Integral and Vector Calculus
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Lecture – 26
Rectification

Hello students. So, in the last class, we looked into the concepts of area of plane regions; where we calculated the area of curve given by the equation either y is equals to fx or r is equals to f theta which basically mean that, we either calculated the area bounded between a curve and two points or the sectorial area where, we also worked out several examples. And I will also try to include some examples in your assignment sheet so, that they can practice them.

Today our topic in agenda is Rectification. So, what do we mean by rectification of a curve and how do we calculate it. So, we will look into that we will look into the definition of a rectification of a curve and then we will work out few examples just to make the concepts clear related to this topic.

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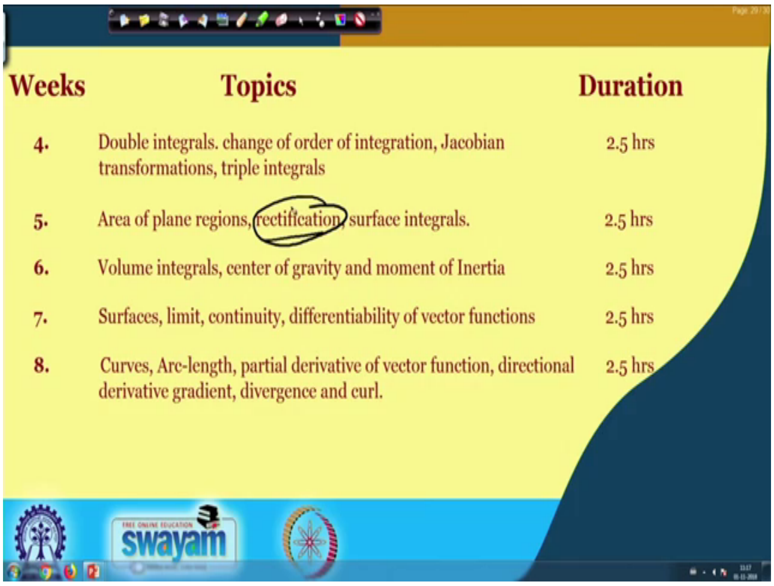

So, let us start. So, today we will start with rectification all right.

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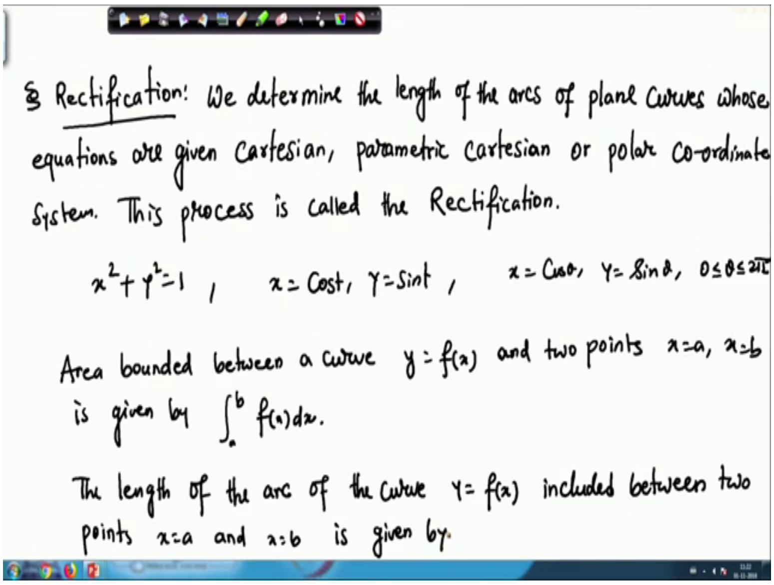

So, let us start; so, rectification alright. So, what do we mean by rectification? Rectification basically mean that, we will determine basically so, we determine the length of the arcs of plane curves of whose equations are given in Cartesian, parametric Cartesian which means x equals to phi t y equals to phi t; so, parametric Cartesian or polar co-ordinate system or polar co-ordinate system alright.

So, this process and this process is called the rectification. So, the process of determining the length of the arcs of plane curve whose the equation is given, either it can be a Cartesian equation. So, something like x square plus y square equals to 1 which is basically the equation of a circle with unit radius or parametric Cartesian.

