Integral and Vector Calculus

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Lecture – 27
Rectification (Contd.)

Hello students. So, in the last class we started with Rectification of a curve which basically means that calculating the arc length. And we worked out few examples we also give looked into several formulas, some of which we are going to implement today and we worked out 1 or 2 examples where we saw if we are given a curve let us say y equals to fx, then in that case how we can calculate the arc length between 2 points x equals to a and x equals to b. So, today we are going to do few more examples and today we will start with a parametric Cartesian a form where we calculate the arc length.

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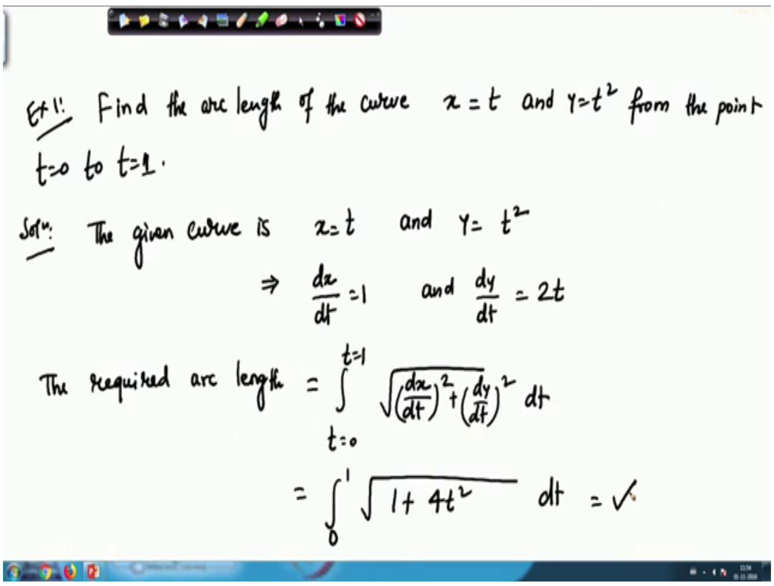

So, let us start with our problem. So, example 1: so, here our problem is find the arc length or length of an the length of the arc its the same thing the arc length of the curve x equals to let us say t and y equals to t square from the point t is equals to 0 to t equals to 1 all right. So, if we are already given the curve and we are also given the point where we need to do the integration, then we need only to draw the curve because we are already provided enough information to perform that calculation of the arc length.

So, here we are given the curve as the given curve is x equals to t and y equals to t square. So, from here we can write dx dt as 1 and dy dt as 2 t. So, therefore, the required arc length is integral t running from 0 to t running to t to 1, because that is the limit for the parameter and then we know the formula is dx by dt whole square plus dy by dt whole square times dt. So, here we have t running from 0 to 1 dx by dt is 1 plus dy by dt is 4 t square dt and now this has become our simple and integral calculus problem. So, we know that from one of those formulas we can be able to write this write this integral into a nice formula here, and then we just substitute t is equals to 0 and t equals to 1 and that will give us the required answer.

So, I will give you that task up to the students, because this is from here it is pretty much straight forward. So, this is how you calculate the arc length if we are given the Cartesian parametric Cartesian system. We can consider an another example of similar source.

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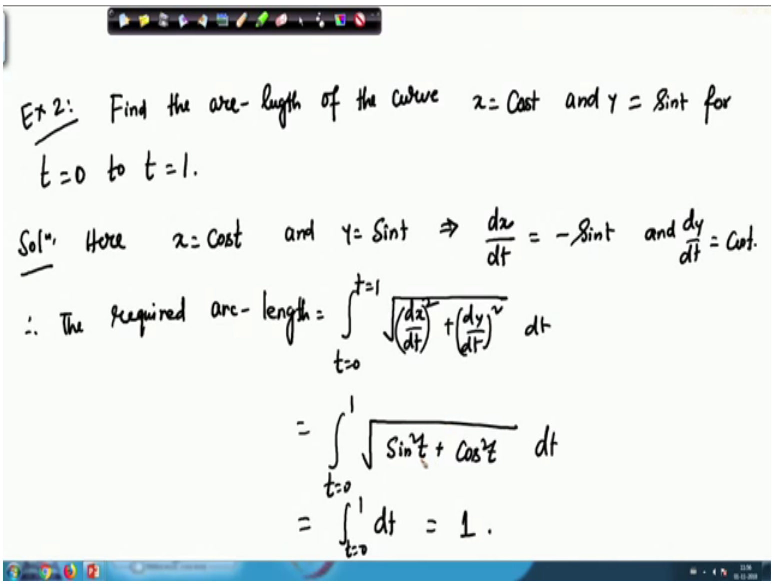

