Integral and Vector Calculus

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Lecture – 28
Surface Integral

Hello students. So, in the last class we practiced rectification of curves which basically mean that means that, we calculated the length of the arc and we also saw that we can calculate the length of the arc for different types of course; in given in different forms actually. So, either you will be given y is equal to f x form or x equals to f y form or r is equals to f theta. So, whatever the form maybe we can implement that particular type of formula, to calculate the length of the arc between certain point to a certain point.

So, those kind of examples we practiced and we also saw that in what situation what kind of formula we can use. So, today we will start with basically an application side of integral calculus. So, when I say application, it basically means that we will be doing some line integral surface integral and the volume integral. So, that is the final or maybe 1 chapter before the final chapter. So, the final chapter is basically calculating the moment of inertia and things like that, but before that we will practice this line integral, volume integral and surface integral problems.

So, today we will start with the line integral. This is also motivated from the vector calculus as well. So, when we go through the topics on vector calculus, we will come across the similar topics where there also we will practice line integral, surface integral and volume integral, but that will be more towards the geared towards the vector calculus part. And today and probably in the next lecture we will look into more from the integral calculus perspective.

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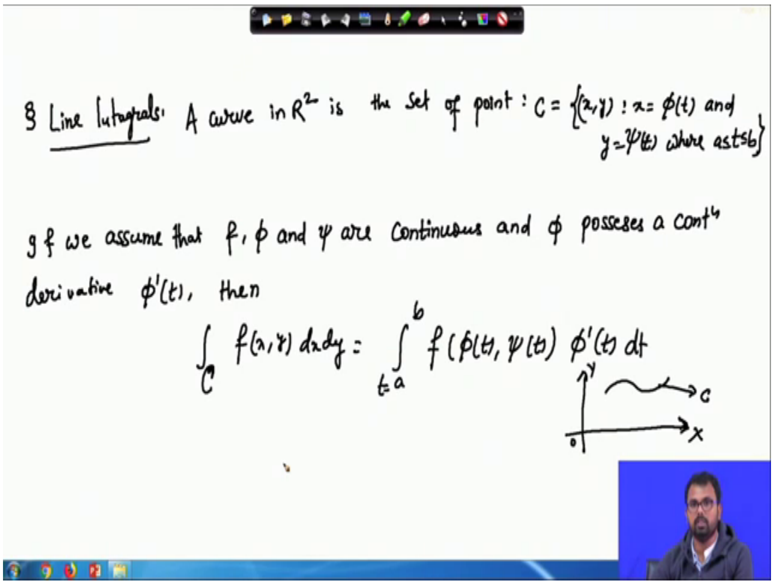

So, let us start with line integral. So, line integrals; so, the line integrals basically mean that a curve. So, we give a small definition. So, a curve in R 2 is the set of points is the set of points c which is equals to all such x and y such that x equals to some phi t and y equals to some psi t where t is lying between a to b. So, for example, circle is a curve in R 2 and its equation can be given by x equals to cos t and y equals sin t for t running between 0 to 2 pi.

So, that circle is a curve which can be expressed in this form and we say that if we assume that let us say and if we assume that, if we assume that the function f phi and psi are continuous and phi possesses a continuous derivative phi dash t, then integral from a to b f x y or in fact, we can write the curve actually. So, the integral over the curve c f x y d x d y is equals to integral from a to b f phi t, psi t times phi dash t d t.

So; that means, basically what we are doing is, we have a curve let us say; we have a curve let us say something like this and we are basically calculating this integral, integral of this function along this curve c. So, the name this curve is basically our curve c. So, with hit here in this definition of the curve, we can write x equals to some phi t and y equals to some psi t for a certain values of t and we replace this value, the x and y in this integral on the left hand side with phi t and psi t and this here is basically the range for t, then this integral is equal to the integral of this function here in of course, parameter t and t is varying from a to b.

So, that is what we basically mean by the calculation of the integral of a function f along this curve c from let us say point t equals to a to, from a point a to point b. So, we have a point certain point here let us say x equals to certain some point and y and here x equals to some point, and then this will be given by that a comma b to c comma d then in that case, the integral of the curve, integral of the function f along this curve from a comma b to c comma d will be given by the formula on the right hand side, where we have to express now, the equation of the curve in terms of a parametric equation.

So, that is the main task I would say here is to find the parametric equation of the curve and based on that we can calculate the limit, from where to where the t is varying and then just substitute on the function and then we can calculate the right hand side, and that will give us the required how to say line integral of the function f. And next this is somehow also how to say motivated from the vector calculus sense, but we will work on that part when we actually look into the vector calculus part. So, how do we calculate these type of integrals? So, the integral calculation can be done in the following way.

