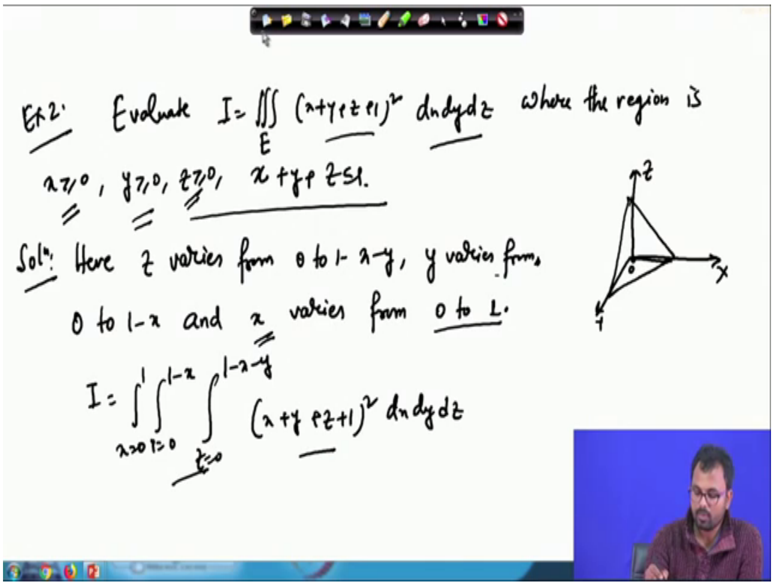
Integral and Vector Calculus
Prof. Hari Shankar Mahato
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 23
Triple Integral (Contd.)

Hello students. So, in the last class, we started with Triple Integral, and we also looked into the basic definition, and the concept of triple integral. We also started with one example which turns turned out to be a little bit long. So, we how to say stopped it in the midway. So, we will continue with the similar example, and we will try to finish it in this lecture. And then we will also look into some more examples motivated from triple integral.

(Refer Slide Time: 00:51)

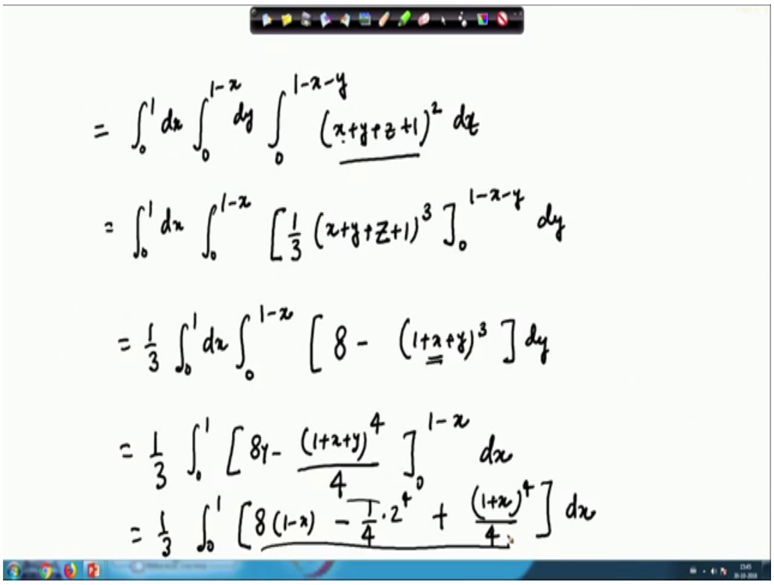


So, we started with basically this example here. Evaluate I equals to x square plus y square plus z plus 1 whole square dx, dy, dz, which is basically the volume element where the region is given by this here. So, this one was basically our region of integration. So, from here we calculated the range for z. So, obviously the z is varying from 0 to 1 minus x minus y, because we start with all the points which are lying on the positive side of the z-axis.

Similarly, y will vary from 0 to 1 minus x, because we again start with the points which are lying on the positive side of y-axis. And similarly, we start our x will vary from 0 to

1, because we are starting for all the points which are lying on the positive side of x-axis. So, now that we have the limit, we can substitute for this for these values for on this on this region of integration it is better to write a here, just to signify that we are talking about a region.