So, example let us say 2. So, find the arc length of the curve let us say x equals to cos t. So, and y equals to sin t for t equals to 0 to t equals to 0 to t equals to 1. So, here also not only that we know this curve, we also have all the vital information to calculate the arc length. So, here we have x equals to cos t and y equals to sine t all right. So, from here we can very easily calculate our dx dt which is basically minus of sin t and our dy dt would be simply cos t and now we are supposed to calculate the arc length.

So, the required arc length is basically integral t running from 0 to 1, and then we have dx dt whole square plus dy dt; it is better if you put it in the bracket times dt. So, this is integral t running from 0 to 1, dx dt whole square is sine square t and dy dt whole square is cos square t. So, this one we can integrate we dont have to put it off. So, sin square t plus cos square t is 1 and there is square root square root of 1 is 1 dt and t. So, basically we will have t at 1.

So, 1 the square root of 1 can also be taken as minus 1 because we have plus minus 1 as the square root of 1, but we cannot take minus here because length is a positive quantity. So, we cannot have minus 1. So, minus 1 does not make any sense. So, that is why we have taken only 1 as the square root of 1 all right and therefore, we got the length as a as a positive number. So, this is a how to say a small essence of mathematics I would say, where you have to be a little bit cautious what you choose as a value and here is the one such example that we cannot take the negative value we always have to take the positive value. So, this is how we calculate the arc length of a curve which is given in parametric Cartesian system and of course, the points are given.

So, the calculation with is fairly simple. Next type of formula is basically of polar coordinate system. So, let us work out few examples based on the polar coordinate system. So, will do we proceed in the following way.

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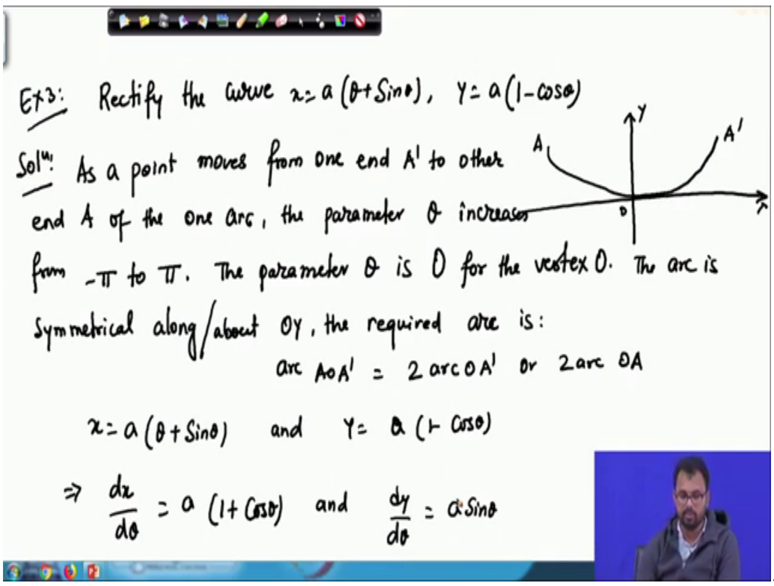

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So, example let us say 3. So, rectify or calculate the arc length whatever. So, rectify the curve x equals to a theta plus sine theta and y equals to a 1 minus cos theta. So, how do we do the rectification?

So, this is our given curve right my figures are little bit how to say upside down mainly because I am not a good draw good painter in a way or good at drawing. So, let us draw this curve. So, it looks something like this. Now about the drawing of a polar of the polar coordinate curves or any Cartesian curves that is a different chapter. So, you may have to look into a chapter called how to draw the chapter on how to draw the curves. So, there are formulas or methods where you learn about how you can draw a curve, but that is not basically our concern here. So, here we will we are assuming that the reader is already familiar with the concepts of how to draw a curve. So, this is how our curve will look like.

