**Integral and Vector Calculus**
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**Lecture – 25**
**Area of Plane Region (Contd.)**

Hello students. So, in the last class we looked into the concepts of how calculating area on Area of Plane Regions where we have also looked into the formula how do we calculate the area which is bounded between 2 different curves, we also worked out few examples. Now in area of plane regions there are several other problems which we can actually work out with and today we will look into some more examples where to just to make the concepts of this area of plane regions a bit clear. So, just as a recapitulation we will start with an another example.

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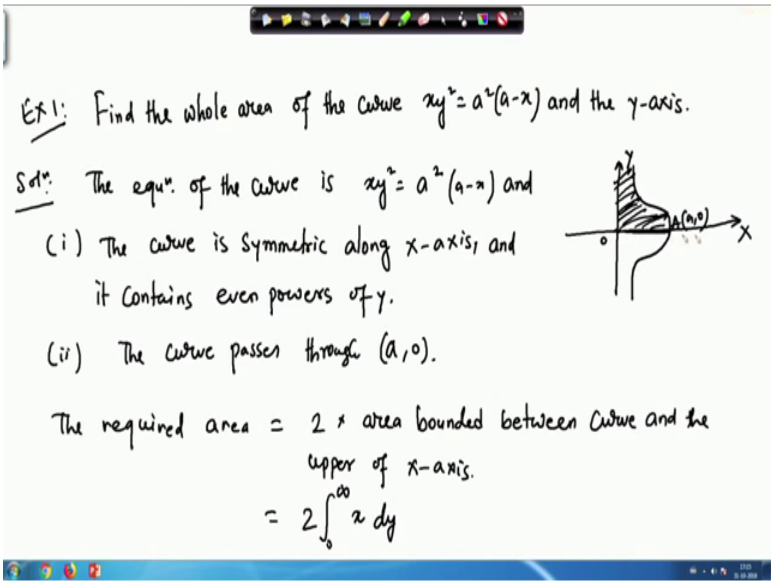


So, let us consider an another example. So, let us say example 1 for today and this example says that find the whole area of the curve xy square equals to a square a minus x and the y axis and the y axis. So, here in this case the curve is the given curve here is basically ah is this one and we have to calculate the area between this curve and the whole of y axis. So, this curve would more or less will look like something of this type.

So, it will look like this is x axis and this is y axis and then it would look somewhat of this type. So, this is our y axis and we have to calculate the area between this curve and the entire y axis between this curve and this entire y axis. So, of course, this curve is symmetric at the point along the x axis. So, it is a so, what we have to do? We have to calculate the area of this region up to y axis and then we just multiply it by 2. So, that will give us the area between this curve and the y axis.

So, there are certain properties we can write. Say the curve the equation of the curve is xy square equals to a square times a minus x and first property is the curve is and the first property is it is the curve is symmetric along or about x axis and it contains even power of y. The curve passes through what is the point that is that it is passing through it is passing through a comma 0 and we so, these are the 2 properties of this curve. So, then to calculate the area, so, the required area is equal to 2 times area bounded between the curve and the upper half of x axis. So, this is this area basically.

So, that is the area bounded between this curve and the upper half of x axis and we can calculate this one in the following way. So, in this case since we are calculating from how to say from 0 to infinity in a way because y is varying from as x is varying from actually 0 to infinity because we are going along the x axis; I sorry we are going along the y axis. So, the equation of y axis is x and y axis is x equal to 0 and here we have x will go until infinity. So, we can write integral from 0 to infinity x dy.

