Integral and Vector Calculus
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Lecture – 24
Area of Plane Region

Hello students. So, upon till last class we looked into the triple integral, we also tried to worked out few examples related to triple integral and how one can basically how to say evaluate these triple integral on a bounded region in 3 dimensional geometry. So, now, that we have covered our double and triple integral, we will go to the application part of integral calculus, where we calculate the area of plane regions. So, we do some rectification, we calculate surface integral, volume integral and things like that.

So, today we will start with area of plane region. So, this is basically our. So, this is basically our new topic today, we will start with our new topic and excuse me.

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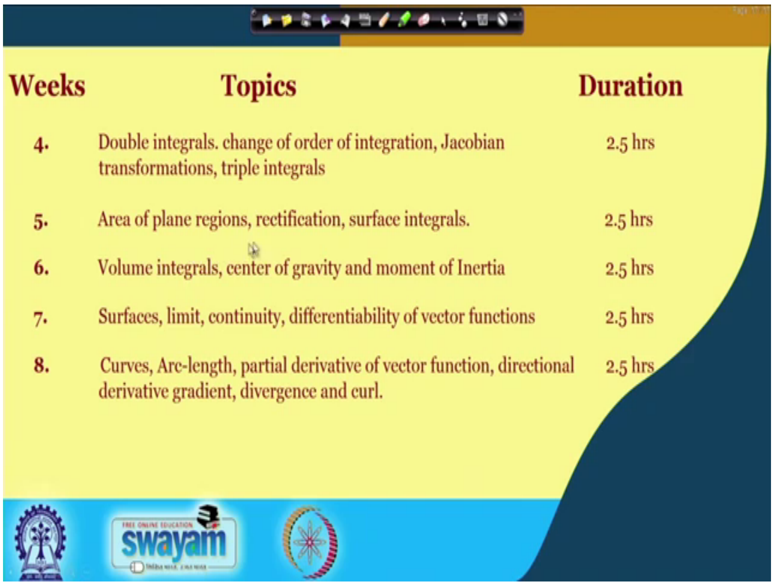

So, this is basically our the topic our require how to say today's topic which we start with. So, let us let us go to the introduction part. So, what do we mean by area bounded between a curve bounded between two points and a curve? So, we know that in our unusual integral calculations when we say that the area bounded between two points.

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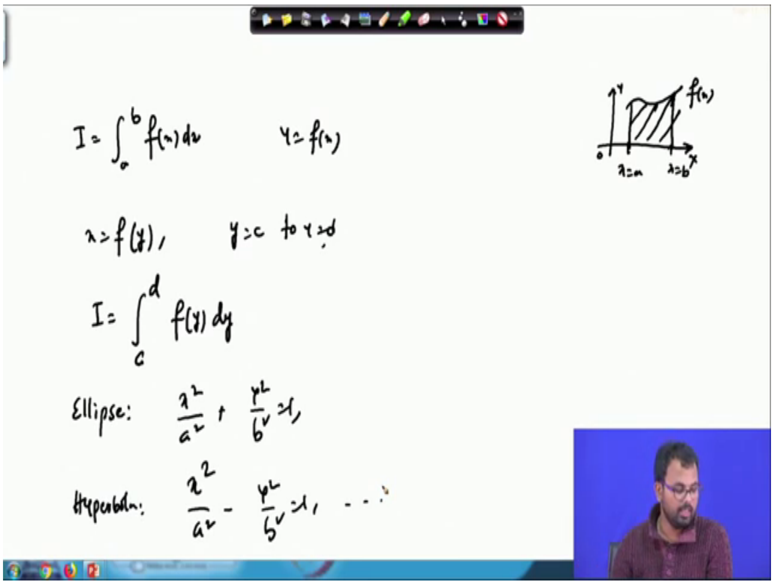

Let us say x equals to. So, this is our y axis this is our x.