So, this is A and this point is A dash all right. So, now, we first have to guess the limit because if we have equation of a curve given in terms of polar coordinate system. So, the rectification formula involves the integration for theta equals to alpha to theta equals to beta; that means, we need to identify these 2 values of theta these 2 angles we need to identify so, that we can do the integration. So, here we will try to do the same thing we will first try to identify the range for theta that is from where to where it is varying alright. So, as a point let us say any point on this arc, as a point moves from one end A dash to other end A of the one arc, the parameter theta increases basically; increases from minus pi to plus pi of course, it will increase from minus pi to plus pi yes and the parameter theta is 0 for the vertex O all right.

So, this is vertex O that is here and 0 is like 0 real number all right. So, as the arc is symmetrical about o y. So, the arc is I did not draw the symmetric figure, but it is symmetrical. So, the arc is symmetrical along O Y or about O Y whichever you prefer to use about or along O Y. The required arc length is basically our arc AOA dash which is 2 times arc, O A dash right or 2 times arc OM.

So, any one of the arc and then you multiply by 2 and that will give you the length of the whole arc. So, we can do the calculation for any one of the let us say sub arcs all right. So, now, we have x equals to a times theta plus sine theta. So, then in that case if we are calculating the calculating the arc length for any one of these sub arcs, we can basically have theta running from 0 to pi all right. So, theta a times theta plus sine theta and we have Y equals to theta a times 1 minus cos theta. So, from here we will have dx d theta equals to a times 1 plus cos theta and d y d theta equals to a times minus cos theta. So, basically sine theta, is not it? So, that is what we get doing the differentiation and now we know our formula where we have to substitute all of these. So, the formula says that.

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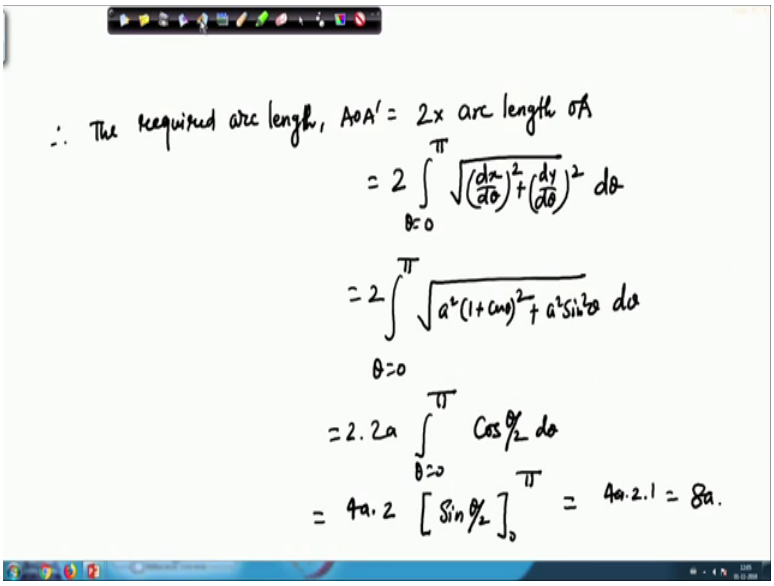

So, the formula says that the required arc length the required arc length AOA dash equals to 2 times the arc length let us say OA. So, then in that case it is 2 times integral from 0 to pi. So, theta is running from 0 to pi a square root of dx d theta from the formula, dy d theta times d theta. Now this is 2 times integral theta running from 0 to pi d x d theta is a square 1 minus what is that? A square 1 plus cos theta; yes a square 1 plus cos theta whole square and this 1 is a square sine square theta. So, if we how to say if we calculate this whole thing, then it will turn out to be 2 a theta running from 0 to pi cos theta by 2. This involves some very simple trigonometrical calculation and I am pretty sure you can be able to do that.

So, ultimately we will be able to obtain to a cos theta by 2 and we have another 2 here. So, this will be 4 a and once we integrate this thing. So, it will reduce to 1 by 2. So, there will be a 2 here and cos theta by 2 will reduce to sine theta by 2 integral from 0 to pi. So, when theta is pi when theta is pi, then in that case this sine pi by 2 and when theta is 0 then sine 0 is 0. So, 4 a times 2 and then sine pi by 2 is 1. So, 1; that means, 8 a. So, that is the required arc length from here to here although we calculated just from here to here by varying the theta from 0 to pi.

