Integral and Vector Calculus
Dr. Hari Shankar Mahato
Department of Mathematics
Indian Institutes of Technology, Kharagpur

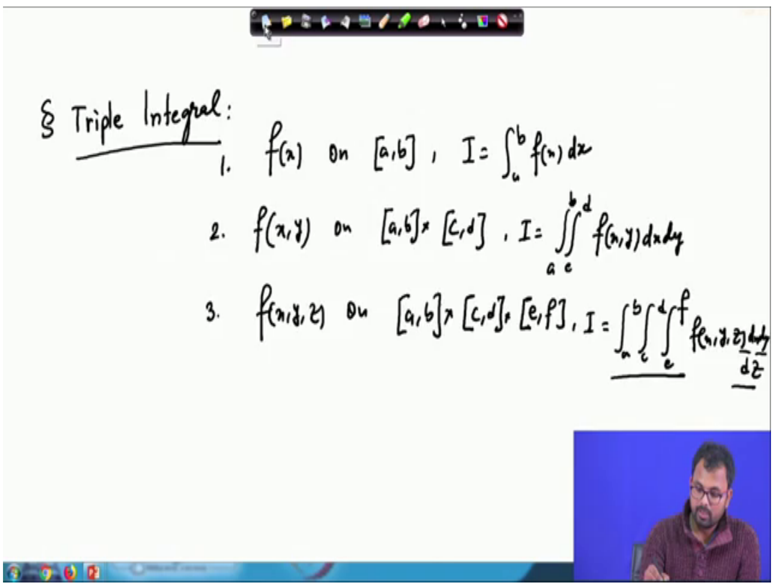
Lecture – 22
Triple Integral

Hello students. So, up until last class we looked into change of variables in an integral and also change of order of integration, where we can reduce a very complicated integral to a rather simpler form by using either change of variables or change of order of integration. Sometimes it is useful, sometimes it is not. So, most of the time we have to judge from our intuition, that whether changing the order of integration or changing the order of variable would be helpful or not.

Sometimes it also looks like the integral is quite complicated and there is no way we can be able to integrate using the traditional method of integration. Then, in that case changing the order of integration or changing the variable not the integration would be the only way out there.

So, we also worked out few examples in the previous class and today, we will start with something called Triple Integral. So, we know that in a; so, this is accordance with your according to your according to your syllabus. So, if we look into your syllabus then today we are going to cover the triple integral.

(Refer Slide Time: 01:37)



So, today we will be covering basically the triple integral. So, today we will be basically covering the triple integral and so, let us go back to the topic. So, we will start with Triple integral alright. So, now, like in case of function of one variable let us say f x which is defined on an interval a to b; then, in that case we can be able to talk about integral of this type or if we want to integrate along a curve, then we can also do that here. Now, if we have a function let us say of two variables.

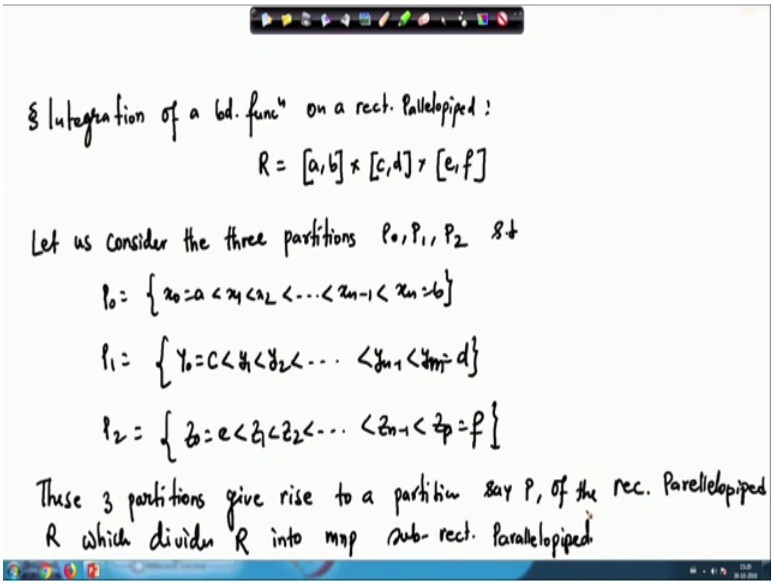
So, this was the first type of integral we saw. Now we have the second type of integral which is basically if we have a function of two variables say x and y on a rectangle let us say a to b cross c to d; this can also be a region. So, it can happen that instead of rectangle we have a region. However, the concept would work similarly. So, the integral can be defined as integrals integration from a to b and integration from c to d f x y dx dy. Now, how do we do the integration that is something we have already seen.