So, let us say x equals to a and x equals to b and if this is a curve then let us say fx then the area bounded between these two points and the curve is basically this area here and it can be given by this integral a to b fx dx. So, that is our usual definite integral formula for the area bounded between two points and the curve. And similarly if we have a curve of let us say this type. So, this one was the curve of y equals to fx, now suppose if you have a curve of this type x equals to f y and if we if you want to calculate the area bounded between the points let us say y equals to c to y equals to d, then in that case your da your area bounded between these two points and the curve can be given by integral from c to d f y d y.

So, regardless whether you are integrating with respect to x or integrating with respect to y, it just depends on the context what kind of curve you are given and what type of points you are given all right. So, this is just to how to say a definition, where we calculate the area between two points and a curve. Now suppose you are given curves like. So, we are we might be given curves like ellipse, let us say x square by a square plus y square by b square equals to 1 or we can be given a hyperbola. So, this is basically a hyperbola x square by a square minus y square by b square equals to 1 and similarly several other curve.

So, what kind of area is bounded by this curve here and between two points? Let us say x equals to given two points actually or for this hyperbola. So, we can be given any curve and some points where we from using that we can be able to calculate the area bounded between the points and the given curve. So, that is what we basically mean by areas of plane regions actually. So, we will perform we will do several examples, where we calculate the how to say area bounded between these curves and the points and we will see how much is that area basically.

So, for example, area bounded between an ellipse. So, inside the ellipse, the area of bounded by this ellipse is actually phi a b. So, we will see that how we can calculate that.

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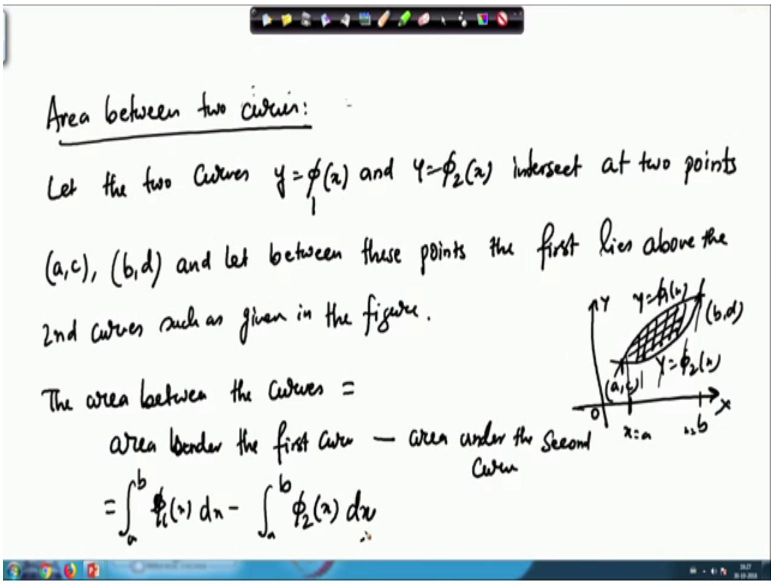

Now we can also calculate area between two closed curves, between two closed curves or simply curves they do not need to be closed. So, let us say two curves. So, between two curves; so, let the two curves, I have to write the definition. So, let two curves y equals to phi x and phi 1 x and y equals to phi 2 x intersect at two points at two points say intersect at two points say a to c ac and b d.

So, they are intersecting at two points and let the and let between these points, the first curve lies above the second curve above the second curve. So; that means, we have x axis we have y axis. So, that is my x equals to a this is my x equals to b and suppose we have a curve like this and then we have a curve like this. So, this point here is basically b comma d and this point here is basically a comma c and this is the. So, this is basically y equals to 5 1 x and this is y equals to 5 2 x and that is the area bounded between these two curves all right.

So, if this is the area bounded between these two curves, then it can be given by. So, such as such as this, such as given in the figure in the figure then the area the area between the curves is equals to area under the first curve under the first curve. So, we first calculate the area under the first curve and then we subtract the area under the second curve and that will give us the area between them.