(Refer Slide Time: 02:13)



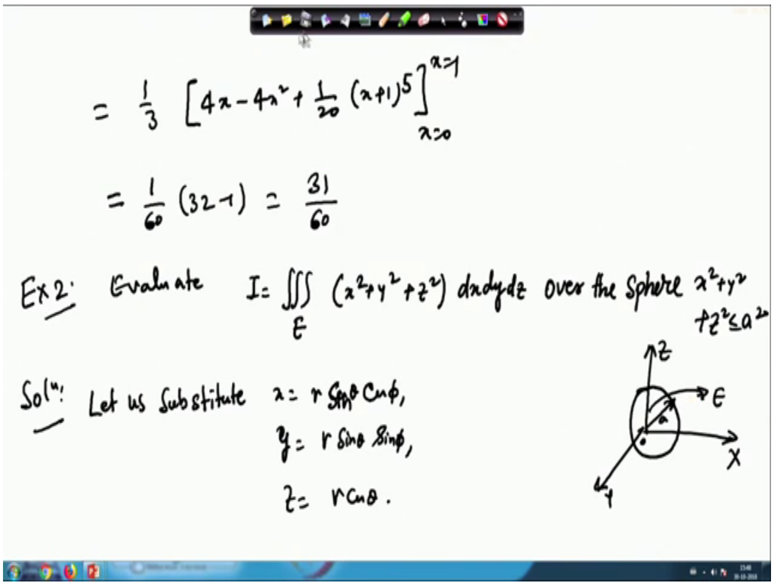
Now, we have substituted the limits as you can see here, and then we will integrate one by one. So, first we will start with integrating with respect to variable and z. So, let us start with integrating with respect to variable z. So, this is basically integral from 0 to 1 dx integral from 0 to 1 minus x dy and then integral from 0 to 1 minus x minus y x plus y plus z plus 1 whole square dz all right.

So, when we are integrating this integrand with respect to z that means, x and y are both treated as constant here, so because there is no there is no x and y variable involving in this in this element dz, so that means we treat x y, and of course 1 as a constant here, and then this is basically a plus x whole square definitely integral. So, for such integrals we know how to do that how to do the integration, so this will reduce to 1 by 3 x plus y plus z plus 1 whole cube, and then 0 to 1 minus x minus y all right. So, this is this will be the integral of this integration of this integration here.

Now, I take 1 by 3 out, and then this one is 0 to 1 dx, and then this is 0 to 1 minus x, and now we substitute the value. So, when z is 1 minus x minus y, so then this will be basically two right, so 8 minus when z is 0, then this is basically 1 plus x plus y whole to the power 3 dy. Now, we integrate with respect to y only. So, again when we integrate with respect to y, then x will be treated as constant, and 1 and 8 are of course treated as constant.

So, we integrate so let us integrate, then this will be 8 y minus 1 plus x plus y to the power 4 divided by 4 integral from 0 to 1 minus x dx all right, and then we substitute the value. So, this will be basically 0 to 1 8 1 minus x minus 1 by 4 this will be if I substitute 1 minus x, then 2 to the power 4, and then minus minus plus, this will be 1 plus x divided by 4 whole to the power 4 dx all right.

(Refer Slide Time: 04:56)



So, we will basically obtain after simplification, we will basically obtain 1 by 3 4 x minus 4 x square plus 1 by 20 x plus 1 to the power 5 integral sorry the value of the function of the integral evaluated at 0 to 1. So, here basically we calculate 1 plus x to the power 4 using the binomial theorem, so it will be x to the power 4 plus 4 x to the power 3 and so on.

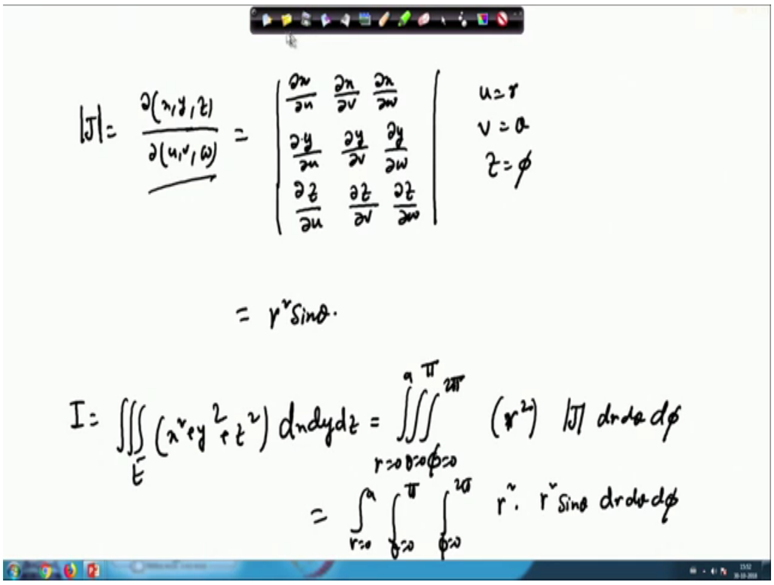
And then we integrate with respect to x, so this is a fairly easy how to say integral to evaluate. So, I am pretty sure you can be able to do this integral, because it is a basically an algebraic expression. And after you do the integration, you will obtain a result like this. And then we substitute the value, and it will result in 1 by 60 times 32 minus 1. So, basically 31 by 16, so that is the value of this integral evaluated between this I mean this region actually.