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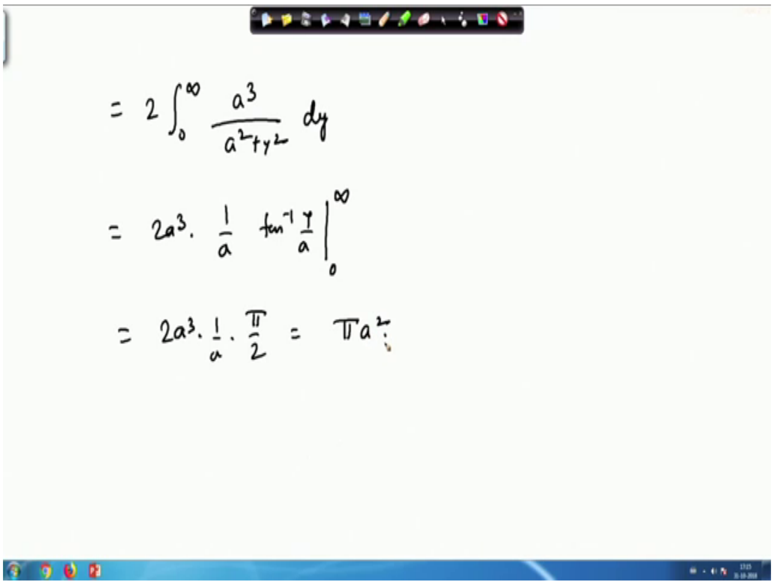


And this can be written as 2 times integral from 0 to infinity x value can be calculated as a cube by a square plus y square and from here we can write dy. Now from here this will reduce to 2 a cube times 1 by a and then here we will have tan inverse y by a where y is running from 0 to infinity now when tan inverse is infinity then this value will be pi by 2 and when tan inverse 0 then in that case the value will be 0. So, we will have 1 by a tan inverse infinity.

So, that is pi by 2. So, the value will be pi a square. So, this is the required area that is being bounded between this curve. So, we can write this point as a 0. So, between this curve and the origin and to calculate that we just have to calculate the upper half and multiply it by 2 because this curve is symmetric along the x axis. So, this is another example where we calculated the area of a plane region bounded between 2 curves. We can consider an another example.

So, to talk about an another example we start with let us say a curve like this; so, example 2.

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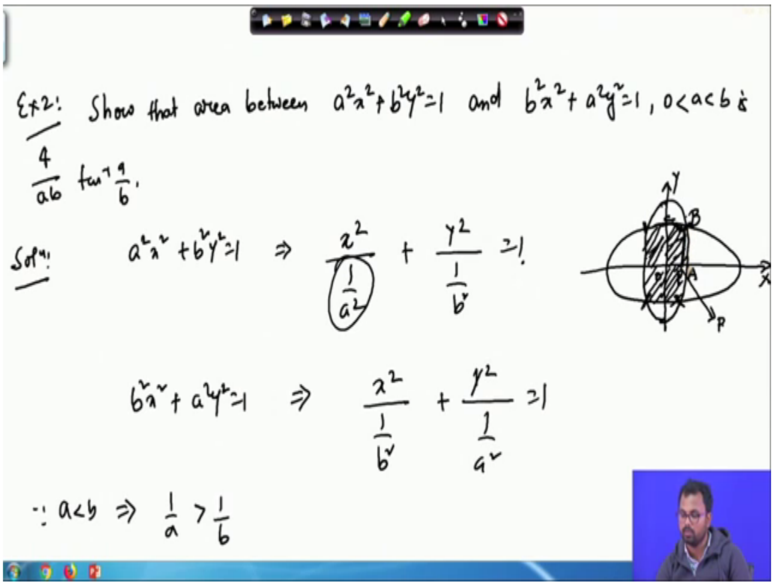


So, this that show that the area between a square x square plus b square y square equals to 1 and b square x square plus a square y square equals to 1 where 0 less than a less than b is 4 by ab tan inverse a by b. So, here we have to be a little bit careful. So, it is not that; so, first of all we have to see what kind of curve they are.