So, this is area under the second curve right. So, how do we calculate the area under the first curve? So, it is between these two points and then phi 1 x and the area under the second curve is again between these two points and then phi 2 x dx all right and then we just calculate the value of the individual sub integrals and subtract them and that will give us the area bounded between these two curves.

So, that is another way to calculate the areas between these two curves. So, now, let us consider an example and see how we actually calculated.

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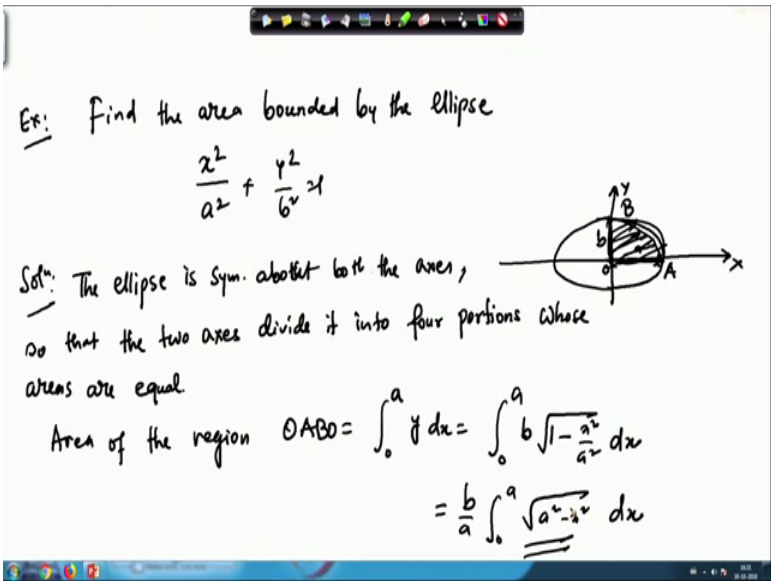

So, example 1, let us say find the area find the area bounded by the ellipse x square by a square plus y square by b square equals to 1. So, we know that if we substitute a equals to b, then an ellipse reduces to a circle and for a circle the area is phi a square; however, what can be the area for an ellipse? As I told you that the answer is phi a b, but how do we actually calculate it we will see just now. So, first of all let us draw an ellipse.

So, this is our x axis, this is our y axis and if we draw the ellipse, as I told you I am really not good at drawings. So, this is our ellipse, that is basically the semi major axis this is semi minor axis or this length is a this length is b. Now we know that an ellipse is a symmetric along both the axises.

So, if we can be able to calculate if we can be able to calculate the area in any one of these quadrants, then we just multiplied by 4. So, whatever the area is here it will be the similar area here, it will be similar area here and then it will be similar area here. So, we do not have to calculate over the entire ellipse we just calculate in for one of these portions of the ellipse and then we just multiplied by 4 all right. So, let us see how we can do that. We can write it that the ellipse is symmetrical about both the axes about both the axes both the about both the axes or coordinate axes so, that.

So, that the two axes divide it into 4 portions 4 portions, whose areas are equal whose areas are equal. So, now, we will basically integrate this region. So, we will basically integrate the area bounded between this curve and these two points. So, x equals to 0 to x equals to 1. So, the area let us say this is O A and B; so, the area of the region. So, area of the region OABO is equals to integral from 0 to a y dx and our y is basically b times one a square root of 1 minus x square a square dx right. So, this will be b by a integral from 0 to a square root of a square minus x square dx. Now this we know that from our usual integral calculus formula that is this will somehow be I mean the answer will involve sin inverse x.

