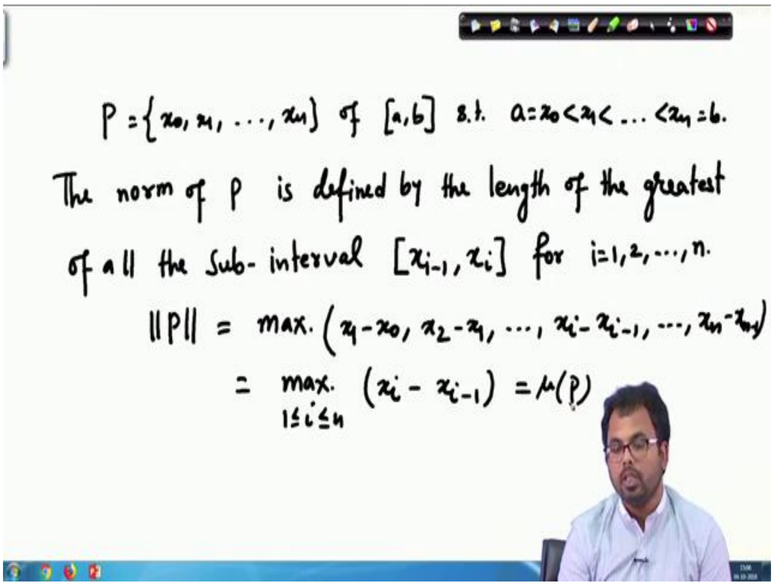
Integral and Vector Calculus
Prof. Hari Shankar Mahato
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 03
Condition of integrability

Hello students. So up until last class we saw the definitions of partition of a set and we also looked into the in how to say, Riemann integration and Riemann integrable functions, upper limit, lower limit sum, upper limit sum, and things like that.

So, today basically we will extend these concepts of Riemann integrable functions; there are some how to say integrability conditions for a function to be Riemann integrable and we will also try to look into a fundamental theorem of integral calculus if time permits in today's lecture.

(Refer Slide Time: 01:12)



So, we in the last class we saw that a partition P which is basically given by x_0 x_1 up to x_n of a closed interval [a,b] such that $a = x_0 < x_1 < x_2 \,.\,.\!< x_n = b$. So, this is one such partition of the set of the of the closed interval [a,b]

Now, for this partition P we can define the norm of this partition. So, the norm of this partition is defined as; so the norm of P is defined by the length of the greatest interval greatest of all the subintervals of all the sub intervals $[x_{i-1}, x_i]$ for i running from 1 2 3 up to n so; that means, norm of P, so norm is usually denoted by $||P||$ how to say bar notation, so norm of P is basically the greatest. So, we can write as

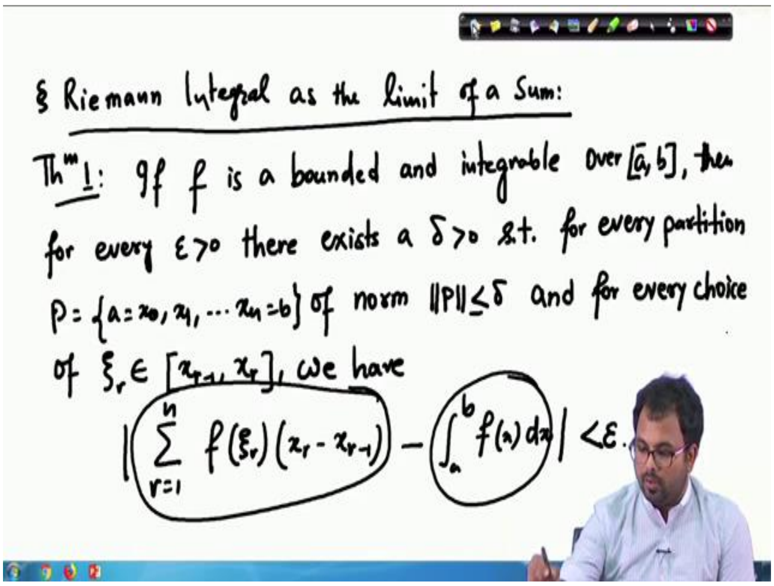
$$. \left||P|\right| = \max\left(x_1 - x_{0,} x_2 - x_1, \ldots, x_i - x_{i-1}, \ldots, x_n - x_{n-1}\right)$$

$$= \max_{1 \leq i \leq n}(x_i - x_{i-1)}$$

So, this is the way we define the norm of this partition P, you can also use a notation something like $\mu(P)$. So, we can write it as $\mu(P)$ and the norm of the partition P you can be denoted by this symbol $\mu(P)$ as well