Now, suppose we know that we also have three-dimensional geometries and three-dimensional figures and in such cases we can also talk about triple integrals. So, triple integral deals with the surfaces which are in 3 D. So, here in this case we have the range or the domain of integration are basically two-dimensional geometry. So, instead of this rectangle we can also have a region which is in 2 D setting, but instead of 2 D setting, we can also have a 3 D setting.

So, suppose if we have a function f x y z defined on a rectangular parallelopiped a to b c to d and e to f. So that means, now we have a rectangular parallelepiped. Here, in this case also we can talk about a triple integral. So, we write it in this fashion; f x y z dx dy dz. So, triple integral comes from the fact that we have used 3 integral symbol. So, hence the triple integral; sometimes people also prefer to call it as volume integral, this is also fine and this dx dy dz is basically the volume element.

So, this dx dy and dz are the volume element and here of course, how we evaluate the triple integral we will see in a minute. Now, we know that in case of function of one variable, we could be able to define that upper integration sum, lower integration sum; from there we defined the upper integral and a lower integral accordingly and if the upper integral and the lower integral are same. Then, we can say that the function is integrable or Riemann integrable. In case of function of two variables, we saw that similar concept works. So, in case of function of two variable we can also be able to calculate the upper integral sum and lower integral sum and from there we can calculate upper integral and lower integral and if they are same; then the function is said to be integrable. On that rectangular region are now in case of function of 3 variable, the concept still holds and it goes in the following way.

(Refer Slide Time: 05:40)



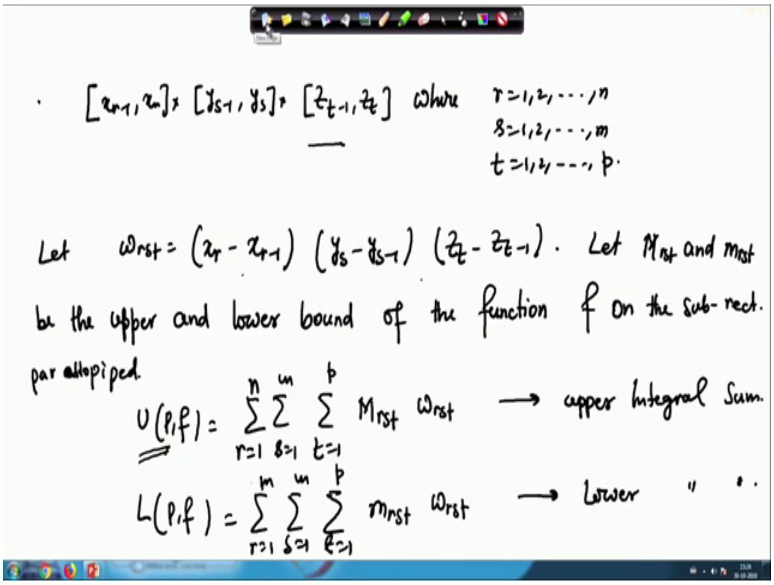
So, the definition we will start basically with the definition. So, in integration of a bounded function; integration of a bounded function on a rectangular, let us say parallelopiped. So, it is given by R equals to a to b c to d and e to f. By the way this f here and the function f, they are two different things. So, this f here is basically a real number and that f denotes the function of x y z alright and suppose we consider.

So, let us consider the 3 partition let us consider the 3 partitions; the 3 partitions say P 0, P 1 and P 2 such that we have P 0 equals to x 0 equals to a less than x 1 less than x 2 dot dot up to less than x n minus 1 less than x n equals to b. Similarly, we can write P 1 equals to y 0 equals to c less than y 1 less than y 2 dot dot up to less than y n minus 1 y n equals to d. And similarly, P 2 equals to let us say z 0 equals to e where this one less than z 1 less than z 2 dot dot up to less than z minus 1, then z n equals to f.

Now, these 3 partition; now these 3 partition; these 3 partitions give rise to a new partition; give rise to a partition say partitions say P of the rectangular parallelopiped of the rectangular parallelopiped which divides rectangular parallelopiped.

So, ped; R which divides R into. So, by the way we can have this one as x n; otherwise it will be a square. So, we take it as a rectangular parallelepiped. So, x n y m and z p into m n p sub rectangular parallelopiped; so, parallelopiped. So, it will divide the whole rectangular parallelopiped into m n times p sub rectangle parallelopiped.

(Refer Slide Time: 09:30)



And where; and where sub rectangular parallelopiped; x r minus 1 to xr times y s minus 1 to y s times z t minus 1 to z t, where r running from 1, 2, 3 up to n; s running from 1, 2, 3 up to m and t running from 1, 2, 3 up to p.