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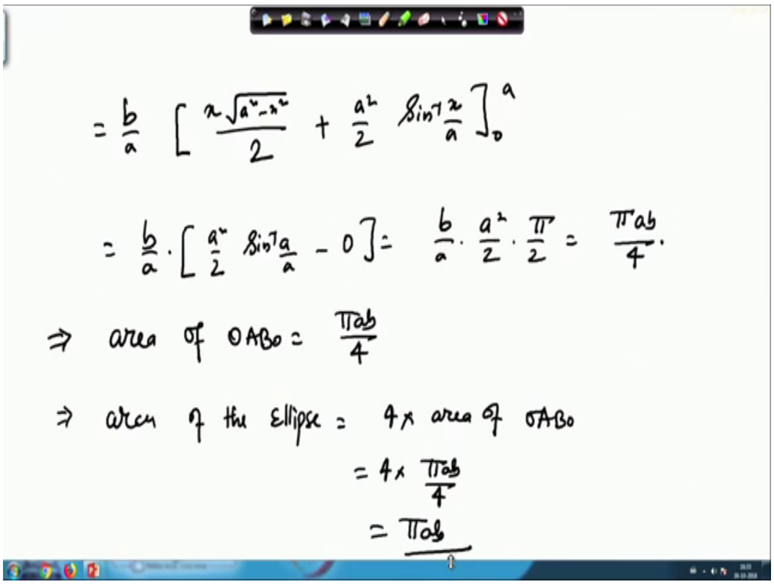

So, we can write now the value of that integrals. So, it will be x times the square root of a square minus x square divided by 2 plus a square by 2 sin inverse x by a integral from 0 to a. So, now, if I substitute the values then this will reduce to. So, when x is a this is 0 when x is 0 then this is again 0 now this one is a square by 2 sin inverse a by a minus sin inverse 0. So, ultimately this is 0 this is sin inverse 1; sin inverse 1 is pi by 2. So, ultimately we will have b by a times a square by 2 times pi by 2. So, this will be pi a b by 4. So, from here the area of OABO is basically pi a b by 4.

Now since we just talked about that ellipses symmetric along both the axes. So, it is symmetric along the both the axes so; that means, if we calculate the area of any one of these portions, then we just multiply it by 4 and then it will be the entire area of the ellipse. So, to calculate the area of the ellipse, so, this implies the area of the ellipse of the ellipse is equals to 4 times area of the portion OABO so; that means, 4 times pi a b by 4 so; that means, pi a b all right. So, that is basically the required area of an ellipse our area bounded by the ellipse.

So, this is one such example where we can calculate the area of a function or of a given curve bounded between two points and then we multiplied by 4 and that that will give us the area of the entire portion or bounded by that curve. So, that is how we calculate the area of this plane region, next we consider an another example.

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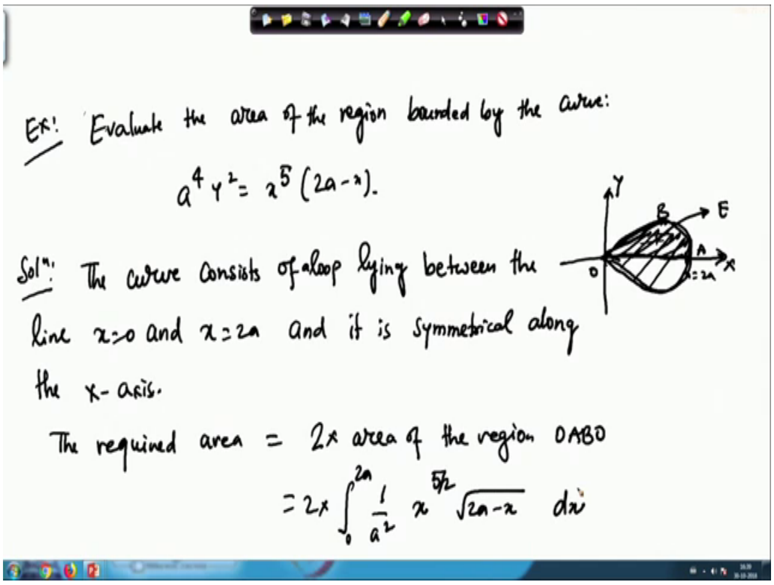

So, prove that prove that or we can basically say evaluate. So, we will just evaluate that will make it more interesting. So, evaluate the area of the region bounded by the curve a to the power 4 y square, x to the power 5 2 a minus x right.