So, we can write, x equals to cos t and y equals to sine t or polar co-ordinate system. So, for polar coordinate system, we can write x equals to cos theta, y equals to sine theta, where 0 less or equal to theta less or equal to 2 pi. So, any one of them can be considered and this process is called the rectification of a curve.

So, like the way we calculated the area bounded between a curve and two points. So, the area bounded between a curve so, the area bounded between a curve let us say, y equals to f x and two points and two points x equals to a, x equals to b is given by we know that, integral from a to b, y dx and y equals to basically fx dx. So, that is how we calculate the area.

Now; similarly, the rectifier how to say the rectification or the length of the arc can be calculated in the following way. So, the length of the arc of the curve, y equals to fx included between two points say, x equals to a and x equals to b is given by so, this is our x axis, this is our y axis, that is the point let us say, x equals to a that is the point let us say, x equals to b and this is our curve.

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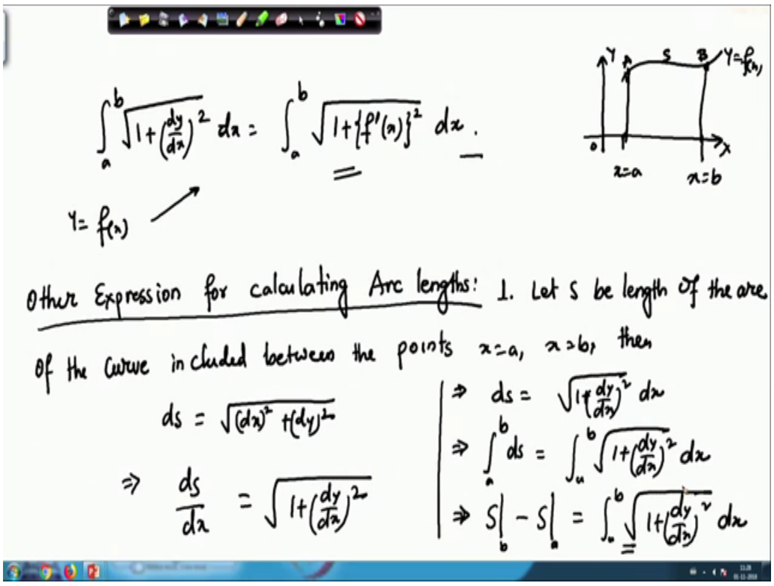

So, we are basically calculating the length of the arc. So, just this length and this length of this arc can be given by integral from a to b, 1 plus integral from a to b square root of 1 plus dy dx whole square dx or we can also write integral from a to b, 1 plus f dash x, whole square dx. So, that is the formula for giving the arc length of this curve y equals to f x.

So, you remember for so, that is x equals to a and for this point, this is x equals to b; so, the area is basically given by integral from a to b, fx dx and the length of the arc will be given by integral from a to b, then square root of 1 plus f dash x whole square dx. So, that is the formula for calculating the arc length. The proof is slightly big and we will ignore the proof here, we are basically interested in the formula and then working out some examples.

So, if we have an equation of a curve given in the Cartesian co-ordinate system; y equals to fx, then the formula to calculate the length of the arc will be given in this fashion. Now, we just studied that the equation cannot be given always in the Cartesian co-ordinate system; they can also be given in the parametric Cartesian or polar coordinate system so, how we can calculate the arc length.

So, there are formulas for such co-ordinate systems as well. So, we write other expression for calculating arc lengths. So, what are those other formulas? So, the first one is let s be the length of the arc; so, this is basically s length of the arc of the curve included between the points, x equals to a, x equals to b, then what we will have? We will have we know that from co-ordinate geometry, that or and also from the differential calculus, that ds equals to square root of d dx square plus dy square; so, this is something we already know.

So, we can write d s by dx equals to 1 plus dy by dx whole square and this can actually be written as so, from here we can also write let us say, d s equals to 1 plus dy dx, whole square times dx. And now, if we integrate from point a to b, d s equals to integral from point a to b, 1 plus dy dx whole square dx.