So, we can see clearly that we if we divide both sides by a square by b square a square b square if we divide both sides by a square b square then this will be x square by b square plus y square by a square equals to 1 by a square b square and similarly this will be x square by a square plus the y square by b square equals to 1 by a square b square. So, then in that case it is basically an equation of an ellipse actually. So, this is our equation of an ellipse and we can actually write instead, in fact, we don't have to multiply anything with we can write x square by 1 by a square and y square by 1 by b square and that will be a sufficient.

So, we don't have to divide anything. So, it is pretty much obvious that if we can write it as a square x square plus b square y square equals to 1 then from here it will be x square by 1 by a square plus y square by 1 by b square equals to one. So, this is basically this will basically be our how to say square of the length of the semi-major axis and square of the length of the semi minor axis. Similarly, we can write this one as so, this is our another equation of an another ellipse. Now we note that a is less than b.

So, since a is less than b from here it implies that 1 by a is greater than 1 by b. So, then in that case in the first case now we are ready to draw our area bounded between these 2 curves. So, this is our x axis this is our y axis that is the origin. So, if 1 by a is greater than 1 by b then in that case this will act as the semi major this will act as the length of the square of semi-major axis and this will act as the semi minor axis. So, our first ellipse will ellipse will look like this and since 1 by a square is greater than 1 by b square, so, this ellipse will be inverted. So, something like; so, something like this.

So, the area bounded between them is this much and since they contain x square and y square; that means, they are symmetrical in a way. So, it basically means that we have to calculate the area of any one of these parts and then multiply it by 4. So, we don't have to calculate the area of all these sections altogether, we just have to calculate the area of any one of these small how to say sections and then that will give a multiplied by 4 and then it will give the area total area bounded by these 2 ellipse. So, we can call it as A B C. So, o a b c area of o a b c will be the multiplied by 4 will be the area of the entire region.

So to calculate that ah we can calculate this point of intersection and if we calculate that point of intersection, so, what we basically do? We substitute the value of x square from here to here and we calculate the value of y and similarly we substitute the value of y then we get the value of x.

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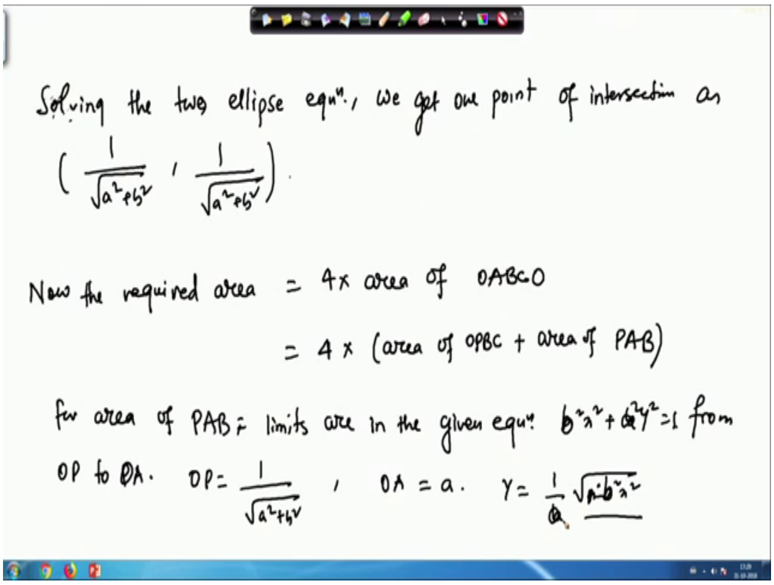


So that means, the point of intersection the so, we get first point of intersection. So, we get solving the 2 ellipse equation solving the 2 ellipse equation 2 ellipse equation we get one point of intersection as 1 by square root of a square plus b square and 1 by square root of a square plus b square.

So, these are the 2 point of intersection and these are the one point of intersection up there of these 2 ellipse. Similarly, we will get the point of intersection of this point of this point, point of intersection of this point and point of intersection of this point and it will surely will give you if you solve these 2 equations it will surely give you the 4 different point of intersection because once you are substituting the value of x here then in that case you are getting the 2 roots of y then 2 roots of y; that means, you are getting 2 values of y positive and negative.