So, now that the definition of norm of this partition is cleared, we will look into 1 or 2 integrability condition or to be very precise Riemann integrability condition in this case. So, let me start the new page.

(Refer Slide Time: 04:05)



So, the first one is: so Riemann integral as the limit of a sum. So, this particular definition or the representation of a Riemann integral will give motivation to the definite integral. So, we will see that what do we mean by it. So, the statement goes like this, so we can write a small theorem.

So, the theorem says that if f is a bounded and integrable function over [a,b] then for every ϵ positive there corresponds or there exists a δ positive such that for every partition $P = \{a = x_0, x_1, \ldots x_n = b\}$ of norm $||P|| \leq \delta$; and for every choice of

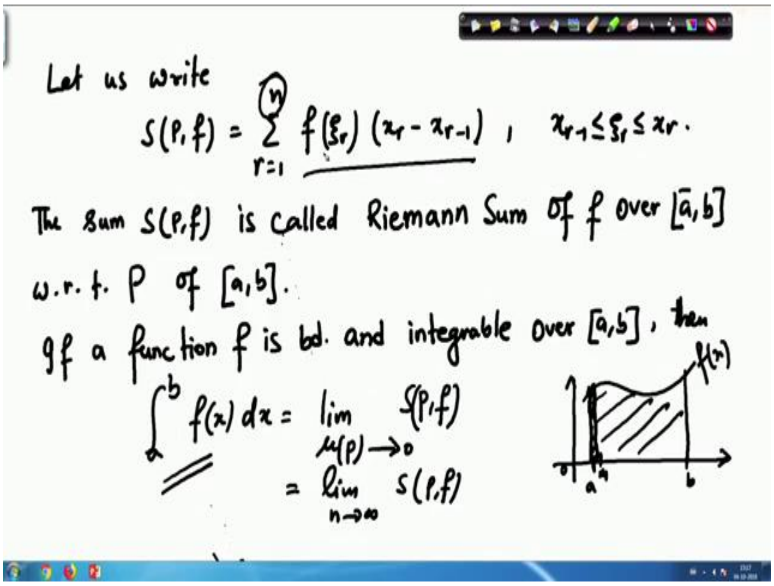
$\xi_r \in [x_{r-1}, x_r]$. So, we choose this point ξ_r in any r-th interval $[x_{r-1}, x_r]$. We have, what do we have? So, we have

$$\left| \sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1}) - \int_a^b f(x)dx \right| < \epsilon$$

So, what does this theorem actually saying? So, this theorem says that, if we have a bounded function f which is also integrable, Riemann integrable over the interval [a,b] then for every ϵ positive. So, this ϵ is an arbitrary chosen small positive number there corresponds a $\delta > 0$ such that for every partition P whose norm $||P|| < \delta$ we can find a point ξ_r in the r-th interval such that this value is less than ϵ.

So, you see ϵ is an arbitrary chosen positive number. So, this integral is nothing but it can be expressed as $\sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1})$. So, this integral here it can be represented as $\sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1})$. The proof of this theorem is a little bit long. So, we will not get into the proof of this of such theorems since, it is a course meant for BSc and BTech students. So, we will not get into the proof, but the flavor of this theorem, so what does it mean I will try to explain that.

(Refer Slide Time: 08:53)



So, let us look into this some. So, let us write; so let us look into this some. So, let us write, let us say S(P,f) So, sum over the partition P for the function f we write it as $\sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1})$ So, this sum where this ξ_r is any arbitrary point between

$[x_{r-1,}x_r]$. So, the sum basically this sum S(P,f) is called the Riemann sum I beg your pardon. So, this sum is called the Riemann sum of f over [a,b] with respect to the partition P of [a,b].

