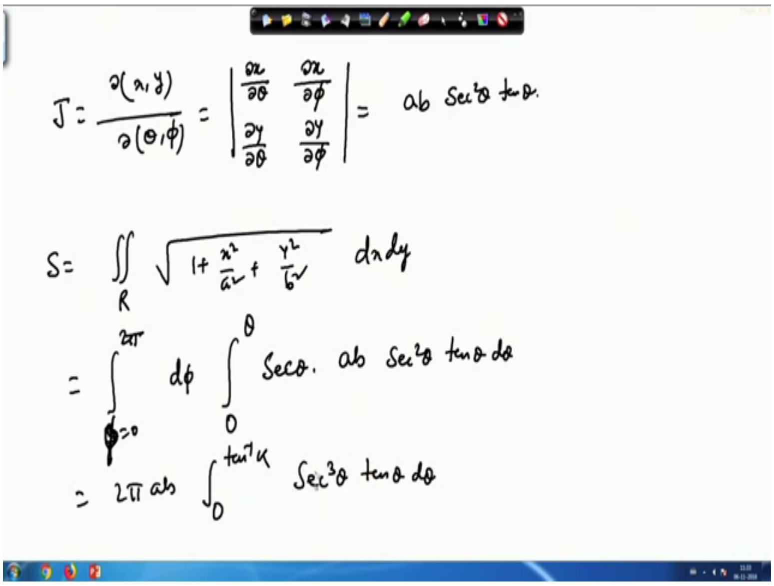

And now, if we do the change of variable then we also have to calculate the Jacobian you remember. From the change of variable in our how to say change of variables part basically, where if we wanted to change the variable we can change it, but we also have to introduce a Jacobian.

So, we can calculate the Jacobian and Jacobian is basically del x y del theta phi. So, this is basically del x del theta, del x del phi, del y del theta, del y del phi. So, if we calculate this whole thing then it will be a v sec square theta tan theta, all right. So, we have got our Jacobian we have got our how to say this integrand as sec theta tan as sec theta and we have also got our theta. So, now, let us go back to our required surface integral.

So, this is this was our required surface integral and we have basically integral over the region R 1 plus x square by a square plus y square by b square times dx dy. So, now we are doing the change of variable in x y plane. So, in x y plane our theta would vary from 0 to 2 pi and this the phi will vary from 0 to 2 pi. So, this is the limit for the phi and our theta would vary from 0 to theta, sec theta times ab sec square theta tan theta d theta. So, our limit for theta is basically 0 to theta, all right.

So, now, here what we can do when we integrate this one. So, this will become 2 ab. So, this will become pi basically 2 pi ab because this will be just phi and phi at 0 and phi at 2 pi. So, that will give us 2 pi and then I am substituting this a and I am putting this ab from here. And now, this is integral from 0 to theta is basically tan inverse k; we will have sec cube theta, tan theta, d theta.

And here we will do some method of substitution, so if I substitute z equals to say tan theta then it will sec square theta d theta or if we substitute say z equals to sec theta then it basically the tan theta. So, by doing some method of substitution this integral would reduce to basically integral sec square theta evaluated at 0 to tan inverse k.

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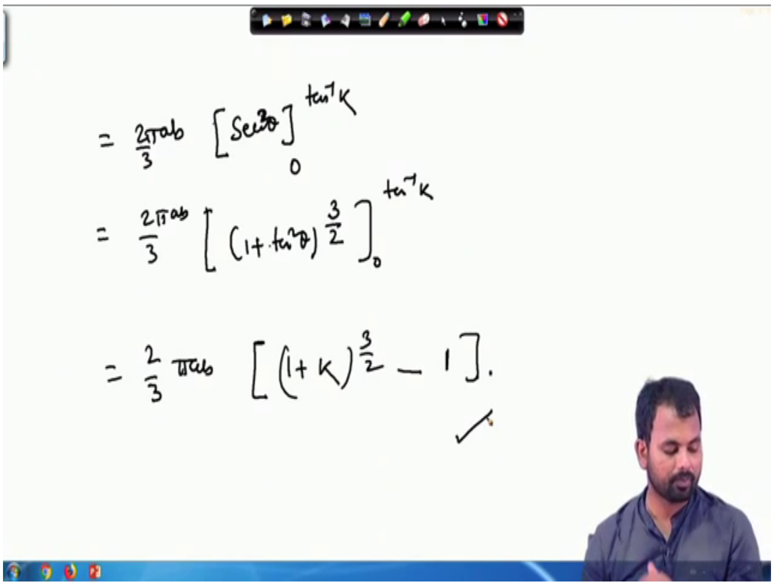

So, from here to here is basically method of substitution. So, we will keep sec theta tan theta together and then we substitute sec theta equals to z. So, that will give us sec theta tan theta and then this whole thing will reduce to just d of sec square theta and then that will give us the required integral.