So, this is actually s at ah, the value of s at b value of s at b minus value of s at a equals to integral from a to b, 1 plus d by dx whole square dx. So, this is basically arc a. So, this is basically our arc of how to say so, if that is the length if that is the length so, here basically the value of s at b will be the whole length and the value of s will be actually at x equals to a is 0.

So, this is basically the arc of let us say, we can call it as a b; so, this is basically our length of the arc a b equals to this formula. So, that is basically this formula here. So, this is equals to basically this formula and we can see that, the calculation for the arc length which is basically s can be given by this formula.

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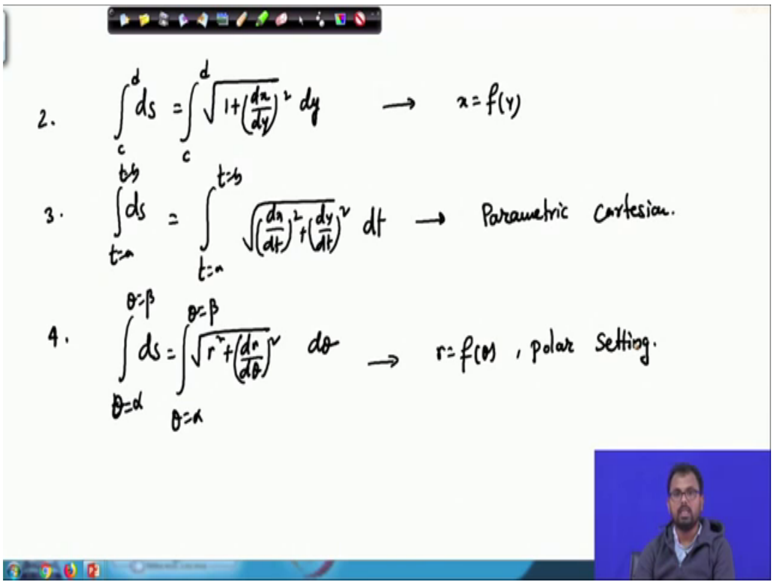

Similarly, if we express here instead of dividing it by dx, we can divide it by dy and then we will still obtain the similar formula. So, the second form will be ds equals to integral from d s equals to 1 plus dx by dy dy and we can integrate from c to d and that will give us the arc length for the curve. We can also have in terms of d s dt so, this will be integral from let us say, t is equals to a to t, is equals to b. This we have to determine what are the limits and in case if the given curve is in parametric form.

So, then in that case, the parametric form will be dx by dt whole square plus dy by dt whole square times dt. So, this is this formula is for the so, this is when the given curve is of x equals to fy form, this is for the parametric, this is for the parametric Cartesian and the last form; the fourth form basically will be for the polar one. So, if we have ds, then we know that ds square equals to r square, d r square ah. So, r dt square plus da square. So, we divide both sides by d s dt theta. So, this will be d theta, whole square.

And then here we will have a d theta and we can integrate for theta running from I do not know, certain point alpha to theta equals to beta let us say, and this will give us the arc length of the curve for the which is given by the equation r is equals to f theta which is actually the polar setting.

So, depending on the type of the equation that is given to us, we can use either one of the formulas. Say, if it is y equals to fx, then we use the formula 1, if it is x equals to f y we use the formula 2, and if it is x equals to some phi t and y is equals to psi t, then we will use this formula and if we have r is equals to f theta; that means, if we have some polar co-ordinate expression, then in that case we will use the last formula. And based on that, we can be able to calculate the length of the arc between two points.

So, earlier we calculated the area between the two points, now we are calculating the length of the arc of which is between two points actually. So, let us work out a few examples so to do some example, let us go to a new page.