Now, the above theorem can be seen in this following way that if a function f, or in other words it can be seen as if a function f is bounded and integrable over the interval [a,b], then what we have is we have integral sorry; what we have is we have $\int_a^b.$ beg your pardon, f (x)dx is basically $\lim_{\mu(P)}$.or the norm of P going to 0 such that this limit S P f is actually the value of that definite integral.

So, if you make the how to say $||P|| \to 0$; that means, $||P||$ or the how to say, the length of those intervals when we say that it is going to 0; that means, you are actually taking the number of subintervals as very small. And then in that case if you look at this figure, let us say this is my f (x) and that is x=a and that is x equal x=b. So, doing this definite integral is basically we are calculating this area. if we are doing integral from a to b.

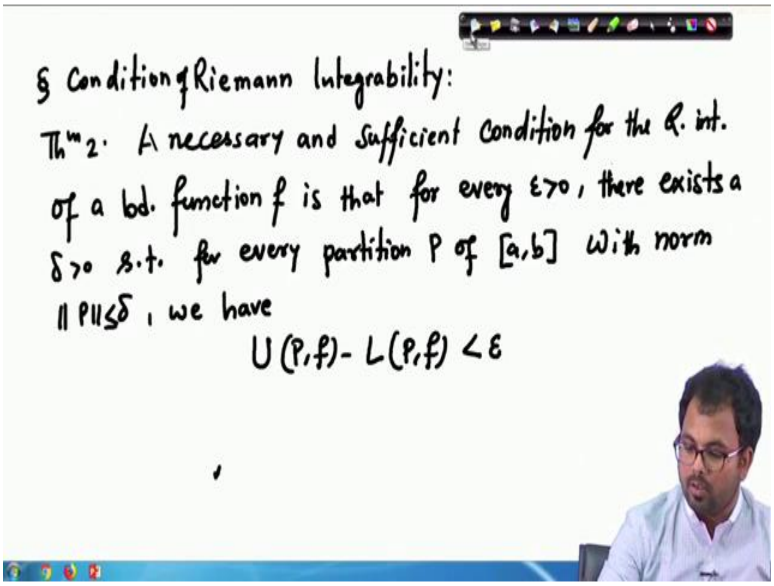
Now, if I take $||P||$ norm of partition of P going to 0; that means, I am basically making the number of subintervals really small and small. That means, this n here these number of subintervals are basically I mean getting larger and larger. So that means, I am taking how to say many large,a really large number of subintervals and if you take really how to say if you take your x_1 here then, in that case what I am doing is the this area here., This area here it is basically if you take x_1 , if you take a really large number of subintervals then in that case these small areas are actually converting into a rectangle. And here in this formula here $f(\xi_r)(x_r - x_{r-1})$ this is basically if you forget about the summation then this is basically the area of 1 such rectangle.

So, you are actually dividing integral $\int_a^b f(x)dx$ into some of the areas of really how to say large number of rectangles. And if you sum all those areas of rectangles then that will actually give you the $\int_a^b$.. So, that when you make $\mu(P)$ goes to 0 you are actually making n tends to infinity. We are actually making here n tends to infinity and if you make n tends to infinity then basically you are increasing the number of subintervals. And then in that case this in on every sub interval you are basically calculating the area of a rectangle. And if you sum all those areas and that will give you the area of the curve of the function f, between the points x=a to x=b. And that is what we mean by this integral on the right hand side.

So, if you make $n \to \infty$ of this Riemannian sum then that will actually give you the definite integral or the integral of the function $\int_a^b f(x)dx$ and this is the how to say the vital point which we wanted to make that how Riemann sum how this sum here is related to the definite integral. So, the proof of the theorem is a little bit complicated, but you can be able to find the proof in any standard integral calculus book where they have shown the proof with the help of upper sum and lower sum it is not that difficult and you can be able to go through that proof. However, for our lecture we would try not to get into those details, but just to give you an idea how Riemann sum is in related to definite integral that is what I was trying to do.

