Integral and Vector Calculus
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Lecture – 29
Surface Integral (Contd.)

Hello students. So, in the last class we started with the line integral and today we will continue with probably with our last example which we started and then, we switch to a Surface Integral.

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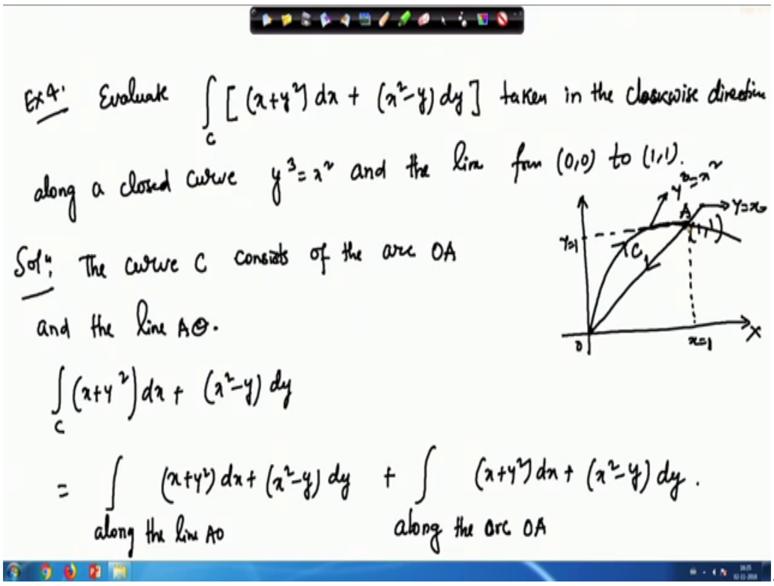

So, in the last class we started with this example where we needed to evaluate this integral along the closed curve given by this curve here and the line from 0 0 to 1 1. So, we can actually draw the figure and the line from 0 0 to 1 1 is a straight line passing through the origin. So, this is our line and the curve is given in this fashion and they intersect of course at the point 1 1.

So, we if we substitute y equals to x here, then it will be basically x cube minus x square. So, from there we can be able to calculate the point of intersection as 0 and 1 and when x is 0, then y is 0 and when x is 1, then y is 1. So, these are the two points where they intersect. Now, that we have the curve and the we have the line, so we basically have our closed curve and now we do that calculation. So, how can we do that? So, of course here the line integral can be done in two steps. So, first we do the line integral along this curve and then, we do the line integral along this curve. So, let us see. So, we have x plus y whole square sorry x plus y square dx plus x square minus y dy. So, we can write this one as the line integral along the line AO x plus y square dx plus x square minus y dy plus integral line integral along the arc OA and this will be x plus y square dx plus x square minus y dy right because we have to divide it into two segments. Now, along the line y equals to x I can substitute along the line y equals to x I can substitute y equals to x here and y equals to x here and x will vary from basically 0 to 1. So, let us do that.

So, I can do that along the line. So, along the line AO I can do 1, so, x is varying from 1 to 0.

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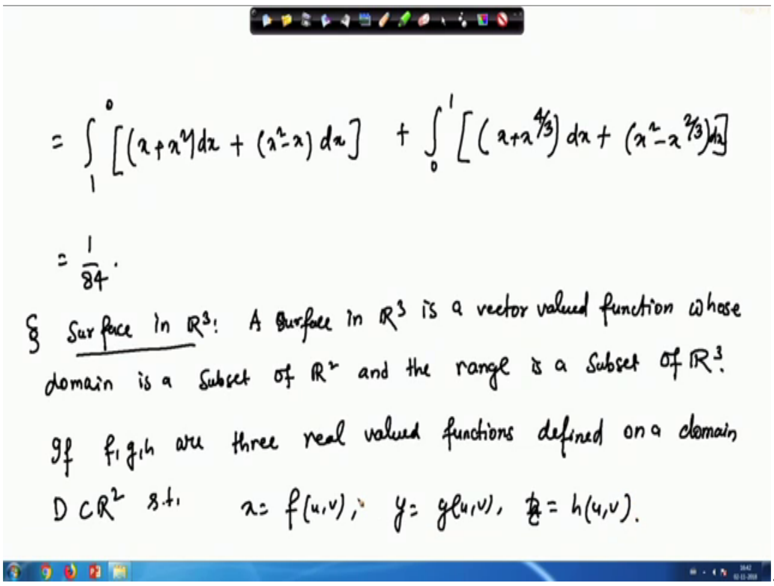