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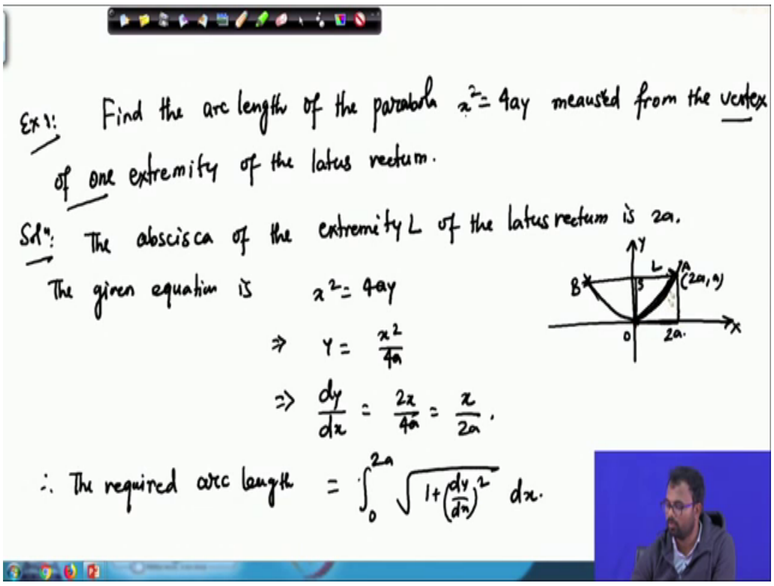

So, suppose our given equation is example 1. So, the problem is find the arc length or length of the arc, it is the same so, arc length of the parabola y is equal to y square equals to, our x square equals to 4 ay. So, let us say we have x square equals to 4 ay measured from the vertex of one extremity of the latus rectum, all right. So, in this case, the abscissa or the x point basically, the abscissa of the extremity L, of the latus rectum is 2a.

So, the latus rectum is basically given by this way. So, this is our x axis, this is our y axis, that is the origin and this is our parabola passing through the origin all right and this is our lattice rectum. So, L and that is 2 a so, this point is a and this is our s. And let us say, we have this point as A and this is our point B all right. So, this is so, we have to calculate the arc length from here to here. So, and this is basically the our latus rectum extremity L of the latus rectum.

So, now, we have to calculate the A and the length of the arc from this point to this point. So, we can see that clearly, that here the required length will be given by so, the required length will be given by because the equation is of type x equals to f y. So, the given equation the given equation is x square equals to 4ay. So, we can write it as y equals to x square by 4a; so, from here, our dy dx will be 2 x by 4a. So, this is basically x by 2 a.

So, the required length so, the required arc length will be from 0 to 2 a; so, integral from 0 to 2 a, 1 plus so, this is basically 2a that is a. So, it is 0 to 2a and that is 1 extremity and this is our arc length. So, 0 to 2 a dy dx measured from the vertex, from the vertex of 1 extremity. So, the vertex from so, this is this is the whole extremity L of the latus rectum. And here we are measuring from the vertex to 1 extremity so; that means, we will calculate basically this arc length all right. So, that is why we are integrating from 0 to 2 a ok.

So, this is this is a hidden part in the question. So, vertex of 1 extremity actually; so, we have to start from the vertex here and then that is the length we need to calculate so from 0 to 2 a. And now, we have dy dx as x by 2a, we substitute there. So, this will be integral from 0 to 2a we will have 1 by 2a, 4a square plus x square dx.

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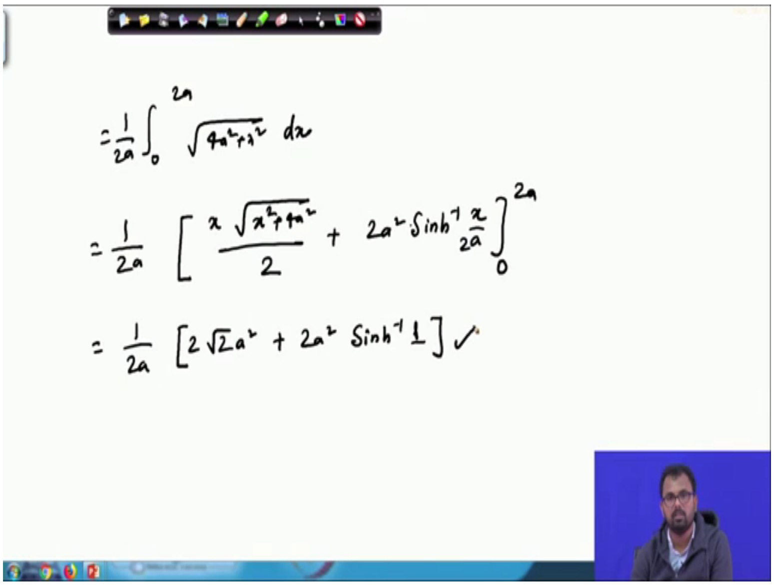

And we can calculate this one; so, this will be 1 by 2a x times, x square plus 4a square by 2, plus 2 a square sine hyperbolic inverse x by a, integral x by 2 a, integral from 0 to 2 m. So, this is some kind of integral calculus formula.