So, this is one way where we can calculate the arc length who is how to say polar Cartesian a polar representation is given. Although it is not entirely polar it is actually parametric Cartesian representation because the equations are given in x equals to phi theta and y equals 2 sine theta. So, instead of t they are using theta and the limits for theta, we are calculating basically just looking at the curve. So, it is not entirely a polar coordinate representation its actually a parametric Cartesian representation ah. So, this is one such example where we can use that formula. Now let us actually work out an example where we will use polar for a polar coordinate system. So, let us go to an another example.

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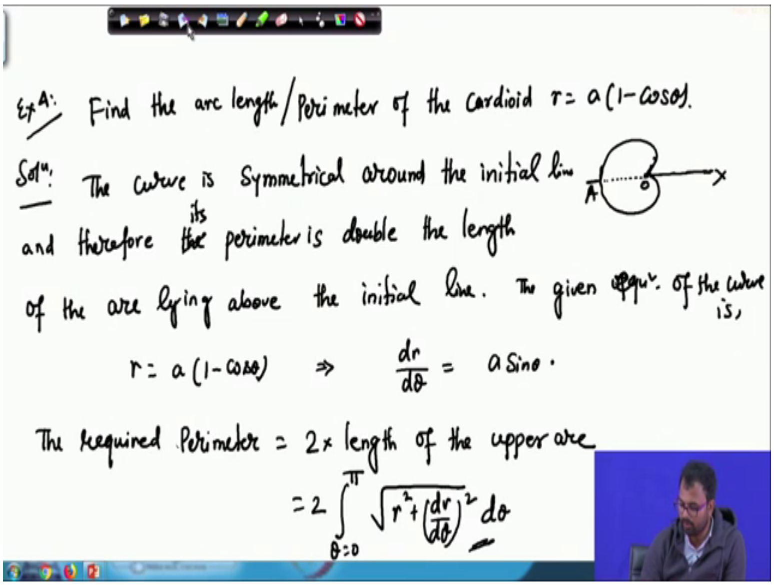

Example I know 4. So, I lost the track. So, it could be example 4. So, find the perimeter or arc length or rectify they are all the same thing

So, find the arc length or let us say perimeter. So, perimeter means actually the whole arc length in a way. So, the entire arc length so, perimeter of the cardioid cardioid r equals to a times 1 minus cos theta. So, the solution; so, how to draw the cardioid that is a different issue and that is a different topic to be studied. So, I recommend to read some books where they have addressed this issue that how you can draw a curve. Now the cardioid will look something like this, this is our A that is 0 all right now this curve. So, this curve is symmetrical around the initial line around the initial line OX initial line ox and therefore, the perimeter is double the length of therefore, its perimeter therefore, its perimeter is double the length of the arc the arc lying above the initial line.

So, we just calculate the arc length above the initial line and we multiply multiplied by 2 and that will give us the perimeter or the entire arc length of the cardioid of the initial line. So, the given equation is r equals to theta times 1 minus cos theta. So, from here we can calculate our dr d theta and dr d theta is a times sine theta its better to write some sentences. So, the given equation the given equation of the curve is of the curve is this one and then we do the differentiation alright. And theta is basically varying from 0 to pi. So, 0 to pi to cover the entire arc length. Now the required arc length is the required arc length or perimeter. So, let us use the term perimeter.

So, the required perimeter, the required perimeter equals to 2 times length of the upper arc length of the upper arc. Now the length of the upper arc will be calculated by theta running from 0 to pi, square root of r square plus dr d theta whole square. This is where we are using the polar coordinate form for calculating the arc length. So, we have used that formula here all right. Now we substitute these 2 values. So, let us substitute the values this will be this will be r square.