So, here first we had to find out the range for the variable x, y, and z. And afterwards we just integrated with respect to z first, with respect to y second, and then with respect to z x third. So, this is the order which is preferred by a most of the people, so they initially try with integrating with respect to z, and then y, and then x. So, we will work out few more examples for you to get a custom with triple integral.

So, let us go to our second example all right. So, the second example is second example is evaluate so evaluate I equals to integral over the region E x square plus y square plus z square dx dy dz over the sphere x square plus y square plus z square is less or equal to a square. So, we have to evaluate this integral here, so that is the integrand basically, so this integral here over the region E. And this region E is basically that sphere which is of course a bounded domain, when a is finite.

And if we want to draw this sphere, so I am not a really good how to say really good at drawing, so this is basically 0, y, z, and x, so that is my radius a, and this is this rectangular I sorry this is this region E, where we have to perform the integral. So, now in this region now in this region if we substitute, so let us substitute let us substitute x equals to r cos theta cos phi, y equals to r sin theta ok. We will start with r sin theta cos phi, r sin theta sin phi, and z equals to r cos theta. So, if we take square of x square plus y square plus z square, then this will actually reduce to this sphere x square plus y square plus z square equals to a square.

(Refer Slide Time: 09:05)



Now, to do the change of variables actually, we know that we need to calculate the Jacobian. So, for the function of two variable, the Jacobian was given as del x, y del u, del v. So, further that was for the function of two variable.

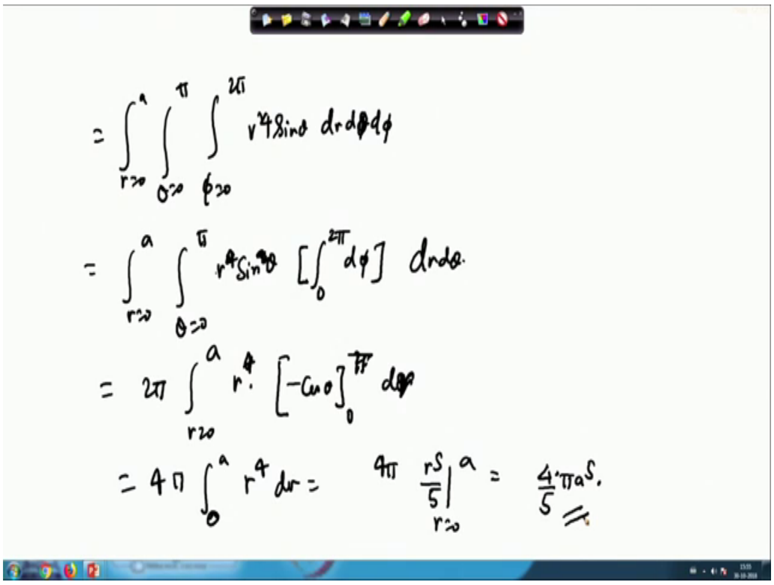
Now, if you want to have the Jacobian for the function of three variable, then in that case we basically write x, y, z, and then del u, del u v, and w. So, this is how to say a standard way of converting a change of converting a one function of one variable to the function of second variable in the integral, and for in order to do that we need to calculate the Jacobian which is given in this fashion. So, this is basically del x, del u, del x del v, del x, del w. And then we have del y del u, del y del v, del y del w, then del z del u, del z del v, del z del w.

Now, here u is basically r, v is basically theta, and z is basically phi. So, if we integrate if we differentiate x, y, and z with respect to r theta and phi, then basically we can calculate this whole determinant, and it will result in r square sin theta. So, this task is very simple. And therefore, I leave this one to the students to calculate. So, we will basically obtain r square sin theta.