So, let us write the solution. So, that is my x axis, this is origin and that is my y axis. Now we have x to the power 5 2 a minus x so; that means, and here we have y square so; that means,; obviously, the point 00 lies on this curve because its if we substitute x equals to 0 and y equals to 0. So, the point 0 0 lies on this curve. So, that is pretty much straight forward. So, one of the points on this curve is 0 0 now we have x equals we have 2 a minus x. So, when y is 0 then basically x is equals to 2 a. So, the curve, so, this is basically x equals to 2 a. So, of course, it passes through the point 0 0 and then it also pass through the point 2 a 0 because when y is 0 x is 2 a.

So, it passes through the point x equals to 2 a and y equals to 0 and now we have y square. So, for y negative and y positive, it will have the same value so; that means, it also how to say it also lies I mean in a way both sides of x axis in a way so; that means, it forms basically a loop. So, what I am trying to say is that it forms some kind of some kind of looping. So, it will look something like this and not exactly of this figure, but it will looks something. So, a little bit down here and then up here and then a little bit down here and then up here.

So, that is how it will look like and this is the region bounded by this, let us say let us call this as E. So, this is the region that will be bounded by this by this curve. So, the curve we can write now the curve consists of loop consists of a loop lying between the line x equals to 0 and x equals to 2 a and it is symmetrical along the x axis because the values of y are minus of a square root of a minus plus a square root of x to the power 5 times 2 a minus x so; that means, y is lying both above the x axis and below the x axis.

So, and that is that is how basically it will form how to say a loop. So, it it has some part lying above the x axis and line below the x axis. So, this is this is what we mean here. So, basically the required area. So, the required area area is equals to 2 times area of the region let us give it give it a name OA say some point that here is B. So, of the region OABO in a way and the area of the region this much only. So, area of this. So, we just calculate the area of this region and then we just multiply by 2. So, the area of this region only will be integral from 0 to 2 a and 0 to 2 a and for this will be two by. So, this will be 1 by a square x to the power 5 by 2 times square root of 2 a minus x d x all right.

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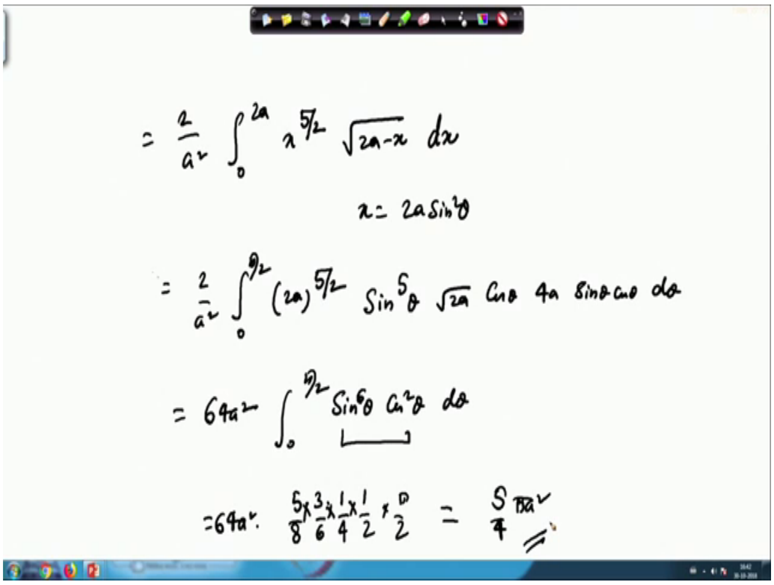

So; that means, we have 2 by a square integral from 0 to 2 a x to the power 5 by 2 square root of 2 a minus x d x all right. And to evaluate the integral now to evaluate this integral we have a how to say square root form. So, we substitute x equals 2 a sin square theta. So, we just substitute x equals 2 a sin square theta. So, that will end up getting some kind of quasi square theta that will how to say neutralize the square root. So, that is what we will do here and ultimately if we do that then the whole thing whole thing will reduce to. So, the whole thing will reduce to the whole thing will reduce to there is no such symbol. So, the whole thing will reduce to 2 a square integral from 0 to phi by 2 a 2 a to the power 5 by 2 then we will have sin to the power 5 theta, then we have a square root of 2 a of course, then we will have cos theta then we will have 4 a sin theta cos theta d theta.