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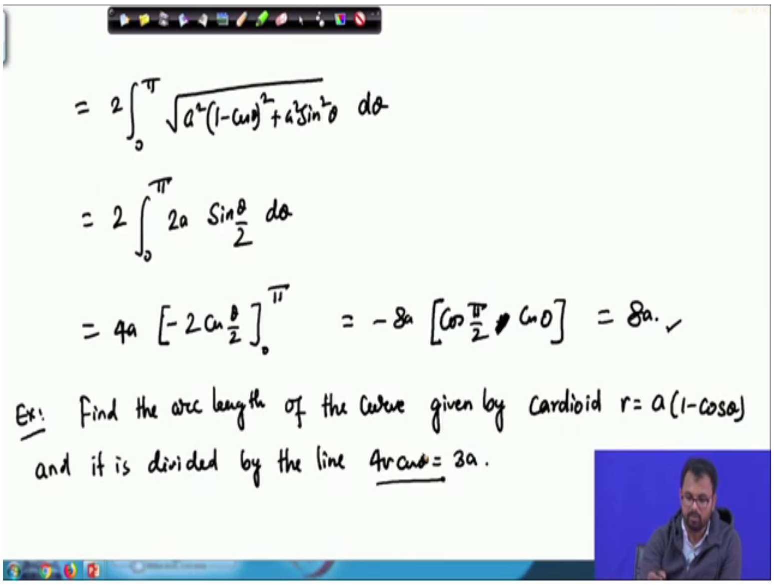

So, in 2 times integral from 0 to pi, we will have we will have a 1 minus. So, we will have a square 1 minus cos theta whole square plus a square sine theta whole square; that means, sine square theta dt theta.

So, from here we will basically obtain 0 to 2 pi, from here we will basically obtain 2 a sine theta by 2. This is a very simple trigonometric calculation which I am leaving up to the students d theta and then this 1 will be 4 a sine theta by 2. So, its integration will be minus 2 cos theta by 2 integral from 0 to pi by 2. So, this will be actually minus of 8 a cos pi by 2 plus cos minus cos 0.

So, cos pi by 2 is 0 and cos 0 is 1. So, this is basically 8 a. So, that is the required arc length or the perimeter of this cardioid this one here. So, here the given equation was actually in polar coordinate system and using this equation, we can be able to. So, using this equation we were able to calculate this arc length here and multiply it by 2, that is here and that will give you the whole parameter. So, that is basically the required perimeter of the cardioid.

So, we can work out an another problem, just to just to get more familiar with th with the with the with this topic actually. So, let me consider an another example. So, another example: so, find the arc length of the curve of the curve given by cardioid r equals to a times 1 minus cos theta and it is find the arc length of the curve given by cardioid and it is divided by the line 4 r cos theta 3 a equals to 3 a. So, this is the required. So, the cardioid basically is divided by the line this one here and then the arc length we have to calculate the arc length of a I mean when this line is dividing the cardioid.

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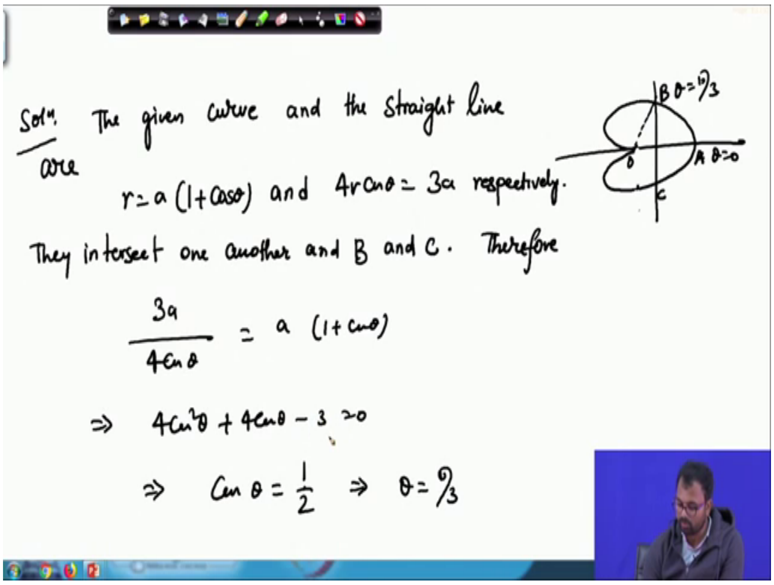

So, if we draw this curve. So, unless we draw this curve we will not be able to understand the problem so, unless we draw this curve. So, let us draw this curve. So, this is the curve. So, that is our origin and this is let us say C B and that is basically theta equals to pi by 3. So, the given curve the given curve and the straight line are the given curve and the straight line r, r equals to a times 1 plus cos theta and 4 r cos theta equals to 3 a respectively. So, basically this is the line and this is the curve and they intersect they intersect they intersect one another at B and C.