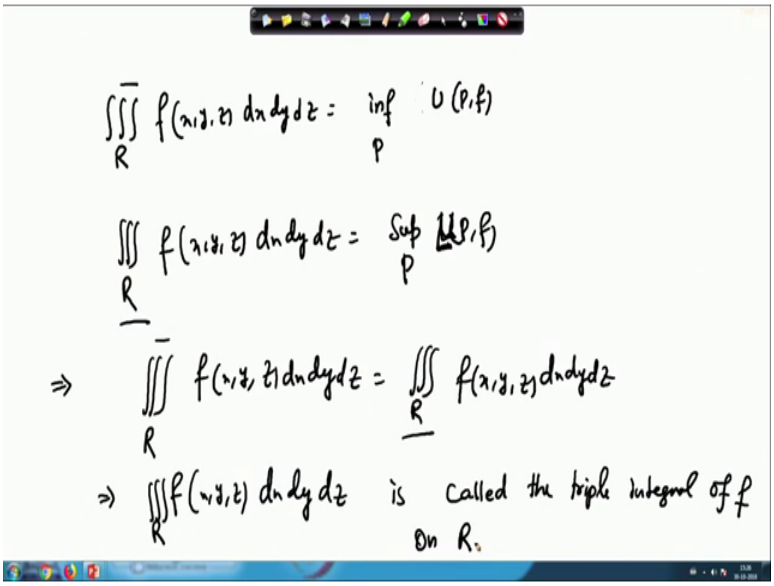
So, this is the number of how to say sub rectangular parallelepipeds by which for this interval this partition p divide actually. Now we write let w rst equals to x r minus x r minus 1 y s minus y s minus 1 and z t minus z t minus 1 alright which is the volume of this rectangle parallelopiped this one. So, that is the volume of this rectangular parallelopiped this here.

So, it is basically length into breadth into height. So, length into breadth into height and now we write let capital M rst and small m rst be the upper and lower bound of the function f on this sub rectangle let us say on the sub rectangular parallelopiped; on the sub rectangular parallelopiped sometimes this is spelling might be wrong.

So, do not worry about it you can parallelopiped. So, you can correct that; so, this spelling is not an issue alright. So, now, that we have the upper bound we have defined, we have denoted the upper bound and lower bound, I can be able to write my upper integral sum which is U p, f which is basically sum of all the upper bounds or I can write instead of writing this notation. So, I was going to write the whole thing together, but let us write it separately; s running from 1 to m and t running from 1 to p capital M rst times w rst. So, this is my upper integral sum.

So, this is upper integral sum; upper integral sum and I can write L p, f equals to some r running from 1 to n; s running from 1 to m and t running from 1 to p. Small m rst times w rst is basically lower integral sum alright. So, this is how we define the upper integral sum and lower integral sum. Now, as for the function of 1 variable or function of 2 variable, the infimum of this set and so, infimum of all such upper integral sum for all the partitions for I mean that is basically called as the upper integral sum and lower integral sum.

(Refer Slide Time: 13:30)



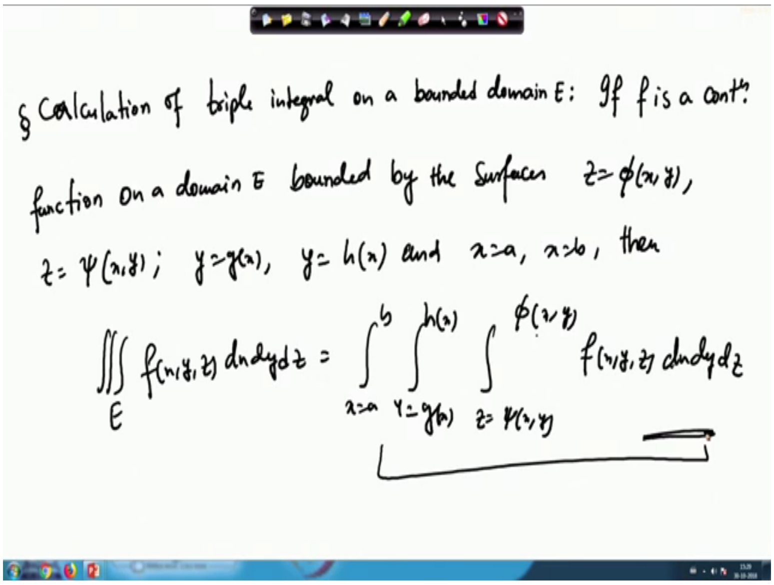
So, basically the upper integral sum on this rectangle let us say R rectangular parallelopiped; let us say R f x y z dx dy dz is basically the infimum of set of all sums.