So, we can write it as 1 to 0 I have x plus y square is that y square. So, y equals to x along this line. So, I can substitute y equals to x square dx plus I have x squared minus y, so, y is again x. So, let us substitute that x square minus x and dx equals to dy equals to dx. So, I like dx here, all right plus for the arc OX is x is varying from 0 to 1. So, I can write again 0 to 1 I have x; I have x plus y square. So, when I do y square, then it will be x to the power 4 by 3. So, I can write x plus x to the power 4 by 3 dx plus x minus y square, I believe yes x square minus y.

So, x square and minus y will be x to the power 2 by 3. So, this will be x square minus x to the power 2 by 3, then here dx and if we integrate the whole thing, so of course integrating would not be that much difficult because we basically have an algebraic expression and I am pretty sure you can be you all can be able to do that. So, we will avoid that doing that here. So, if you do the whole calculation, then you will ultimately obtain 1 by 84, so, this is the required answer. Now, this example is in particular interesting because here we have to divide the line integrals into two segments basically.

So, first we integrated with respect to this line segment and then, we integrated with respect to this arc here and the sum will actually give you the integral along that whole closed curve because they are closed and that is basically our answer is. So, this is how we calculate the line integral. So, you always have to identify the curve in case of line integral. You always have to identify the curve and its parametric representation or the range for the variable x and y and based on that which either in this case in the previous example we did not have to put the parametric representation.

We managed with the x and y representation itself, but if you need to put the parametric representation of the curve, we can also do that. We saw those one or two examples like that and then, you integrate with respect to the parameter t and use the limits of the parameter t, otherwise you can also integrate with respect to x and y and divide the given a closed curve. If it is composed of two different segments, then you divide the closed curve into two segments and then, you do the integration like the previous example.

So, the line integrals are not that difficult and they are also very interesting. So, I am hoping you would how to say find it interesting when you practice it by your own and we will also try to include some formulas or assignments, basically not formulas assignments in your assignment sheet for you to practice, all right. So, more or less most of the examples are of similar types.

So, we will skip these examples and now I am in my lecture note. Now, I am going to start with our next topic which is basically surface integral and in surface integral here will evaluate some surface integral obviously and also we will evaluate some volume integrals ah. In our vector calculus part we will do the same thing actually. So, in vector calculus part also we will evaluate surface and volume integral and of course, those will be motivated from the vector calculus point of view.

So, there we will say Gauss Divergence theorem, Stokes theorem, Greens theorem and here we will stick to our traditional surface and volume integrals. If time permits we might look into the Gauss Divergence theorem or Stokes theorem in the integral calculus itself. If not then we will cover those topics in our vector calculus part which will I am hoping to start it next one or two lectures after actually.

So, let us start with the surface integral. So, surface in R3, so a curve or a surface a surface in R3 is a vector valued function whose domain is a subset of R2 and the range is a subset of R3, all right.

So, if f g h are three real valued functions defined on a domain d subset of R2 such that x equals to f of uv y equals to g of uv and h equals and z equals to h of uv.

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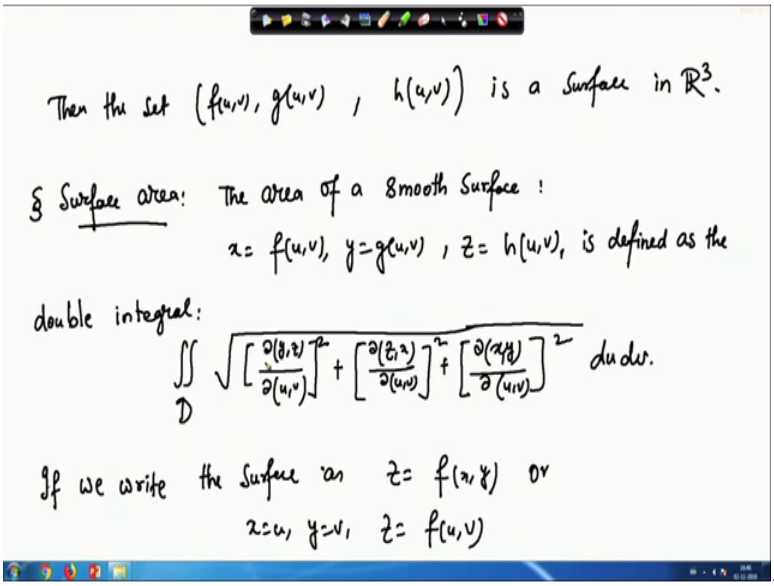