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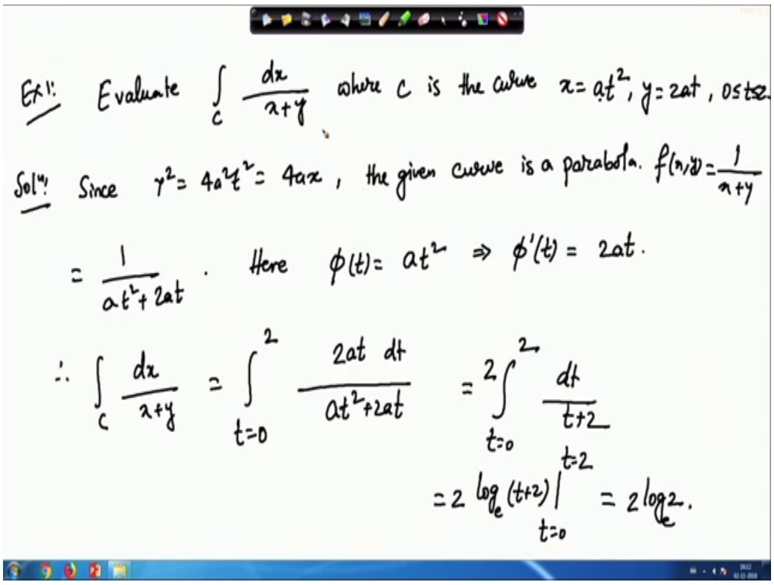

So, let us start with our very first example. So, this example says evaluate integral over the curve c d x by x plus y, where c is the curve and here we have only d x actually. So, here we will have only d x and yes. So, we do not need y actually. So, integral over the curve c f x y d x is equals to integral from a to, from t running from a to b this formula here.

And now d x by x plus y where c is the curve x equals to a t square and y equals to 2 a t where 0 less than t less than 2. So, we already have the parametric representation of the curve and for that we do not need actually the calculation to calculate the parametric representation of a certain curve. This curve is basically as we can see its a parabola because if we do y square. So, since y square equals to 4 a square t square.

So, this can be written as 4 a x. So, the given curve is a parabola and since it is of standard shape. So, we do not need to draw this parabola as well. Now we have our function f the given function f is f x y equals to 1 by x plus y. So, this will be actually 1 by a t square plus 2 a t and if I follow that formula. So, we need to calculate also the phi dash t, in this sense our phi t is basically a t square and psi t is 2 a t. So, that is phi t and psi t the 2 functions.

So, here phi t is basically a t square. So, from here we will have phi dash t is basically 2 a t and therefore, our required integral c d x by x plus y equals to t running from 0 to 2, instead of d x we will have phi dash t d t. So, phi dash t is 2 a t d t divided by f of phi t psi t. So, phi t psi t is a t square plus 2 a t. Now, this will be basically integral t running from 0 to 2 1 a will cancel out, one t will cancel out. So, we will have a t here 2 here and then this will be d t by t plus 2.

So, once we integrate it will be 2 log to the base e t plus 2 and t will be at 0 and t will be at 2. So, when we substitute t equals to 2 then it will be log 4, and when we substitute t is equals to 0 then it will be log 2 and if we do some simple calculation, then it will lead to 2 log 2. So, that is to the base here. So, this is the required answer or the evaluation or the evaluation of this integral along this curve c. So, that is what I mean by line integrals.

So, we are actually integrating along a curve. So, sort of like following a kind of kind of like a line, it is not entirely a line, but the name is somehow motivated about following a certain curve, along a certain curve if I am walking along a certain curve, then what will be the integral of a function along that curve from a certain point to a certain point. So, that is what we mean by line integral and this is what we are doing here. So, here in this case we have a function and we need to calculate its integral along this parabola from the point t equals to; from the point t equals to 0 to t equals to 2.

So, if I substitute t equals to 0; that means, 0 0 so; that means, we are calculating the integral from the origin and when I substitute t equals to 2, then its 4 a so; that means, we need to calculate the integral from 0 0 to 4 a, I hope it is 4 a. So, yeah 4 a to 4 a along that curve. So, that is what we mean by line integral here. We will work out few more examples just to make the concept a little bit clear. So, let us proceed.