Now, here I can substitute 2 pi, ab of course, there will be 1 by 2, 1 by 3 and this will become this will become basically and of course, this will be 3. So, this will become basically 2 by 3 pi a b sec square theta a sec square theta can be written as tan square theta and then whole to the power 3. And then if I write sec theta as 1 plus tan square theta then I have to adjust a 3 by 2 here, yes.

So, this is tan inverse k and now I am substituting the value. So, 2 by 3 pi ab and if I substitute the value then it will be 1 plus tan square theta. So, I am substituting theta equals to tan inverse k. So, this can be written as 1 plus k whole to the power 3 by 2 and when theta is 0, then this whole thing is 0 so minus 1. So, this is the, this will be our required, required answer. So, here it will be tan theta tan square, yes; so 1 plus k whole to the power 3 by 2 minus 1. So, this is what we needed to prove.

The calculation is little bit tricky and but what I am doing here in the calculation is not is not relevant to the complexity of the surface integral. So, this is basically for you to practice how you can do such kind of calculation. So, when you see a complicated integral like this all you have to do is the think of some alternative way. Here in this case we have used method of change of variables.

So, when we change the variables we basically get our variables change and then the Jacobian. So, this is our Jacobian which I have substituted here that is the integrand and then we have d theta d phi where phi limit is 0 to 2 pi and theta limit is 0 to tan inverse k. And then we have done some simple calculation which you can do on your own time.

So, in this surface integral part we have seen certain type of how to say integrals where your one surface is been cut out by another surface. And then surface area which is common between these two surfaces needs to be calculated. So, it is like in our plane curve region where we had one curve is being cut off by another curve, and then we have to calculate the common area bounded by these two curves or enclosed by these two curves. So, its smaller less the same thing but in a higher dimension. So, kind of like a generalization of those problems.

Now, we can work out one last example because this is indeed an interesting topic.

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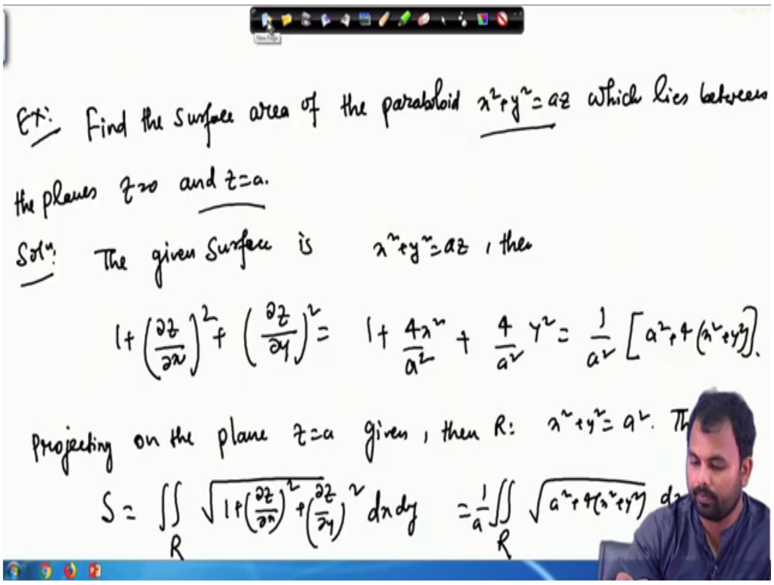

So, let us let us work out our very last example on the surface integral section. So, find the surface area of the paraboloid x square plus y square equals to az which lies between the planes between the planes z equals to 0 and z equals to a. So, of course, the plane is lying is lying in z equals, so the plane is given by z equals to 0 and z equals to a.

So, that is, those are the two points where those are the two planes basically in between we have our parabola then we have to calculate that surface area lying between these two planes. And to do that basically we will have the given surface, yes. So, z equals to 0, z equals to is basically a plane in x y plane and we will basically take the projection in x y plane actually. So, let us do that.

So, the given surface is x square plus y square equals to az, then our 1 plus del z del x, del z del y is equals to 1 plus. So, if I do del z del x then this will be 4 x square by a square and then this one will be 4 a square by y square. So, if I take 1 by a square common and this will be a square plus 4 times x square plus y square, all right. So, this here.