And therefore, I can write my integral I over the region E x square plus y square plus z square dx dy dz. So, if I transform this integral into a polar one, then in that case this will be r running from 0 to a, because that is what our that is the range for the variable r or the radius. And theta will run from 0 to pi. And phi will run from 0 to 2 pi all right.

And this will be x square plus y square plus z square. So, if I substitute for x r square r sin theta cos phi, then it will be r square sin square theta cos square phi. And similarly, it for y it will be r square r square sin square theta sin square phi, and then z will be z will be r square cos square theta. So, we substitute all those things, and then this will be reduced to r square times determinant of z dr d theta and determinant of z dr d theta d phi. And then this will be r running from 0 to a, theta running from 0 to pi, and phi running from 0 to 2 pi, this is r square, and determinant of j is r square sin theta dr d theta d phi.

(Refer Slide Time: 12:37)



And then this whole thing will reduce to integral r running from 0 to a, theta running from 0 to pi, and phi running from 0 to 2 pi r square sin theta dr d theta dr d theta d phi, so the same thing here. And now we integrate with respect to phi first. So, we integrate with respect to phi, and then this will be phi, so the value of phi will be 2 pi basically. So, I am taking that 2 pi outside, and then here we will have dr d theta.

Next we will integrate with respect to we will integrate with respect to theta first, and then in that case this will reduce to minus of cos theta. So, here let us write r running from a to pi, when we integrate with respect to theta first, it will be so r square r square. So, sorry we will have r to the power 4 right yes so r to the power 4.

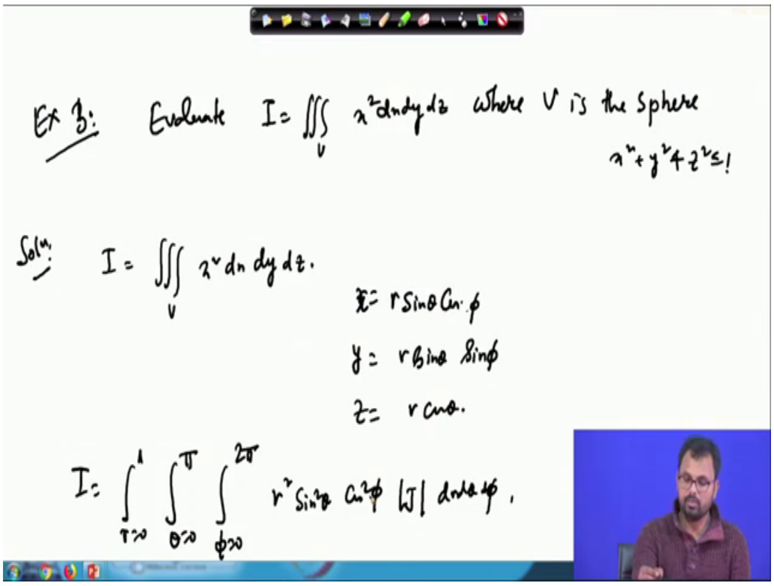
And when we integrate with respect to theta, and this will be a minus of cos theta theta will vary from 0 to pi d theta sorry d r and when cos theta when cos theta when theta is pi, then cos pi is minus 1, then this one will be plus 1. And this one will be cos 0. So, cos 0 is 1, and then minus minus plus. So, this will again be 4 pi times integral from 0 to a r to the power 4 d r. And when we integrate this one, then this will be 4 pi r to the power 5 by 5 r running from 0 to a. So, this will be 4 by 5 pi a to the power 5 4 by 5 pi a to the power 5. So, this is the required answer for the integral for the triple integral which we started with.

And we can see that so here so here we can see that we were given a bounded region first of all. So, from the bounded region, it is also very convenient to use a pull of spherical polar coordinate system. So, here we have I am here we have as sphere basically. And for this is sphere, we can use this spherical polar coordinate substitution, so that is the r sin theta cos phi, r sin theta sin phi, and r cos theta, of course this point lies on the sphere.

Now, we substitute so next we calculate the Jacobian, and while we have the Jacobian value. We can do the how to say change of variable or transformation, and by changing the variable we have next new variable as dr d theta d phi, and then here you will have a Jacobian J, and just substitute the value of the Jacobian. And then integrate with respect to phi first, then with respect to theta, and then with respect to r, and that will be our required answer.