So, then the set then the set f uv g uv and h uv, sorry there is a comma h u, v is a surface in R3. So, all of this, so this will give the point x, this will give the point y and this will give the point cell. So, the collection of all these points will actually represent a surface in R3 and we will actually do the integration on these surfaces. So, let us see how we can do that and in our case whatever surface we work with, they are always smooth.

So, that means all of them will have a continuous partial derivatives so with respect to these variables actually. So, now how can we calculate the surface area of a surface area. So, the formula is the area of a smooth surface; the area of a smooth surface, so, that means they have continuous partial derivatives x equals to f u, v y equals to g u, v and set equals to h u, v. H u, v is defined as the double integral as the double integral over the domain D for u, v and here we will have a square root of the Jacobian of del y, z del u, v whole square plus del z, x by del u, v plus del x, y by del u, v whole square d, u d, v if we simplify this formula. So, consider if we consider the surface. So, if we let us say so these are basically the Jacobians, but we have to simplify this formula.

Now, if I write so if we write; if we write the surface as z equals to f x, y. So, if this is the given surface or we can write x equals to u y equals to v and then z is our f of u, v, so again x is equals to some h u, v, y equals to some g u, v and z is our f u, v, then in that case we can calculate these individual Jacobians. So, calculating these individual Jacobians will not be difficult, we have already done the calculation. So, this for example, is del y del z del y del y del u del y del v, then del z del u del z del v and then, we do the calculation of that determinant that will give us the first Jacobian, then, similarly the 2nd Jacobian, similarly the 3rd Jacobian.

So, if we do all those calculation, if we do all those calculation, then we will basically obtain let us say if we will basically obtain the surface integral which I am going to write as.

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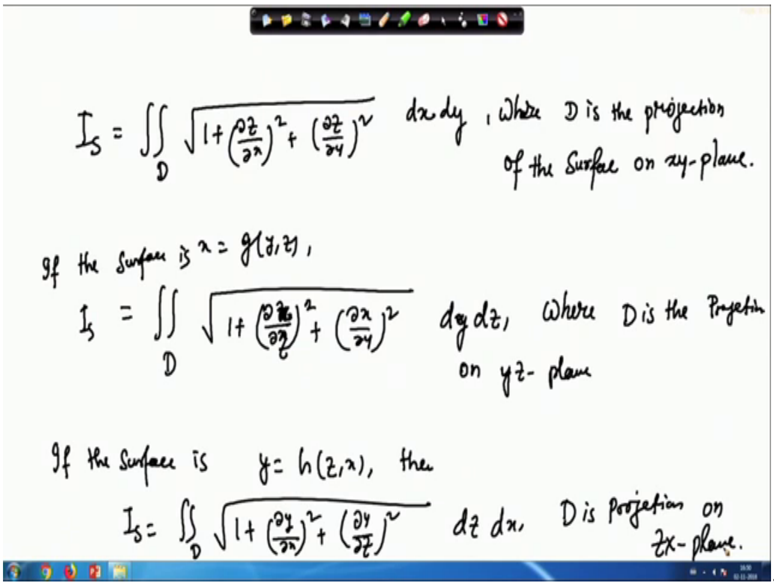

I s is equals to integral over the domain D a square root dx 1 plus del set del x, it is a formula. Basically del set del y where D is; where D is the projection of the surface; projection of the surface on xy plane, right and if I do the projection on let us say zx plane, so then in that case this will be del z del x.