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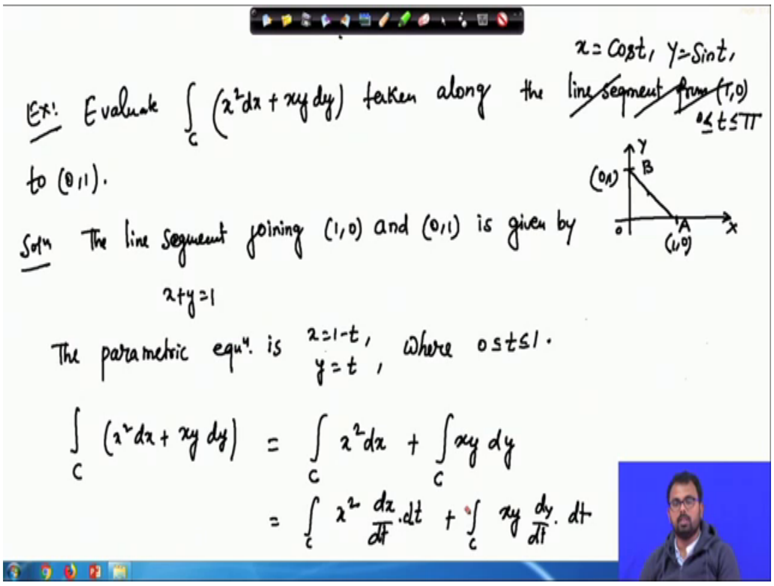

So, here we will have. So, here we have evaluate integral over the curve c x square d x plus x y d y right taken along the line segment line segment from 1, 0 to 0 1. So, we have to calculate the line integral of the curve, we have to calculate the line integral of this curve taken along the line segment 1 0 and 0 1. So, first of all let us draw this curve. So, the equation; so, the line segment is 1, 0 and another to the point 0 1 so; that means, if I join these two points let me call it as A and this 1 is B. So, we are basically calculating the line integral along this line segment.

So, the line segment or the straight line; the line segment or the straight line joining 1 0 and 0 1 is given by, what will be the equation? Is given by x plus y equals to 1 so; that means, this is the equation of this straight line. So, if I put y equals to t then in that case x will be 1 minus t. So, the parametric equation or we do not have to do the parametric equation here, when we are walking along this line or let us do it anyways the parametric equation is let us say y when x is t. So, when x is t then y will be 1 minus t alright where 0 less or equal to.

So, where t is; so, it would be better if we take this one the calculation point of view, you would understand what I am trying to say. So, if I take y equals t then x is 1 minus t where 0 equals less or equal to t less or equal to 1 so; that means, when t is 0 the parametric the point is 1 0. So, this point and when t is 1 then its 0 1. So, this point and all the other points for every value of t will will lie on this a straight line. So, that is the parametric representation.

Now, let us go to this; let us go to this curve. So, here we have x equals to 1 minus t and y equals to t. So, that is the parametric equation. So, to be very precise we can write it, this is not the equation this is just a coordinate. So, this is our parametric equation if we write it correctly. So, that is basically the parametric equation, the earlier what we have written was just a point. So, this is the parametric equation. Now we will integrate the function x square d x x y, d y.

So, first of all we will integrate the first term x square d x integral over c x y d y alright. So, we can write it as integral over c x square I can write it as d x by d t times t d t and integral over c x y d y by d t times d t. Now what is d x d t? D x d t would be minus 1 and d y d t will be 1. So, let us substitute the value of x square value of x y and d y d t and d x d t.

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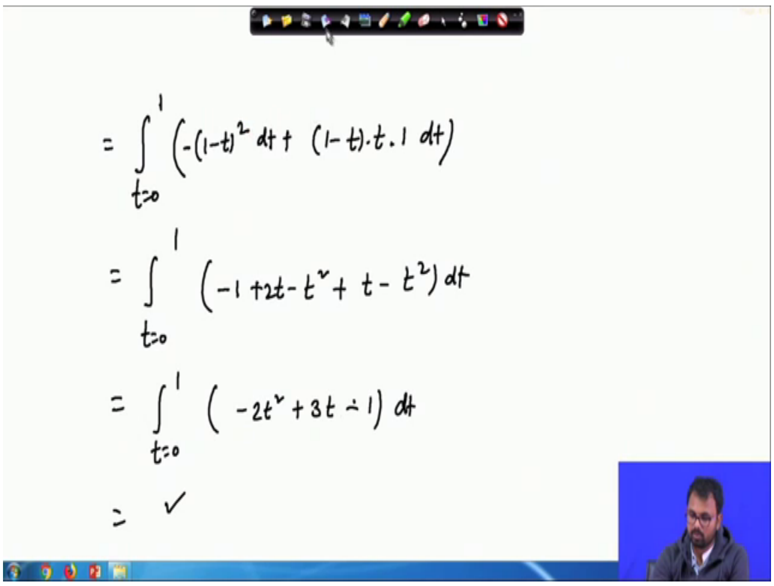