So, ultimately this whole thing will reduce to 64 a square 64 a square integral from 0 to phi by 2 sin to the power 6 theta times cos square theta all right. So, sin to the power 6 theta times cos square theta now this falls into our reduction formula category. So, we have already evaluated integrals of this type in our reduction formula type situation. So, this will be 64 times a square and then we will have 5 times 3 times 1 time 1 and then this will be 5 by 8. So, basically what I am trying to say is that 5 by 8 times 3 by 6 times 1 by 4 times 1 by 2. So, multiplication times pi by 2.

So, if we calculate the whole thing then it will be 5 by 4 5 by 4 pi a square and that is basically our required area of this where is that of this region. So, of this entire region. So, we did not have to calculate the area of the whole region since it is symmetric about the x axis, we just calculated the area of the of the above region and then multiplying it by 2 it will give us the area of the whole region, and that is what we have done and the required answer is 5 by 4 pi a square. So, this is how we calculate the area of this plane region. Next we have a curve that is given something like this.

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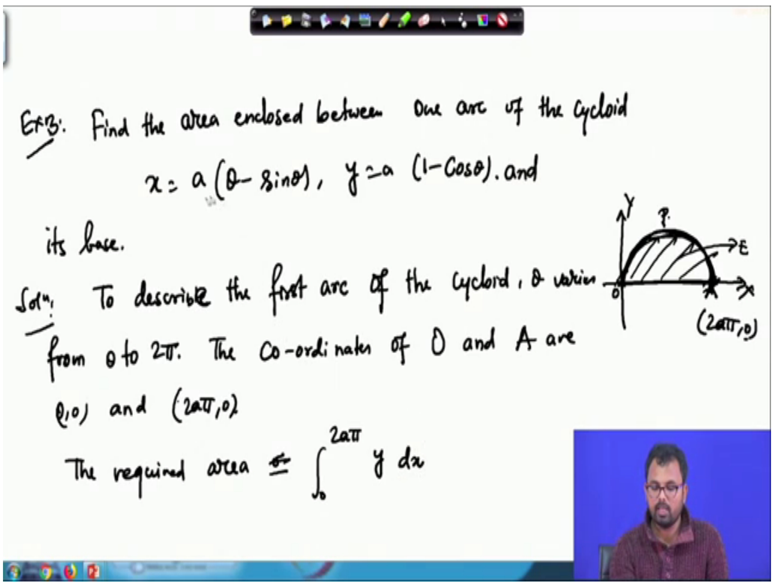

Let us consider an another example. So, sometimes instead of having a pool symptoms instead of having a Cartesian coordinate expression you may have a polar coordinate expression and the curves like a cycloid or lemniscate of Bernoulli things like that, they are expressed in a in can they can be expressed in terms of a polar coordinate system as well.

So, let us look at one such curve. So, find the area enclosed between one arc of the cycloid x equals to a times theta minus sin theta and y equals to a 1 minus cos theta and its base and its base. So, basically if you look into any two 2D coordinate geometry book then in that case there you can be able to see all these nice figures and the figure for cycloid can be given by something like this here let us say A and P. So, just one arc and its base. So, this is the base and that is the one arc all right.

Next, so, this is Y 0 A P all right. So, to describe the first arc to describe the first arc of the cycloid. So, to describe the first arc describe the first arc of the cycloid theta varies from 0 to 2 pi and the coordinates of O and A are and the coordinates of O and A are 0 0 and 2 a pi 0. So, here this point is 2 a pi 0 and this one is 0 0. So, this is basically the two points where we how to say evaluate the area of this cycloid. So, let us do that. So, the required area is basically integral from 0 to 2 a pi y d x.