Now, that is we know how Riemann sum is related to definite integral we will jump to a new or how to say a new result where we look into the condition of integrability.

(Refer Slide Time: 16:01)



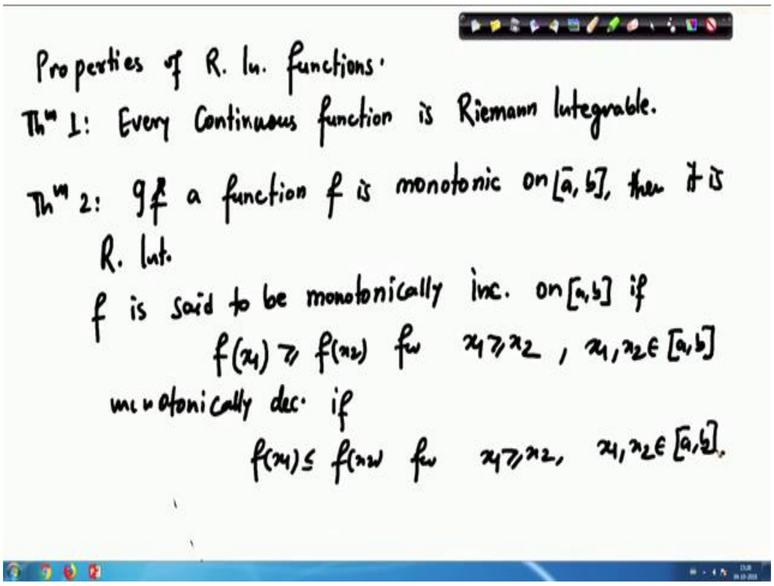
So, condition of Riemann integrability, so we can formulate this in terms of another small result. So, theorem 2 and it says that; a necessary and sufficient condition for the Riemann integrability of a bounded function f is that for every $\epsilon > 0$ there exists $\delta > 0$ such that, for every partition P of [a,b] with $||P|| \leq \delta$ we have U(P,f) which is our upper sum minus L(P,f) which is our lower sum is less than ϵ.

So, that means if you are given a bounded function f and if you would like to see whether that function is Riemann integrable or not then we have to calculate the upper sum and the lower sum. In the last class we have seen that how we calculate the upper sum and lower sum and if the difference of the upper sum and lower sum is less than this chosen ϵ, then in that case that particular function is said to be Riemann integrable. This makes sense because the upper sum U(P,f) if you take how to say the infimum of all the U (P,f) over the partition P, then in that case that will give you the upper integral sum. And if you take the supremum of all such L(P,f) such that now for this partition P then that will give you the lower integral sum and the difference basically we say that the function is Riemann integrable when the lower integral sum and upper lower integral sum and upper integral sum are same.

So, basically if you say that the difference is less than ϵ. That means, the function is Riemann integrable and your lower integral and upper integral are same. So, in a way this condition actually makes sense and based on which we can say that the given function is Riemann integrable or not. Next we have is several properties of Riemann integrable function so for example we know that if a function is bounded. And then if it satisfies this condition $U(P,f) - L(P,f) < \epsilon$ then it is Riemann integrable, but other properties the function needs to have if you want to say that whether the function is Riemann integrable or not.

So, we will look into a few properties and how to say some results related to Riemann integrable function.

(Refer Slide Time: 20:31)



So, let us go to the first property. So, I can write it as properties of Riemann integrable functions. So, the first property is or you can write it in terms of as theorem. So, the first property or the first theorem in this regard is every continuous function continue sorry every continuous function is Riemann integrable.