So, this will reduce to integral t running from 0 to 1 because that is our curve and then we have x square, x square is 1 minus t square then we have d t right. So, x square and then d x d t sorry. So, d x d t is minus 1. So, let us write that minus 1. So, there will be a minus here and then d t then plus x y. So, 1 minus t times t. So, let us write 1 minus t times t and then d y d t is 1 and then again d t. So, this will be integral t running from 0 to 1, we have minus 1 plus 2 t minus t square plus t minus t square d t. So, this will be t running from 0 to 1, we will have minus 2 t square plus 3 t minus 1 d t.

So, from here we can do the rest of the calculation. So, you see we substituted the value of x which is 1 minus t. So, that is what we did and x times y so, y is t. So, 1 minus t times t and then d y d t is again 1 alright. So, we substituted the value. So, this will be x square is d x. So, d x d t is minus of t. So, minus of x square; so, this is minus of 1 minus t whole square d t, plus x is 1 minus t times t 1 minus t times t d t and then we just break this formula. So, its minus 1 plus 2 t minus t square and then this one is plus t minus t square.

So, if we integrate this whole thing then we will obtain basically whatever. So, that will be the required answer. So, this is how we do the line integral for this type of problem, similarly if we want we can also have this curve as. So, if instead of this line segment we can also be given let us say some circle. So, taken along a circle x equals to cos t instead of a line segment, let me say it is a circle x equals to cos t and y equals 2 sin t where t is between 0 to pi let say. So, then in that case you have to substitute x equals to cos t here y equals to sin t here and then do the similar thing.

So, it just playing with what kind of curve that is given to us. So, its not that complicated alright we will work out maybe 1 or 2 more examples. So, let us do that.

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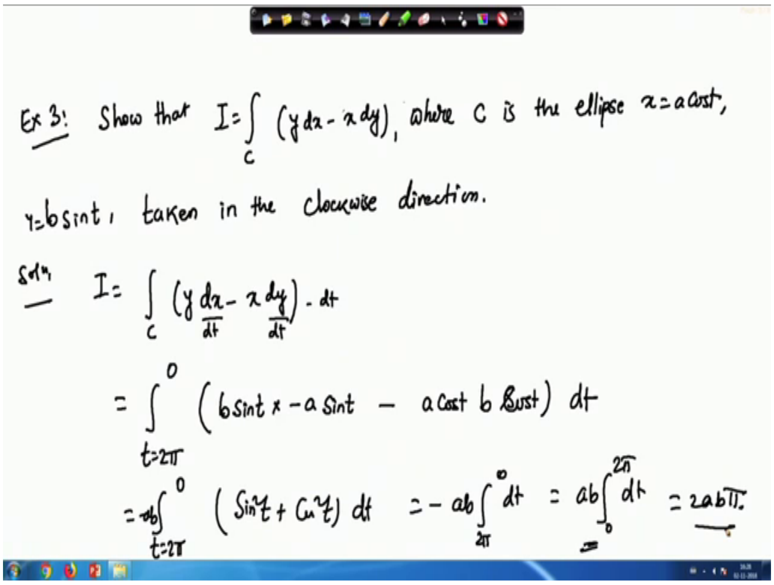

So, here we have example 3rd, show that integral I equals to integral over the curve c, x minus y whole square d x plus x minus. So, this one is x minus y whole square d x, and let us take a simpler examples instead of going straight to the complicated one, we can start with another simple example.

So, this one integral over the curve y d x minus x d y; the reason why I am considering this example is to show you a slight trick here and also the statement of the problem. So, that is why I am choosing this example because its a little bit interesting compared to the traditional examples. So, c where c is the ellipse x equals to a cos t and b equal y equals to b sin t taken in the clockwise direction. So, when we say clockwise direction; that means, the seconds key in the clock it moves or the minute key they move like; they move they move in this fashion.