And this is the area which is basically bounded by sorry this is the how to set the value of the integrand of the integrand given bounded by that region or that sphere x squared plus y squared plus z square is less or equal to a square. So, this was one such example of triple integral.

(Refer Slide Time: 16:27)



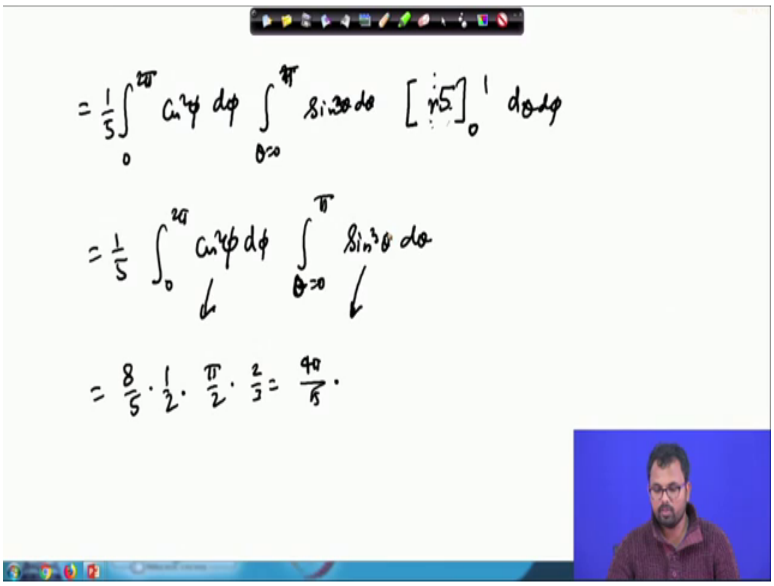
Similarly, we can also have example of type let us say example-3, evaluate where v is the sphere is it 1 yes. So, now again here we are given a sphere, which is basically again a bounded sphere. So, a bounded region we have in our hand, and we need to evaluate this integral here.

So, the integral I is given by integral over v x square dx dy dz. And obviously, v is a bounded region where x, y, and z can be transformed into a spherical polar coordinate system. So, we can have x equals to sorry x equals to r sin theta sin phi sin theta cos phi or sin theta sin phi like here. So, sin theta cos phi, and then sin theta sin phi ok, we can also take that one. So, sin theta cos phi, we can take sin theta cos phi. And then y equals to r sin theta sin phi. And then z is r cos theta.

So, now we can have dx dy dz and I calculated, then on the right hand side we will have dr d theta and d phi calculated. And we substitute the value of x here, which is r square sin square theta cos square phi. So, ultimately this whole thing will reduce to r varying from 0 to 1, and theta is varying from 0 to pi, and phi is varying from 0 to 2 pi, then we have x square which is basically r square sin square theta cos square phi. And then dx dy dz, which can be transformed into a determinant of J dr d theta and d phi all right.

So, now we can calculate this Jacobian here. So, in order to calculate this Jacobian, we will do the similar trick, what we did in the previous example, and this will be basically r square sin theta was it the same thing yes. So, we will obtain r square sin theta. So, substitute the value of r square sin theta, and then this whole thing will reduce to a basically a trigonometrical expression. So, this whole thing will reduce to basically a trigonometrical expression, and then we just integrate with respect to phi first, and then with respect to theta, and then with respect to r.

(Refer Slide Time: 19:52)



So, let us try to do that, so what we do? We first separate the function phi. So, cos square phi d phi, and then integral from theta running from 0 to 2 pi 0 to pi sin square theta d theta. And if we integrate with respect to r first, then this will be 1 by r to the power 5 0 to 1, and then I will have a 1 by 5 here, and this is basically d r so this is basically d theta d phi.

So, this is basically so what we have here, so we have r square sin theta. So, we have r square sin theta. So, this will be sin cube, and we will have sorry so we will have 1 by 5 r to the power 5. So, 1 by 5 is already outside, so we will have r to the power 5 all right.

So, now we do substitution for r equals to 0; and r equals to 1, so this term will remain as it is and then here we will have a integral from 0 to 2 pi cos square phi d phi integral from theta running from 0 to pi sin cube theta d theta. Here we can either use the induction formula or we can use the trignometrical formula, where this will reduce to sin 3 theta minus 3 sin theta and from there we can do some calculation, and we can be able to find out the value of this integral with respect to theta.