Similarly for each value of y then we will get 2 values of x and then we have basically 4 combinations.

So, x and y both positive, x is negative y is positive then x and y both negative and then x is negative y is positive. So, we will basically get 4 different types of combination here; that means, 4 different types of point of intersection all right. So, we are just focused in this first quadrant because we will multiply the area of the first quadrant by 4 ok.

So, now, that we have the point of intersection we will write now the required area now the required area is equals to 4 times area of I have named it OABC or so, let me say it OABCO. So, that is the area and area of OABCO is basically 4 times area of if we call this point of intersection as B, if we call this point of intersection let us say B then in that case it will be it will be so, then in that case it will be area let us say of this curve and then this curve.

So, suppose this point is P, I am calling this point as P so; that means, we are basically taking the area of the entire region as the area of 2 different area bounded by these 2 curves because this is the first curve and then this is the second curve. So that means, we will integrate first from here to here and then here to here. So, 2 x points; so that means, we will get basically area of what is that? Area of OPBC; so OPBC plus area of PAB OABC and then area of PAB, so, area of PAB right. And for area of PAB we have to now we will calculate the area individually. So, for area of PAB here this is the part of so, PAB is basically this 1 here and this is the part of our inverted ellipse and from here we have to calculate the value of y actually.

So, if we calculate the value of y then that will be so, for the area of PAB the limits are given in the equation limits are in the given equation. For area of PAB limits are in the given equation from; so, what are the limits? So, the limits are from this point to this point; that means, this point is basically A and this point is basically OP is basically a 1 by a square plus b square. So, from O P to PA right; so, from here to here and then here to here. So, the limit will be from OM to OC and so, this one is basically O O P to O A. So, the limits are basically OP to OA and OP is basically 1 by square root of a square plus b square and OA is basically our a and from here we can solve and from here we can solve y. So, y is basically 1 by b is 1 minus a square x square. So, this is our y actually. So, now, that we have y and we have our limits, so, we can calculate the area of the region PAB.

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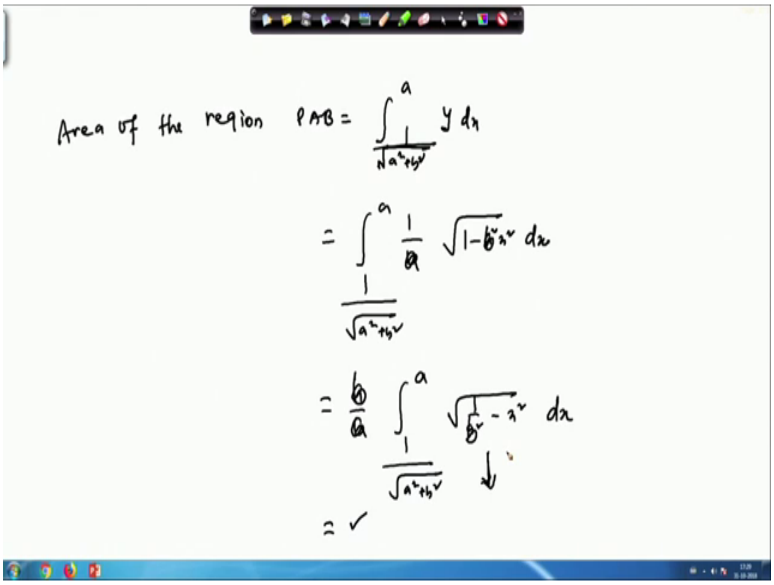


So, the area of the region PAB is basically integral from integral from a square 1 by a square plus b square to a y dx. So, here we will have 1 by a square plus b square to a and our y is this one; 1 minus a square x square dx and similarly the area of this region, similarly we will calculate the area of this region by integrating from 0 to 1 by a square plus b square and then we will take the curve y from here and now we have to find the integration of this thing here. So, we have to calculate the integral. So, this is pretty straightforward. So, we will take out a square from here.