So, that is our clockwise direction and if it rotates in the in the opposite direction then its called the anti clockwise direction. So, that means, t is actually instead of moving from 0 to 2 pi it is moving from 2 pi to 0 and having this particular thing written here anti clockwise or clockwise and the in its plays a very important role. So, you have to pay very close attention what is the range I mean in which range t is varying, whether it is 0 to 1 or 1 to 0? So, I mean of course, when we say that 1 to 0 theoretically or conceptually it does not make sense because it will always go from 0 to 1 not from 1 to 0, but it does happen and from and when we actually use those limits in your integral you may have to play with the integral to correct that limit or; that means, you can do 0 to 1 at some point.

Because you might get a minus of integral and then you can use the formula which is minus of integral from a to b f x d x equals to integral from b to a f x d x. So, some kind of integral calculus formula implementation might be there, but you always have to play very close attention; always you always have to pay very close attention to this statement, whether it says clockwise or whether it says anti clockwise. So, here in this case it says clockwise and that was the intention of taking this example.

Now, we will see how we can solve this line integral. So, to solve this we just write I equals to integral over the curve c y d x minus of x d y and x is given. So, we can easily calculate d x d t, we just write the limit first from 2 pi to 0 because it is clockwise. Now y is b sin t and when we do d x d t, d x by d t times d t. So, d x d t will be minus of a sin t minus x will be a cos t then b sin the differentiation of sin t would be cos t alright and then we will have d t.

So, what we have done is basically we have done something like this alright. So, that is what we are doing at the moment and then it is integral t running from 2 pi to 0 minus a b minus a b so, that both everything will come outside and then it will be sin square t plus cos square t, d t. So, this is basically minus of a b integral 2 pi to 0 d t. Now as I was saying we can use some integral calculus formula to revert the limit. So, its a to b 0 to 2 pi d t and this is the kind of correction I was talking about. So, although you had higher value to lower value this in the limits at some point you might be able to correct that and this is what we will get. So, the answer is 2 a b pi.

So, that is actually the line integral of this function or of this integrand along this ellipse alright. So, this is what we mean here now we can consider a complicated example, afterwards we will close this line integral chapter.

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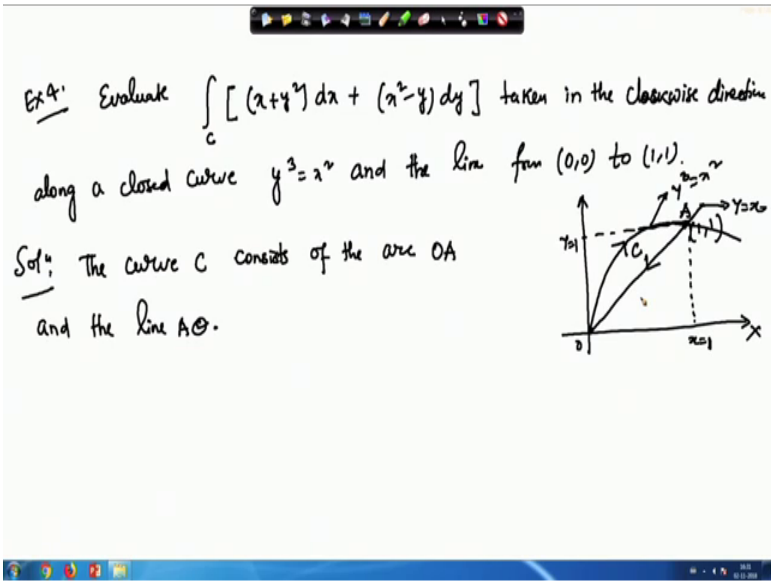

So, the fourth example is, evaluate integral over the curve c x plus y square d x plus x square minus y d y taken in the clockwise direction; in the clockwise direction, along a closed curve y cube equals to x square and the line from 0 0 to 1 1.

So, first of all we have a closed curve between this and this closed curve and the line this. So, let us first draw the curve. So, first of all we have a closed curve between this curve and the line. So, let us draw the line. So, here somewhere we can have the point 1, 1 and y cube equals to x square this curve will be given by something like this. So, you can look into any book how to draw this curve. So, this is our y cube equals to x square and this is our y equals to x and since we are in the clockwise direction. So, if we are in the clockwise direction; that means, we are moving from this direction to this direction and this is our closed curve.

So, this is our closed curve alright. So, the curve c consists of consists of the arc OA. So, let us call this point OA and the line AO and the line AO. So, now, that we have our curve this is our curve c, we will stop here for this problem, we will continue with a similar problem in our next class, where we see how we can evaluate this integral and we will. So, this will involve a slight how to say kind of a trick or conceptual knowledge in a way. So, we will see that in our next class and will stop here today.

Thank you.