So, this is the projection on del z plane. Now, if the given surface is if the surface is x equals to say g y, z, then again I can substitute y equals to u and z equals to then I can substitute y equals to use that equals to v, then it will be x u, v and therefore, the required form will be del x del x del z and then, here it will be del x del x del z and then, del x del y whole square dx. Sorry dy dz where d is the projection on yz plane and similarly if the surface if the surface is; if the surface; if the surface is let us say x equals y equals to h zx, then I s is integral over the domain d 1 plus del y del x plus del y del z n dz dx, where D is the projection is the projection on zx plane.

So, here it also depends on in what form you are given the equation. So, if we have the given surface as x equals to f y, z, then we basically. So, if the given formula is z equals to f xy, then we basically obtain the first formula and then, we use the first formula. Actually if we are given the equation of the curve as x equals to g yz, then in that case we will use the 2nd formula and if it is given y equals to h zx, then we use the 3rd formula. So, it also depends on the fact that in which form you are given the surface and based on that we can use either one of these formulas. So, I think it was a little bit abstract. So, let us start with the first example.

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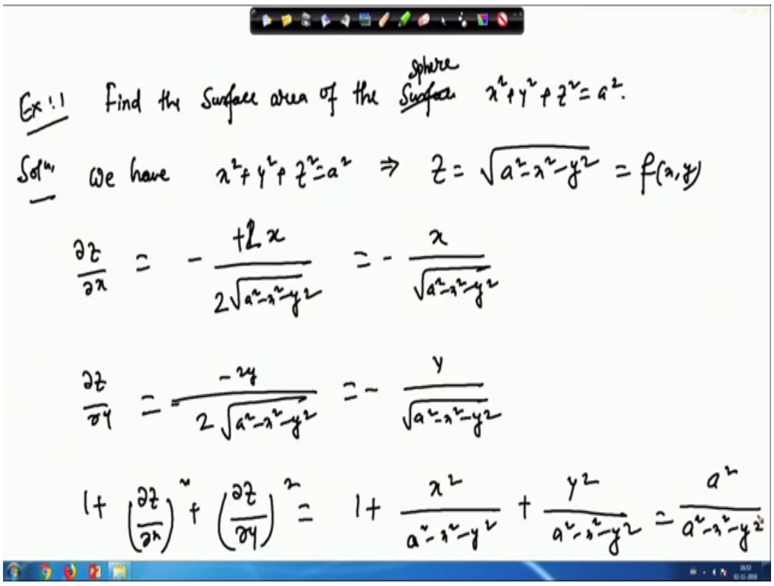

So, example 1; find the surface area. Let us find the surface area of a curve which is or a surface which is already known to us. So, find the surface area of the surface x square plus y square plus z square equals to a square. Now, just looking at this curve or at the surface we can easily say what kind of surface it is. It is our sphere, all right. So, we know that. So, instead of surface I can write now sphere now that we all know. Now, we know that surface area of a sphere is 4 pi a square. So, let us see whether we can be able to obtain using these integral calculus formula or not.

So, let us start. So, we have x square plus y square plus z square equals to a square. So, from here I can be able to write z equals to a square minus x square minus y square, right. I am not taking the negative value, so, let us let us take only the positive value of z, right. So, this is basically my f x, y isn't it and now if I have the formula in terms of z equal to f x y, then we can be able to obtain the 1st formula for the surface integral. So, for the 1st formula I need to calculate where is that del z del x and there is a del y. So, let us calculate del z del x. So, this will be minus of 1 by square root of a square minus x square minus y square and then, this is minus 2 x.

So, this is basically 2 x and there is a 2 here, so, 2 2 will get cancelled. So, this is basically minus of square root of a square minus x square minus y square. Similarly I can calculate del z del y equals to minus of 2 y two times square root of a squared minus x squared minus y square. So, this is minus of y a square minus x square minus y square, right. So, now based on this formula; based on this formula I can be able to calculate 1 plus del z del x whole square del z del y whole square.

So, this is basically 1 plus x square by a square minus x square minus y square 1 plus y square by a square minus x square minus y square. So, if I take this if I calculate this, then this will be a square by a square minus x square minus y square.