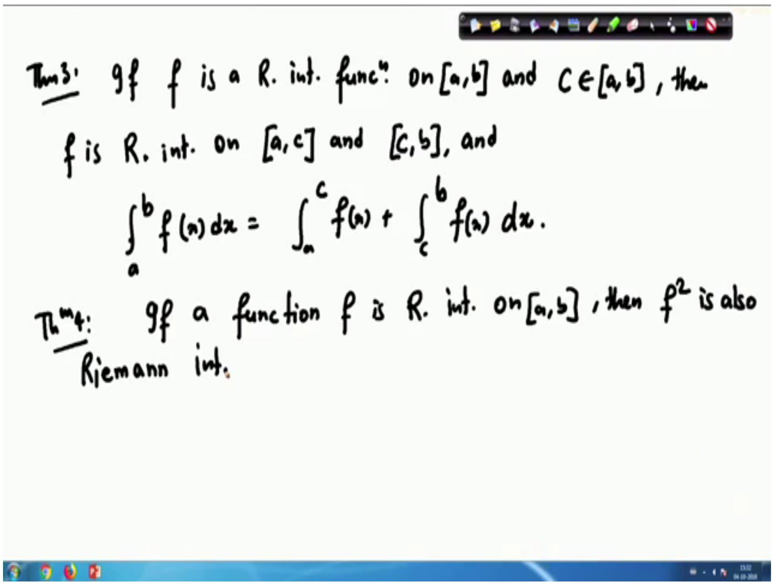
So, this makes sense because every continuous function is always bounded and if the function is bounded then all we have to show that whether $U(P,f) - L(P,f) < \epsilon$ or not and if it does then in that case the function is Riemann integrable. The proof is a little bit how to say well it is not long it is just that for the proof you can look into any book here I am just how to say listing some of the important properties of Riemann integrable functions. For the proofs I would recommend to look into the references which I have shown you in the first class and there you can be able to find the proofs of these theorems.

The second property is if a function f is monotonic on [a,b] then it is Riemann integrable. So, I am hoping that all of you know what does a monotonic function mean. So, a monotonic function f a function f is said to be said to be monotonic. So, a function can be monotonically increasing or monotonically decreasing or it can be strictly increasing or strictly decreasing. So, here monotonically increasing on [a,b] if for if $f(x_1) \geq f(x_2)$ for $x_1 \geq x_2$ where x_1 and x_2 are both 2 points in [a,b] monotonically increasing and similarly we can define the monotonically decreasing part.

So, a function is said to be monotonically decreasing mono sorry so it is said to be monotonically decreasing if $f(x_1) \leq f(x_2)$ for $x_1 \geq x_2$ where x_1 and x_2 are 2 points in this interval [a,b]. So, here we can see that if a function is monotonic. So, let us assume that it is monotonically increasing on [a,b], then in that case of course, it is bounded. And if it is bounded then all we have to show is that whether $U(P,f) - L(P,f) < \epsilon$ or not and then that will ensure the Riemann integrability.

Similarly we can assume that the function is monotonically decreasing and if it is monotonically decreasing, then of course in that case also it will be bounded. And we again try to show that $U(P,f) - L(P,f) < \epsilon$ and then the function will become Riemann integrable. So, the proof of this theorem can also be found in those in those books which I have listed as reference next property is I can list it as let us say theorem 3.

(Refer Slide Time: 25:26)



So, if f is a Riemann integrable function on [a,b] and c is any arbitrary point between the interval [a,b] then f is Riemann integrable on [a,c] and [c,b] and we can write the integral $\int_a^b f(x)dx$ as the sum of these 2 integrals. So that means, you can take as many points as you please in your closed interval [a,b] and the sum of those integrals on each one of those sub intervals will be the integral of the how to say whole of the function f on the whole interval [a,b]. So, here not only c you can take as many points as possible in between that interval [a,b] and the result would still be true. Next theorem we have or the property we have is if a function f is Riemann integrable on the closed interval [a,b] then f^2 is also Riemann integrable.

So, if you have and similarly you can continue, so if f^2 is Riemann integrable and if then in that case you can be able to show that f^3 or f^4 they are also Riemann integrable so that means, this property carries forward. And again for the proof of this theorem I would recommend you to look into the references which I have mentioned.

Next, as we have seen that for the Riemann integrability, how we can connect the Riemann integrability with Riemannian sum and how it is connected with the definite integral. We also saw that the necessary and sufficient condition for a function to be Riemann integrable and some properties associated with Riemann integrable function.

So, we will stop this lecture at here. And in the next class we will again continue with the properties of Riemann integrable function.

Thank you.