So, then it will be if I take a square from there then it will be integral 1 by square root of a square plus b square to a and it will be 1 by a square minus x square and then we will use the known formula of integral calculus and if we integrate this then this will actually reduce to something called then this will actually reduce to and the first part will reduce to sin into x x times a square 1 by a square minus x square by a plus sin inverse x by 1 by a. So, this can be evaluated very easily and similarly we can evaluate the area of this part as I showed you that we have to now integrate from the origin to this point and this point is 1 by a square plus b square and then we will use this curve.

So, the value will come from here. And in this case now in this case it is b square x square and then a square y square. So, this is 1 by b square x square and we will divide it by a and sorry so, this one is b square x square and we have divided by a. So, this is b by a and this here is b square. So, ultimately we will integrate with respect to x here and then that will give us the required value of the integral and similarly we will integrate this part and that will give us the that will give us the area of this part and then we will sum them. So, then it will give the area of the whole ellipse of this section and then we will multiply by 4.

So, when we multiply by 4 that will be the required area of the whole region bounded by the ellipse.

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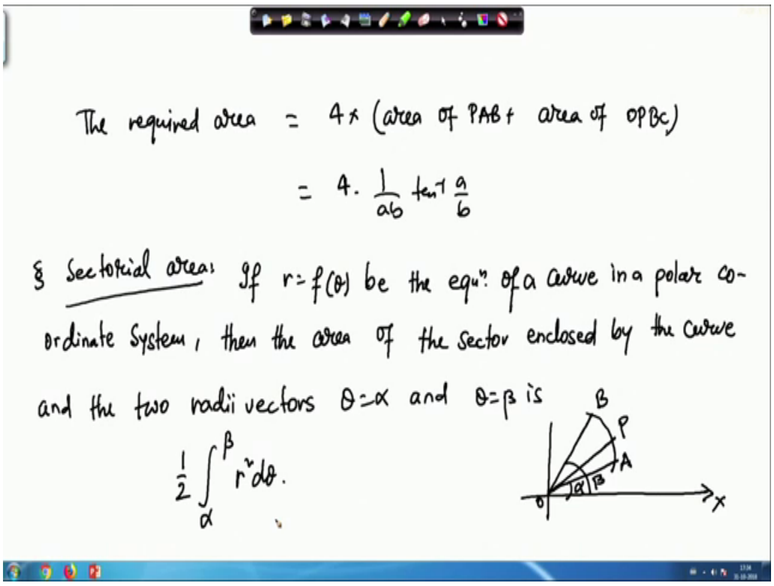


So, the required area would be 4 times area of PAB plus what was that another one was area of OPBC area of OPBC and that will give us actually 4 times. So, this will come out to be 1 by a b tan inverse a by b. So, it is a bit lengthy example ah, but I am pretty sure you got the idea what I was trying to say. So, this is an another way to calculate the area of 2 of the of the region bounded by 2 plane curves.

So, here in this case the 2 curves were ellipse and you just have to in always remember that in this case in these type of examples you always have to draw the figure first. So, if you can be able to draw the figure correctly then first you have to notice where are they intersecting. If they are intersecting then calculate the point of intersection or at first you can solve them and then you will get the point of intersection.

And once you get the point of intersection you sort of can draw the figure and if you draw the figure then you will be able to see that whether you need to calculate the area only above the one curve or you have to involve both the curves and if you are involving the both the curves and you have to guess the limits correctly like we did in this case and once you have the limits correctly then calculating the am area would be pretty much straight forward all right.

So, next ah point we have in agenda is sectoral a sectorial area; so, sectorial area. This is motivated from a polar coordinate system. So, sectoral a sectorial area means if r equals to f theta be the equation of a curve in a polar coordinate system in a polar coordinate in a polar coordinate system, then the area of the sector enclosed by the curve and the two radial vectors or two radii sometimes they are also called as radial vectors theta equals to alpha and theta equals to beta is 1 by 2 integral from alpha to beta r squared d theta.