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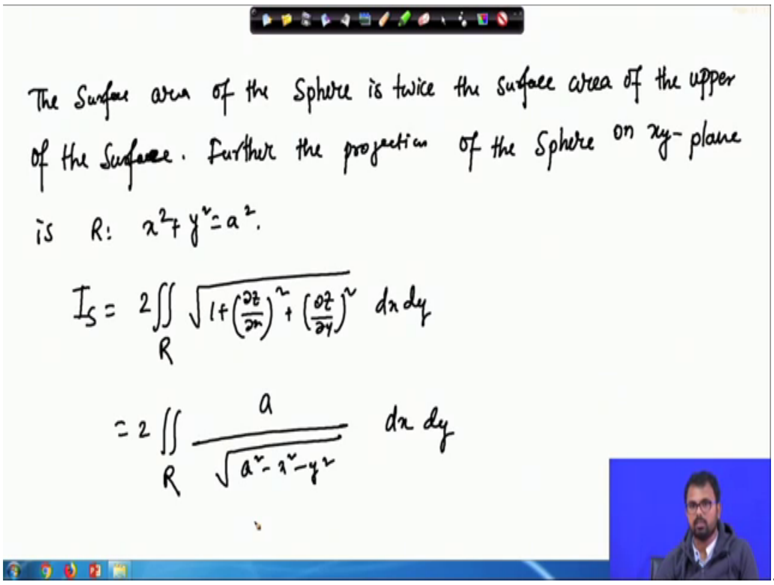

So, now the surface area, so, the surface area of the sphere is n twice the surface area of the upper half, right upper half of the sphere; upper half of this sphere, right and further the projection of this sphere; projection of the sphere on xy plane is, right. So, we have taken the upper half because the surface area of the whole sphere would be just the double of the surface area of the upper half and if we do the projection on xy plane, then it will basically be a circle.

So, initially it is an upper half of this sphere, but once we do the projection then in that case it is basically a circle which is given by this equation. So, the radius will remain same, it is not like if we do the projection, then the radius would change. So, the radius would still remain same and this is the equation of our circle. Now, our required surface area will be two times d is the R is the projection. So, let us write R 1 plus del z del x whole square plus del z del y whole square dx dy. So, this is basically two times integral from 0 integral over the region r square root. That means, this is a and then, this one will be square root of a square minus x square minus y square dx dy.

Now, this is just our plane integral calculus for the function of two variable. This is this we have already done. So, this type of integral calculation of integral for the function of two variable we have already done. So, here we basically take help of the; take help of the how to say polar coordinates actually.

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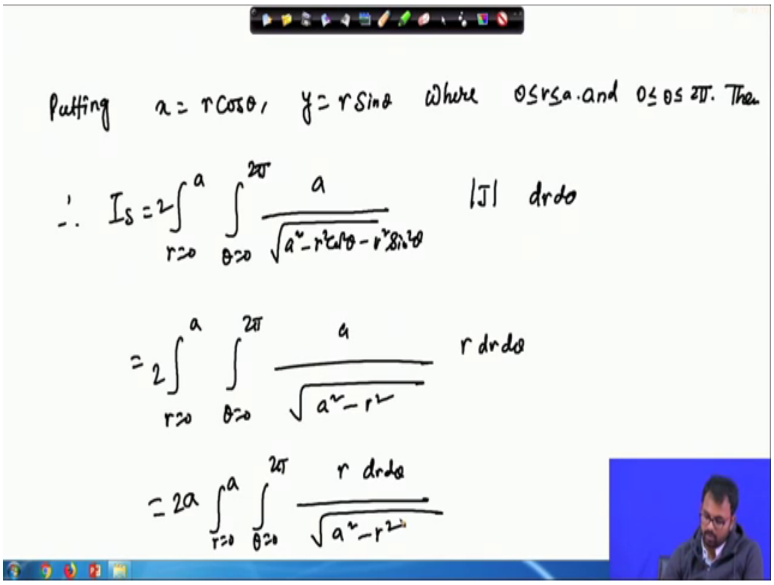

So, let us substitute putting x equals to r cos theta and y equals to r sine theta, where 0 less or equal to r less or equal to a and 0 less or equal to theta less or equal to 2 pi, then the surface integral I s would reduce to r running from 0 to a theta running from 0 to 2 pi n. I have a divided by I will have a square minus r square cos square theta minus r square sine square theta and then, Jacobian times r Jacobian times dr d theta.