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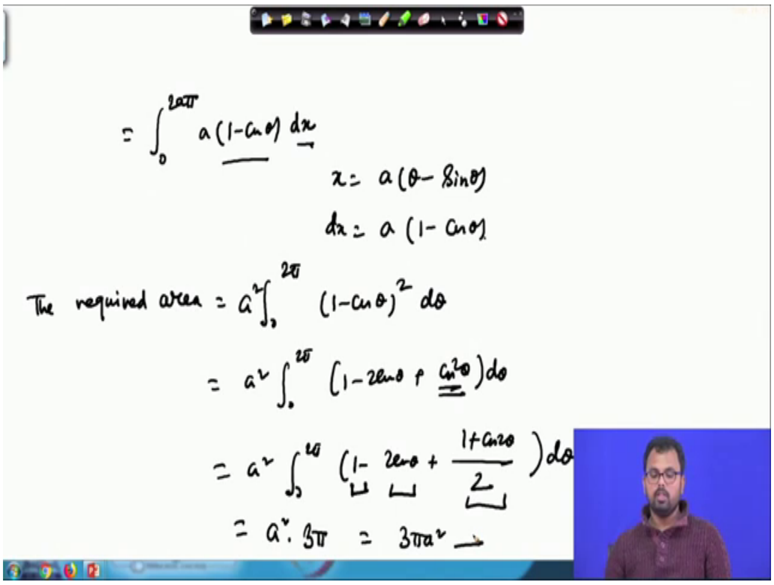

So, we can write it as 2 a pi. So, we have basically. So, we have basically y equals integrals. So, this is basically 2 a pi. So, we have y equals to a times 1 minus cos theta right and then we have d x, but since this is in theta and that is in d x. So, we have to somehow transform this x into a theta form. So, we know that x is basically a times theta minus sin theta all right. So, theta minus sin theta. So, from here we have d x equals to a times 1 minus cos theta is that correct and when x is 0 then in that case theta is 0 and when x is a then theta is 2 pi. So, we can write this here like which is basically the required area.

So, that it required area is basically integral from 0 to 2 pi a times a. So, this is a square and then we have 1 minus cos theta whole square d theta all right. So, now, we can break this whole thing into integral from 0 to 2 pi 1 minus 2 cos theta plus cos square theta, then we are just a plus cos square theta then we are just a 2 here by adjusting a half and this will reduce to integral from 0 to 2 pi 1 minus 2 cos theta and if we adjust that half then there will be a 2 here 1 plus cos 2 theta d theta and now we can integrate this individual term one by one. So, this is fairly simple and I am pretty sure you can be able to do that.

So, this will whole this whole thing will reduce to a square times 3 pi basically and 3 pi a square is the required answer. So, in this case we were given a cycloid. So, from here first of all we need to find out how does that curve look like. So, sometimes you also need to practice with curves. So, that just looking at the curve we can make out this is the way this curve should look like. And to describe the first arc of the cycloid we can see that theta will vary from 0 to 2 pi, and the coordinates for O and the point A are basically at 0 0 and 2 pi 0 and the required area would be then this one here.

So, this is our required region or the area which we want to calculate. So, of course, a x will vary from 0 to 2 a pi and y dx. So, this is the y part. So, y dx and we substitute the value of y here and dx can be calculated from here which we have done in this and the second page. So, here I have forgotten a d theta. So, d theta and then we just substitute the value of d x here. So, this a square will come here and we will have a 1 minus cos square theta. So, it is just a simple trigonometrical calculation and I am pretty sure you can be able to do that and ultimately we will obtain 2 3 pi a square which is the required area of the cycloid and its base between the cycloid and its base.

So, will stop here for today, but in our next class will start with a new example related to area of plane regions and will how to say practice may be at least one or two more examples to clarify that concept and then will move to a rectification for if time permits. So, thank you for your attention and look forward to your next class.