So, U p, f infimum fall such U p, f where for all partitions P is called as the upper integral sum and similarly the lower integral sum let us write a lower sign at the bottom f x y z dx dy dz equals to supremum over all partitions L p, f. So, this is basically L p, f and this is called as lower integrals a lower integral. Now from our classical theory, we know that the upper integral if the upper integral and the lower integral let us write here.

So, this R means rectangular parallelopiped alright. So, f xy z dx dy dz, if they are equal then this actually implies that the function f is triple integrable over the rectangular parallelopiped and the common value is basically denoted by rectangular integral over the rectangular parallelopiped f x y set dx dy dz is called the triple integral. It is called the triple integral of f on the rectangular parallelopiped R.

So, this is how we basically define in general the triple integral of a function f on a rectangular parallelopiped R like we did for the function on a rectangular domain in R 2 actually. Similarly, instead of having a rectangular parallelopiped, we can have a region. So, we can have double triple integral.

(Refer Slide Time: 16:09)



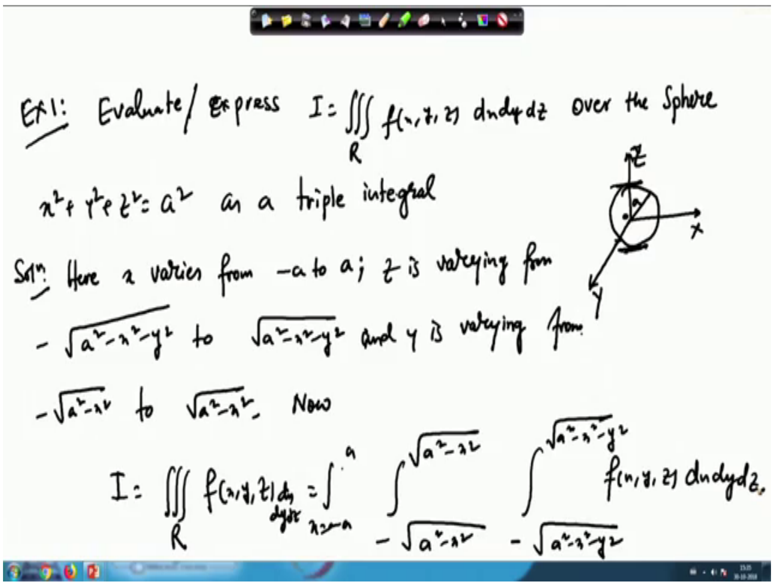
So, triple calculation of triple integral on a rectangular on a on a region actually. So, we can have something like calculation of triple integral on a bounded domain. So, like bounded region, here we have bounded domain E. So, the definition goes like this. If f is a continuous function; is a continuous function on a domain E bounded by the surfaces; bounded by the surfaces z equals to phi x y and z equals to psi x y, then y equals to g x, y equals to h x and x equals to a and x equals to b.

Then, our triple integral over the region E f x y z dx dy dz will basically be integral x running from a to b; y running from g x to h x and psi z running from psi x y to phi x y d x sorry f x y z dx dy dz. So, that is how basically we will calculate the triple integral over a bounded region.

So, first of all we need to identify by what curve the region E is formed actually. So, that can be calculated or that or sometimes it is also given. So, we can be able to find out that and the bounded region and then, the triple integral over that bounded region can be given in this fashion. So, we can see that from this form. So, we can see that from this form which is that first we are integrating with respect to z. So, we are substituting for z and then, we will get a function of x and y only. So, then we will integrate with respect to y to get the function of x only and then, we integrate with respect to x. So, that is how that is the kind of flow chart we are following here, alright. So, these are the two basic definitions in a way for us to calculate the triple integral.

So, either the given domain can be a rectangular domain or it can be a region and if it is a region, then we have to find out and these curves actually. So, these curves here and then, we this is how we calculate the double triple integral. So, now, that we have the definition; we will try to work out few examples.

(Refer Slide Time: 19:35)



So, let us start with our first example. So, the first example is. So, evaluate or express let us say; evaluate or express, better we use the word express. I equals to integral over the region let us say R f x y z dx dy dz over the sphere x square plus y square plus z square equals to a square as a triple integral, as a triple or repetitive repeated integral. So, how can we express this integral as a triple integral? So, first of all the given region is a sphere.