So, this will be integral r running from 0 to a theta running from 0 to 2 pi a divided by a square minus r square because we take r square common and it will reduce to cos square theta plus sin square theta. Jacobian of this will be r dr d theta. We have already seen it many times how we can calculate the Jacobian for this transformation and therefore, we will end up with a times. Yes of course, there is a 2 here, so, 2 a times r running from 0 to a theta running from 0 to 2 pi r dr d theta square root of a square minus r square. So, now we separate theta n r because they are not mixed up together.

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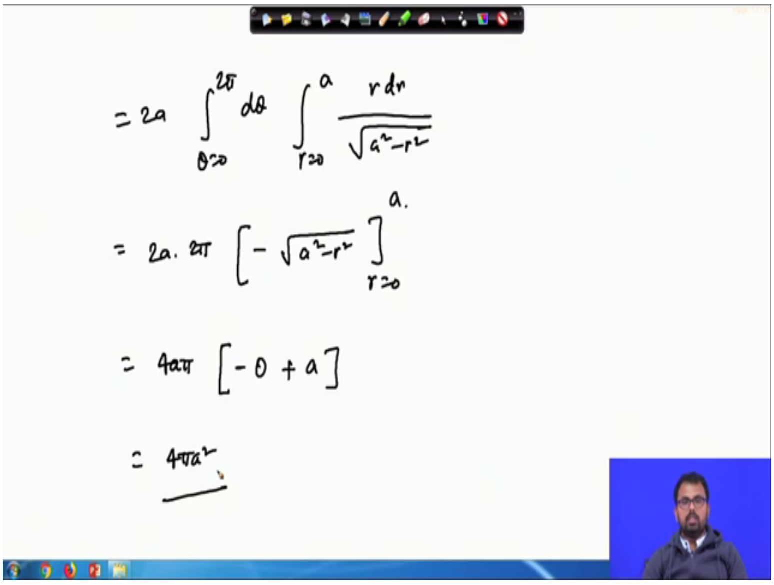

So, 2 a integral theta running from 0 to 2 pi d theta integral r running from 0 to a r dr by a square minus r square. Now, I substitute a square minus r square equals to z and then, in that case we will have 2 r dr equals to some dz and that can be handled very easily. So, calculating this integral would not be difficult. So, this will be 2 a times 2 pi and this integral would be minus of a square minus r square, where r is running from 0 to a. So, this is basically 4 a pi and when we substitute r equals to a, so this will be 0 plus a. So, that means 4 pi a square. So, this is the required surface area of this sphere. So, where is that problem?

So, calculating the surface area of this sphere we tried to express the function z, sorry we tried to express the variable z as a function of x and y. So, now we are in that z equals to f x, y form. So, for that we need a del z del x and del z del y. So, we substituted the value here and then, that is our this wire and then, in the formula also we need to take the projection. So, before doing the projection we first need to find out the surface area. How I mean what kind of concept do we need to use? So, since it is a sphere the surface area of the whole sphere be would be twice the surface area of the upper half.

So, that is what we have done and R is the projection. So, when we take the projection of the upper half on xy plane, it will be a circle with similar radius. So, that is what we have written and then, we are just using this how does a conversion into a polar coordinate system. So, this is x equals 2 r cos theta r sin theta where ad is between 0 to 8, theta is between 0 to 2 pi and then, we just substitute the value, do some simple calculation that will give us the required integral as 4 pi a square.

So, here in this example we saw that how we can calculate the surface area and that calculation does match with others, match with our traditional how to say a concept of surface area of a sphere. So, we will definitely work out one or two more examples on surface on calculating the surface area of a surface. We will probably see some examples where we have two intersecting surfaces and how we can calculate the surface area. We will do all those things in our next class and then, we will switch to volume integral. So, am I thank you for your attention and I will see you in the next class.