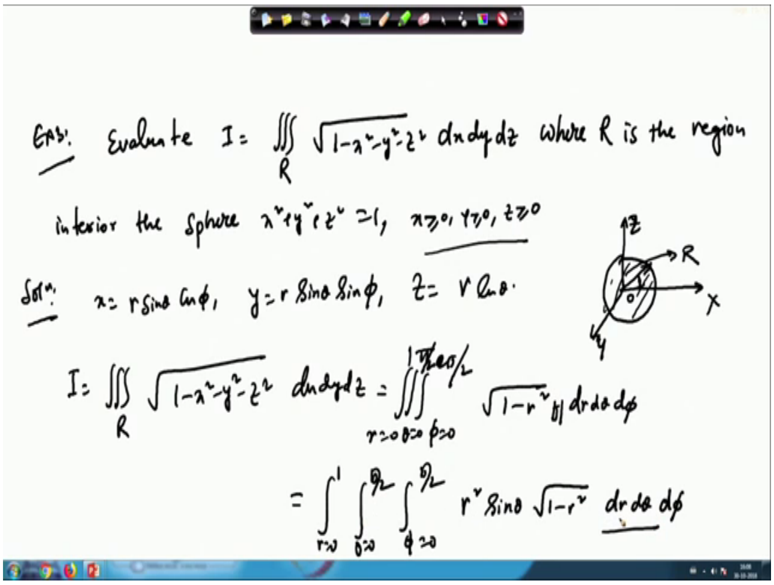
And then we integrate then we are just a factor two here, so that we will get 2 cos square phi, and then we can write 1 minus 2 cos square phi is cos 2 phi minus 1 sorry cos 2 phi plus 1, and from there we can be able to calculate this integral as well or we can use the induction formula. So, on both of them we can either use the induction formula or use our traditional cos 2 phi, sine 3 phi results.

And then after doing the simplification, we can be able to obtain 8 by 5 times 1 by 2 times pi by 2 times 2 by 3, so this is basically 4 pi by 15. And this is the required answer of our problem where is that this here. So, here the given integrand was x square the times dx dy dz, so basically doing this the simple method of substitution. ah

We can be able to reduce the whole integral into a rather simpler one, and then we just use some trigonometrical formula like cos 2 theta or sin 2 theta or cos 2 phi cos 3 theta naught, and that will give us the required value of this integral, because now we are integrating basically two different integrals, because of their integrand.

So, this one we can do that at home, and this one also we can do that basically by ourselves, and just what to say writing cos 2 phi cos square phi as the has that formula expression of cos 2 phi plus 1, and we can write sin to the sin cube theta as sin 3 theta formula we can be able to calculate these two integrals separately. And then substitute them, and that will give you the answer So, you will reduce them to a trigonometrical form, it is not that difficult all right. And this was the second example or third example basically which we covered. ah

(Refer Slide Time: 23:46)



We can work out one more example if you if we have some time, so let us consider an another example. So, and another example could be without having to reduce it into a trigonometrical form could be so we can start with a let us let us consider this problem here. So, example-3 evaluate I equals to integral over the region R 1 minus x square minus y square minus z square dx dy dz, where R is the region interior to the sphere x square plus y square plus z square equals to 1.

So, solution here also what we will do basically is R is the region, which is the interior of the sphere. So, again the region R is bounded by the sphere x here let us say. So, we will we can draw the so this is our z-axis, this is our y-axis, this is our x-axis, and that is my sphere, this is the radius 1, and this whole is the region R so this is the region R.

And if we substitute again x equals to r sin theta cos phi y r sin theta cos phi, y equals to r sin theta sin phi, and z equals to r cos theta, then this will basically this integral I equals to integral over the region R 1 minus x square minus y square minus z square dx dy dz.