So obviously, n a is a finite positive number; so obviously, it is a bounded domain. So, I do not know if we can be able to draw the 3 D figures correctly, but let us try to do that anyways. So, this is basically our sphere X 0 Z and Y. So, this is the radius a and of course, it is a bounded region. So, we have to first find out the limit for x. So, here x is varying from minus a to plus a. So, here x varies from minus a to plus a.

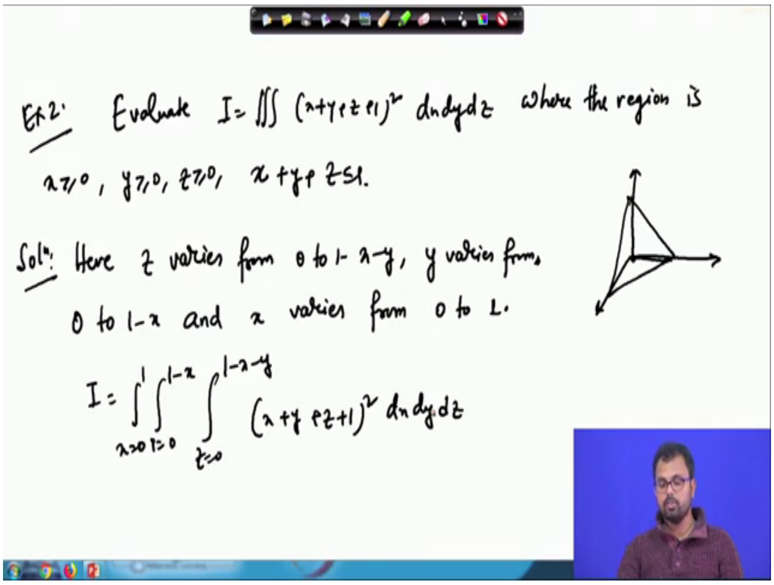
Now what about z? So, z is basically so, if we see the z the sphere and the z axis going through the center of it, then in that case z is basically varying from a square minus x square minus y square a square root to a plus square root a square minus x square minus y square. So, z is varying from this point; so, this point to this point. So that means, this point and this point to this point; that means, its slightly difficult to figure out the 3 D geometry, but its varying from minus of a square root of a square minus x square.

So, minus of a square minus x square to square root of a square minus x square minus y square and y is varying from square root of if we take z equals to 0, then into 2 dimensional geometry is it is basically a circle x square plus y square equals to a square; where, y is varying from minus of a square minus x square, square root to a square minus x square positive basically. So, this is our required how to say range for the variable x y and z and now, we can write that integral. So, now, we can write that integral I equals to integral over the region R f x, y, z equals to integral over the region R which is basically so here we will have dx dy dz.

And this one will be x running from minus a to plus a; y is running from minus of a square minus x square to square root of a square minus x square and then, z is running from minus of a square minus x square minus y square to square root of a square minus x square minus y square f xy z dx dy dz. So, this is the required form for this integral in terms of how to say triple integral.

Well, now we have the limit of the integration. Of course, if we are given a function, then this will give us the volume bounded by this by this region E for that function f. So, basically if we have the function f, then we can be able to calculate this volume integral quite easily. So, now, that we have seen the first example.

(Refer Slide Time: 24:48)



Let us go to our second example. So, here it says evaluate x plus y plus z plus 1 whole square dx dy dz; where, the region is x is greater or equal to 0; y is greater or equal to 0; z is greater or equal to 0; x plus y plus z less or equal to 1. So, this one; now here we have basically a region. So, here we have basically a region which is a x plus y plus z is less or equal to 1. So, we can draw it something like this.

Now, we have to consider the fact that, we have to consider the fact that x is positive. So that means, all the points of the region which are lying on the positive side of the x axis. Similarly, y is positive. So, we have to consider all the points which is lying on the positive side of the y axis and then we have to consider the points z greater equal to 0; that means, all the points which are lying on the positive side of z axis. And then, x plus y plus z equals to is less or equal to 1 is the upper part of the plane that is how to say forming a bounded region basically.

So, here z varies from 0 to 1 minus x minus y, y varies from 0 to 1 minus x and x varies from 0 to 1. So, we have got the limit of the integration and now, we can calculate the integral. So, basically integral triple integral x running from 0 to 1, y running from let us write these integrals a bit apart so that we can write these things neatly; y running from 0 to 1 minus x and z running from 0 to 1 minus x minus y f x y z is basically our x plus y plus z plus 1 whole square dx dy dz and now, what we will do is, now we will integrate this integrand here.

So, this will require some time to and perform this integral. So, we are running out of time in this lecture. So, we will continue with a similar example in our next lecture, where we will evaluate this integral and we can calculate this double this triple integral. So, I thank you for your time today and I will see you in the next class.

Thank you.