So, from here we can there therefore, the point of intersection therefore, the point of intersection can be calculated by substituting r equals to. So, from here we can substitute the value of r here ah. So, that is 3 by 4 cos theta equals to a times 1 plus cos theta and from here we will basically obtain a quadratic equation 4 cos square theta plus 4 cos theta minus 3 and if we solve this equation, then we will basically obtain cos theta equals to half and if cos theta is half then theta is pi by 3.

So, they are intersecting at the point theta equals to pi by 3 and we have to calculate the value of r ya. So, we can substitute for theta equals to pi by 3 here that will give us the value of r and another value of cos theta is ignored because the other value is cos theta equals to minus 3 by 2 we can write the other value.

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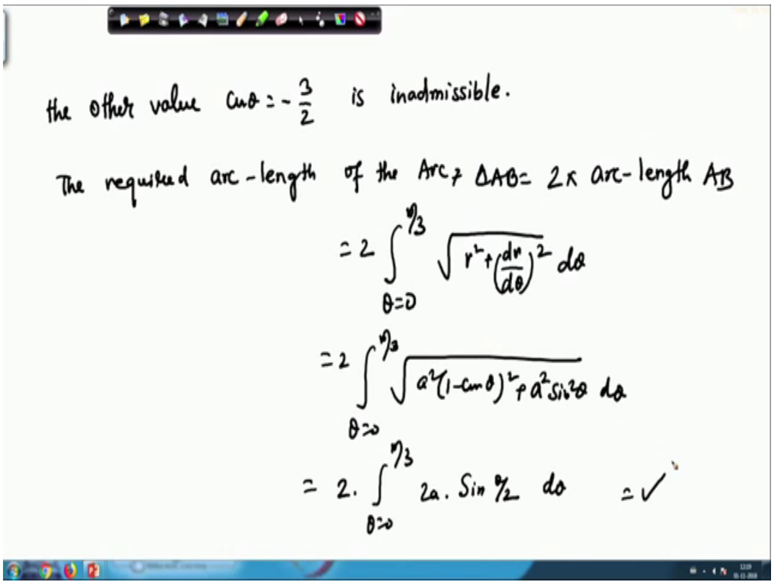

The other value cos theta equals to minus 3 by 2 is inadmissible is inadmissible. So, we have ignored that value and for theta equals to pi by 3 we can also calculate the value of r. So, that is the point where they are intersecting one another and the required arc length of the arc let us say. So, this is the arc for which we have to calculate the arc length.

So, the required arc length ABC we have to write the correct order. So, CAB. So, C A B equals to 2 times arc length what is the arc length then upper half because it is symmetrical. So, the arc length A B. So, since it is symmetrical we just write 2 times arc length A B. So, this is basically 2 times integral theta running from 0 to pi by 3 r square dr d theta whole square times d theta. So, we know that our r square is r square is a square times 1 minus cos theta, whole square and d rd theta is very simple to calculate from here. So, dr d theta is sorry.

So, here it is minus. So, dr d theta is basically a sine theta. So, this is a square sine square theta d theta and if you do the simplification then we will basically obtain here 2 times 2 times theta running from 0 to pi by 3, we will obtain 2 a times I believe sine theta by 2. And we basically do the integration like we did before and then we will get our required answer. So, here we saw that the that we had to calculate the arc length of the curve given by the cardioid and it is divided by the line.

So; that means, that this part of the cardioid this part of the cardioid or this part of the arc we needed to calculate since it is symmetrical. So, we just calculate the one side and multiply it by 2. So, that is what we are doing here and here its fairly just its pretty much just a simple calculation like we did before and that will give us the required arc length between these 2 curves.

So, I tried to cover enough examples from all of these type of coordinate system Cartesian, parametric Cartesian and polar coordinate system and we will continue with our next topic, which is basically surface integral in our next class and I will try to include some examples on rectification in our assignment sheet and I look forward to you in your next class.

Thank you.