We will not go into the proof, but what we mean is. So, suppose this is our x axis in a way so, this is our x axis and suppose this is a given curve, now this point here is the point P and this is our point A and that is our B.

So, this angle XOA is basically angle alpha and XOB is basically our angle beta. So, that sector sectorial area means the area bounded between OA and OB and this curve. So, this area and this radii vector this radial vector and the curve so, that particular area so, this area is basically called as the sectorial area and it is given by 1 by 2 integral from alpha to beta r squared d theta. This is very useful when a curve is given in the polar coordinate system.

We will see that what do we mean by it. So, if a curve is given in a polar coordinate system, we can calculate the sectorial area using this formula.

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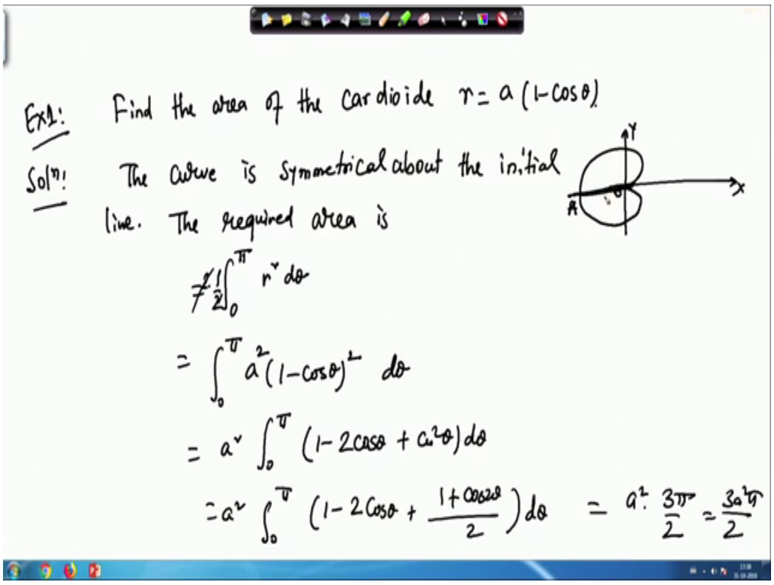


So, let us see an example. So, let us say we have find the area of the cardioide cardioide r equals to a times 1 minus cos theta. So, the curve first of all if we draw the I say a cardioide. So, I am pretty sure you may have seen the figure for the cardioide before. So, it looks something like this all right. So, that is the origin. Let us say this is the point A and that is how it looks like and it is symmetrical the curve is symmetrical about the initial line. So, the initial line is basically theta equals to 0. So, if theta is 0 then in that case we have cos 0 as 1 and this 1 will be 1 minus 1 0 so, r is also 0.

So, that means, that this is our initial line where theta is 0 and the curve is actually symmetrical. So, the required area will then be just theta from 0 to pi because it is symmetrical along this initial line. So, we just multiply by multiply it by 2 and that will give us the area of this entire cardioide. So, the required area is equal to integral from 0 to pi r square d theta and since the formula is 1 by 2 we write 1 by 2 and since the a required area is 2 times of the area of any one of these sections; so, we have to also write multiplied by 2 here.

So, then in that case this 2 and this 2 will get cancelled and we will have 0 to pi r square d theta. Now r is a times 1 minus cos theta whole square. So, here we will have a square here we will have a square d theta. So, this will turn out to be integral from 0 to pi 1 minus 2 cos theta plus cos square theta d theta.

Now I can write 1 by 2 here and then this will reduce to 1 minus 2 cos theta plus 1 plus cos 2 theta y 2. Now evaluating this integral is fairly easy, I am pretty sure you can be able to evaluate. So, this will become theta this will become 2 minus 2 sin theta this will become theta by 2 plus 1 by 2 times integral of this 1 will be sin 2 theta by 2 and then you substitute the value of theta.