Now projecting, now projecting on the plane z equals to a gives. So, if I project if I project this here on the plane z equals to a, then in that case then our region R will be x square plus y square equals to a square because if you project the parabola like this then on x y plane it will be basically a circle, all right.

So, that is the circle we will obtain and thus our surface area will be, integral over the region R we are taking the projection on x y plane. So, always look for the equation. What type of equation is given? Whether it is given in x z variable or y z variable or x y variable or even if the equation is given in terms of planes then in what sense they are meaning and based on that we have to decide or projection. So, here z equals to 0, z equals to a or basically equation of x y plane and therefore, we have to take the projection in x y plane, all right.

So, del z del y, dx dy and this will be integral 1 by a, integral over the region R, it will be a square plus 4 x square plus y square dx dy, all right.

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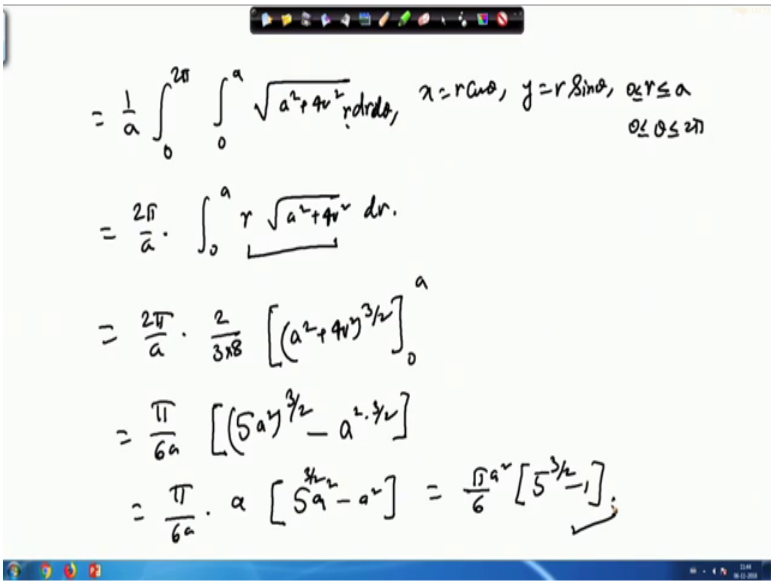

Now, this can be written as 1 by a integral from 0 to 2 pi integral from 0 to a we will have a square plus 4 r square, where x equals to r cos theta and y equals to r sin theta, where r is between 0 to a and theta is between 0 to 2 pi.

Because in this case instead of focusing on one quadrant we can how to say rotate our theta from 0 to 2 pi and that will give us the entire surface area over the entire circle and this will be basically dr, d theta here and since we also have to incorporate a Jacobian then it will be r here.

So, now this is our required changed change a variable its how to say changed integral. So, we have basically changed the variables using x equals to r cos theta y equals to r sin theta. So, this is the integrand, this is the integrand, that is your Jacobian, that is you how to say dr, d theta.

Now, if we perform then basically we take d theta here, so this will be 2 pi by a and this will be integral from 0 to a r a square plus 4 r square dr. And to calculate this integral we just use we substitute r square equals to some z, then this will be 2 r d in dr equals to some dz and then we will use some integral calculus formula.

So, calculating this integral is not difficult. You can do that by your own. So, ultimately you will obtain 2 pi by a, 2 by 3 times 8 then this will be a square plus 4 R square whole to the power 3 by 2 integral from 0 to a. So, this will be 4 and then to this. So, this will be pi by 6 a. And here we will have 5, so when a 0 so this will be 5a square whole to the power 3 by 2 minus a square whole to the power 3 by 2.

So, ultimately, we will have pi by 6 a, a common. So, we will have 5a square and 5a square minus a square and 5 to the power 3 by 2 and one is getting cancelled. So, we will obtain pi by 6 basically. So, we will have pi by 6 a square and this will be 5 to the power 3 by 2 minus 1. So, the simplification is up to you. So, how you do the simplification?

Now, this is this is one another interesting example where instead of having it cut off by some other surface we just had two planes in between we have our paraboloid and then we needed to calculate the surface area. So, these are some problems which you can encounter in surface integral part. And I hope I try to cover as many examples as I could. I will also include some examples in your assignment sheet and hopefully you will be able to work on them.

So, I thank you for your attention for today. And, I will see you in the next class.