So, I am pretty sure you have done this before so, you just write the write the whole formula here. And when we substitute, the value as x equals to 2a and x equals to a; so, then this will reduce to 1 by 2a, 2 square root of 2a square plus 2 a square sine h inverse one. So, sine h inverse 1, sine h inverse 0 will be 0; and therefore, we will be left with 4a square to cancel yes.

So, therefore, the that will that will basically give us the required how to say the length of the a length of the arc or arc length. You can further simplify if you want here, but we will just how to say leave the answer up to here. So, one can also write the value of sine hyperbolic inverse 1, but this is this will be the answer.

So, you see the given parabola here, was of inverted shape and we were asked to measure the length of the vertex from the vertex to one of the extremities of the length of the latus rectum. So, this is the latus rectum and this point the extremity point is 2a, a and from there we can calculate the length by integrating from 0 to 2a, by integrating from 0 to 2a and we use this formula. So, that will give us the required arc length of the given parabola.

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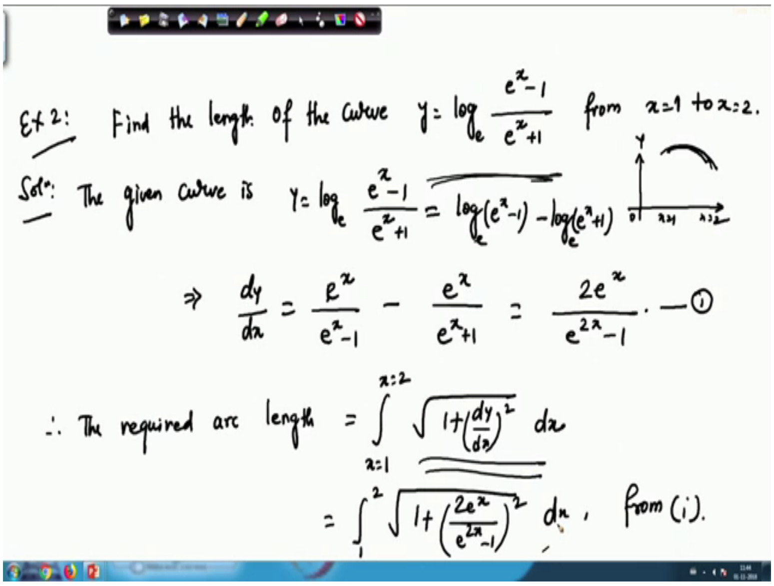

Similarly, we can calculate; we can consider an another example. Example 2 let us say; so, an another example it says, find the find the length of the curve y equals to log of, log to the base e, log of e to the power x minus 1 divided by e to the power x plus 1 from x equals to 1 to x equals to 2.

So, we are here not concerned about the figure because it will take some time to draw the to draw this figure; it is just that we are given the curve and we are also given the points where we have to do the, where we have to calculate the length of this curve. So, this can be a curve of I am not sure, but let us say hypothetically, this can be a curve of something like this and then this is our point x equals to a and this is our point x equals to 1 and this is our point x equals to 2 and we are basically interested in calculating this arc length, all right.

So, to do that, the given curve the given curve is y equals to logarithm e to the power x minus 1, e to the power x plus 1. So, from here we will calculate dy dx equals to so, dy dx will be we can also write this one as log of e to the power x minus 1, minus log of to the power x plus 1.

So, now, doing the derivative would be fairly easy. So, it will be 1 by e to the power x minus 1 and then this one will be e to the power x, minus e to the power x, divided by e to the power x plus 1. So, this whole thing will comprise to 2e to the power x divided by e to the power 2 x minus 1 all right; so, that is the derivative.