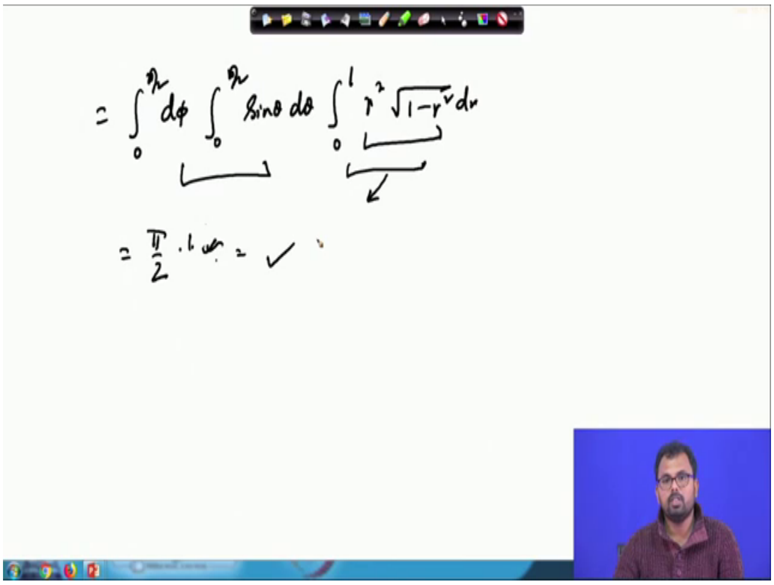
So, when we are transforming it into how to say a polar spherical polar coordinate system or r will run from 0 to 1, theta will run from 0 to pi, and phi will run from 0 to 2 pi, then we substitute for x y axis x y and z and here basically here basically the value of the integral. So, when we substitute R equals to cos square theta R equals to sin square R equals when I substitute x equals to r square sin square theta cos square phi, and y equals to so the value of this integrand basically will be 0.

So, this here, so this problem will not be that much how to say interesting for us. So, what we can do is we can impose conditions like x greater or equal to 0, y greater or equal to 0, and z greater or equal to 0, then in that case we are actually in the first quadrant. And then for the first quadrant, we can have theta running from 0 to pi by 2 and theta running from 0 to pi by 2, and phi will run from again 0 to pi by 2, where is r running from 0 to 1, and then this will be 1 minus r square d x d Jacobian of J dr d theta d phi.

So, we just made the problem a little bit more interestings by imposing these three conditions. So, if we impose these three conditions that means, x will lie in the first quadrant, y will lie again in the second quadrant, and z will also lie in the so y will also lie in the first quadrant, and z will also lie in this first quadrant, so that means, they are both lying in the positive quadrant of x, y, and z-axis.

And then in that case theta will go from 0 to 2 pi, and phi will also go from 0 to 2 pi, so that is what we have written here. And now this Jacobian we know that is basically r square sin theta. So, let us write that and this will be integral from 0 to 1 theta will be from 0 to pi by 2, and phi will be from 0 to pi by 2 r square sin theta square root of 1 minus r square dr d theta d phi. And now we can evaluate these three integrals, so basically the triple integral one by one.

(Refer Slide Time: 29:15)



So, what we can do, we can write integral 0 to pi by 2 d phi, and then integral from 0 to pi by 2 sin theta d theta, and then integral from 0 to 1 r square 1 minus r square dr. So, this is what we will get, and we can calculate this one from our usual trigonometrical results. So, we again substitute r equals to sin you know t, and then from there we can try to calculate this for this function this integral here, calculating these two would not be difficult.

So, this one is basically pi by 2, and this one is minus cos theta, so cos pi by 2 is 0, and this so this basically will give us one. So, the value of this integral is one, this one is pi by 2. And the calculation of this one should be should be fairly easy using some trigonometrical results.

And ultimately, we will obtain the value as pi by 2 times 1 by 4 time four 1 by so pi by 2, this one is 1, and this one is just you just calculate the value and substitute here that will that will give us the required value. So, this is pi by 2, this is 1. And you have to calculate the value of this part to substitute here, and that will be the answer of this integral. So, I am leaving the calculation up to the students, because it is a very fair simple example.

So, here in this case also our given region of integration was this sphere bounded by three other conditions that the all the points should lie on the positive side of x-axis, y-axis, and z-axis that means, we are in the first quadrant. And therefore, your r theta and phi would vary from 0 to 1, pi by 2, and pi by 2 respectively. And then we substituted the values for x, y, z using spherical polar coordinate system. And then we just have to do this simple calculation, so and that will give you the required answer.

So, here in again in this example, we saw that how we calculate the triple integral on a bounded region. And we can also work out several examples of this type. All you have to do is to find out the find out the region or the domain for the variables x, y, and z for which we can do the v integral actually. And once you have the region, then doing the integral or performing the triple integral would be fairly easy as we saw in these examples.

So, we will try to include some more examples in your assignment sheet, so that you can be able to practice them, and be perfect at them. And we will stop our triple integral part here. And in the next class, we will start with our area plane regions. So, thank you for your attention today, and I will look forward to you in the next class.