So, ultimately you will be able to obtain a square times 3 pi by 2. So, this is basically 3 a square pi by 2. So, that is the area bounded by this cardioide between I mean in this region actually. So, you see instead of converting this into a Cartesian coordinate system and then trying to draw this curve and also finding out the limits and all we just had to remember the shape of this curve and we from the shape we can see that the curve is symmetrical about the initial line and then we can calculate the area or the sectorial area in this sense by just writing the limit 0 to pi and then multiplying it by 2 so, that will have the entire area and then you just do some simple calculation and that will give you the required answer. So, this is a very interesting example where we actually did not have to take help of the Cartesian coordinate system. We can consider an another example.

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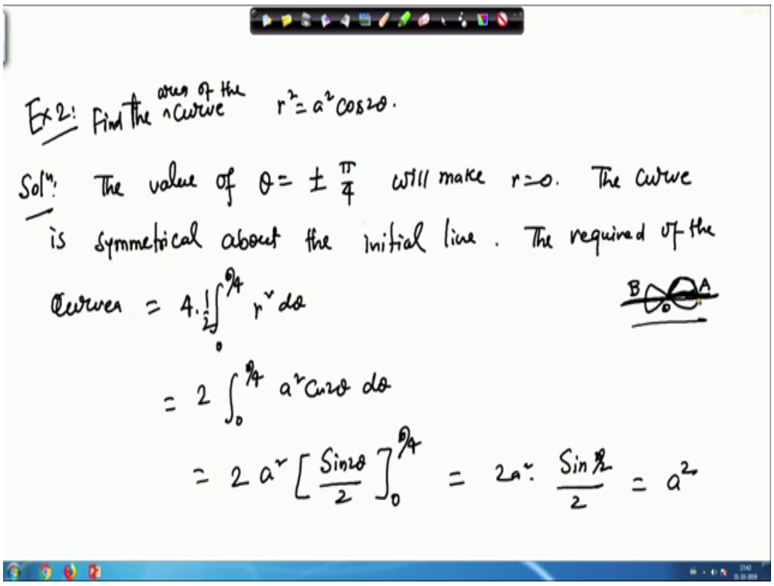


So, let us see. So, find the area of the curve r square equals to a square cos 2 theta. So, the value of theta equals 2 plus minus pi by 4 will make r equals to 0. So, for theta equals to minus pi by 4 it will be 0 r will be 0 and theta equals 2 plus pi over 4 r will be 0.

So, the curve is symmetrical about the initial line and the required area then the required area of the curve sorry of the curve is equals to 4 times because this curve is will look like something of this type. So, this is our initial line. So, that is the point A let us say this is point B this is point and this is point A this is point B. So, we are basically calculating this area and then we are multiplying it by 4. So, 4 times integral from pi 0 to pi by 4 r square d theta and then here we will have 1 by 2.

So, this is basically 2 times integral from 0 to pi by 4 r square is a square cos 2 theta d theta and then this 1 is 2 times if we integrate then this will be a square and then sin 2 theta 0 to pi by 4 and then this will be 2 a square sin pi by 2 by 2. So, that means, this will be ultimately a square yes. So, the required area of this, this is also called as a Lemniscate of Bernoulli and the curve actually looks like this.

So, in order to calculate the area we have to see for what value of theta this is becoming how to say 0 r becoming 0 and that will actually give us the limit from where to where we have to integrate and the area of the entire curve will be given by 4 times the area of these small sections. So, 4 times the area of the entire sector entire oh sorry not entire there of this region is 4 times this region and then the area of the entire Lemniscate of Bernoulli will be a square. So, this is another way where you can calculate the area of a curve which is given in terms of polar coordinates. So, we will stop here for today and we will continue with our further examples in our next lecture.