Now, the required arc length; now the required arc length. So, this will be from our known formula, we have y is equals to fx type situation a situation and if we have y is equals to fx type situation, then the arc length is x running from a to b. So, our a is 1, b is 2 plus a square root of 1 plus dy dx whole square dx, dy dx we have already calculated in equation 1 let us say this is our equation 1. So, from 1, we will have 1 to 2, 1 plus 2 e to the power x divided by e to the power 2 x minus 1 whole square dx. We can write it as from 1 all right.

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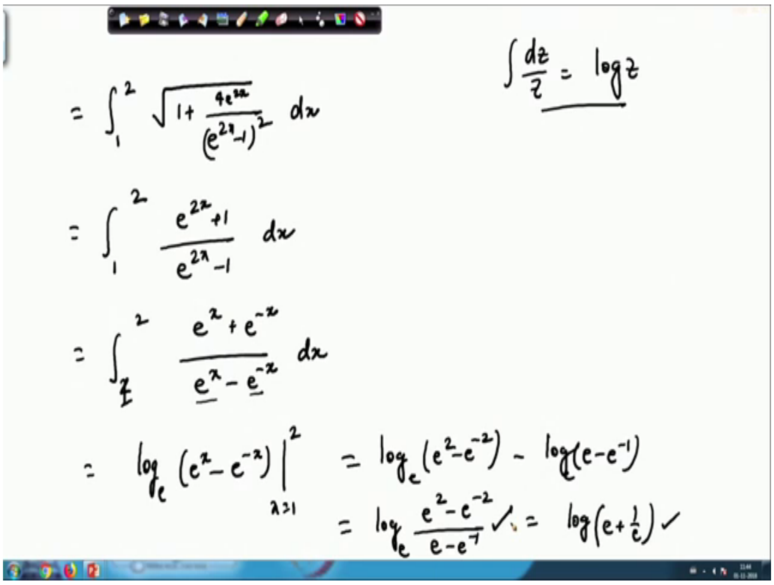

So, if we if we do the square here and then this will reduce to this will reduce to integral from 1 to 2, 1 plus 4e to the power 2 x, e to the power 2 x, minus 1 whole square dx and this can be simplified to integral from 1 to 2 e to the power 2 x plus 1, e to the power 2 x minus 1 dx and this can be further simplified to e to the power x plus e to the power minus x, divided by e to the power x, minus e to the power minus x dx. We are just dividing by e to the power x or we are multiplying both numerator and numerator and denominator by e to the power minus x.

Now, we see that this term here is basically the derivative of this term. So, if we substitute this one equals to z, sorry this is 1, if we substitute this one equals to z, then this will be the derivative; this term will be the derivative of this one.

So, we can simply write, we know that if we have something like dz by z; so, we write log z right. So, that is the formula we are using here. So, log of e to the power x minus e to the power minus x, x is at 1 and 2. So, this will reduce to log of log of e to the power 2 minus, e to the power minus 2, minus log of e minus log of e minus 1 both will have base e.

So, this is nothing but, log of e to the power 2 minus, e to the power minus 2, e minus e minus 1 and if we multiply both numerator and denominator by e to the power let us say 2, then in that case this will reduce to log of, log if we multiply both numerator and denominator by 2, then this will reduce to e to the power 2, e to the power minus 1. So, this will reduce to log of e plus, 1 plus e.

So, this is the required answer; I mean we can leave up to here, it is not a problem and that is how we calculate basically the arc length for this given curve. So, you see you just have to guess the, you just have to guess the form first of all. So, whether it is given as y equals to fx x equals to fy x equals to phi t or psi t or a polar system. And from there you have to find out the points where you need to you need to the points where you need to calculate the length.

So, once you have these ingredients you pretty much know what kind of formula we have to use so, the in this case I calculated dy dx, because the curve is already given in terms of y equals to fx form and from there, this is our required formula to calculate the arc length, we substituted every value and then we just did some integration.

Now, this is just a calculation part, you can leave your answer up to here and that is still correct. Simplifying it to a certain form that is not our concern here, how we calculate the arc length that is our concern. So, this is how you calculate the arc length.

So, we will stop today's class here and in the next class we will start working on some few examples which might also include parametric Cartesian or polar coordinate system and we calculate the length of those arcs. So, thank you for attention and I look forward to you in the next class.