Integral and Vector Calculus
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Lecture – 21
Change of order of Integration

Hello students. So, up until last class we looked into the concepts of changing the variables in a double integral; we also learnt about Jacobian transformation.

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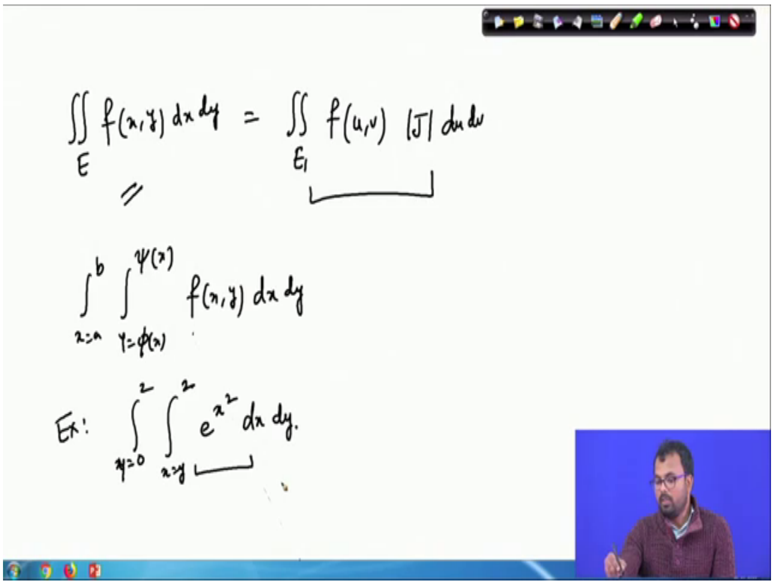

So, we actually looked into the fact that if we have an integral of let us say integral f x y dx dy, where the integration is uncertain region E. Then by using some sort of transformation, we can be able to transform the surface integral to a region E 1 and here we will have f of u v and Jacobian and then d u d v. So, we can change the variables. So the, of the integrand and by doing this change of variables, it will actually make our life a little bit easier because the transformed integral. There is a strong possibility can be a simpler integral and then, it will be easy to compute this integral than computing this difficult one and of course, we have to find out the range for this integration.

So, it will change accordingly based on the transformation and then we just perform, we just calculate the Jacobian and then we calculate this integral. And in most of the cases, this actually comes out to be very simpler integral to compute and the answer would exactly be same. Now, besides change of variable of integrals sometimes it is also wise to change the order of integration. So, suppose if we have range of let us say x running from a to b and then, y is running from some function of x to some function of x f x y dx dy.

So, it can be possible that this integrand is so this it can possible that this integrand is so complicated or how to say not so common that we cannot integrate it properly. So, it is not falling into one of the traditional how to say integral calculation. So, we cannot simply do the integration here. One such example could be let us say we have example we have something like this; integral 0 to 2. So, x is running from 0 to 2 y is run, x is running from 0 to sorry y is running from 0 to 2 and x is running from y to 2 e to the power x square d x d y.

So, now this integral let us say for example, here the integrand is not so common because in order to integrate this even with respect to x or with respect to with respect to y, I mean with respect to y it will be easy; but with respect to x we do not have any anti derivatives here or we cannot express e to the power x square in some method of substitution way. So, calculating this integral will not be straightforward and then, the only option we will have is to transform this integral into a simpler integral by doing the change of order.

So, if by doing the change of order, if we can be able to obtain a rather simpler integral; then, that can be evaluated and of course, the value of those two integral would be same. So, this is one such scenario. Now, let us work out an example where we can see that by changing the order of integration, we will still obtain the same answer.

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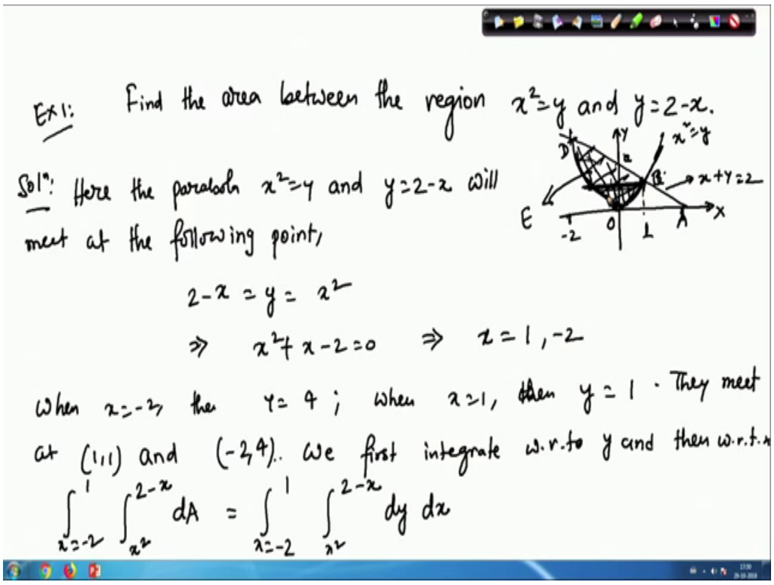

So, let us start with the following example. It says find the area between the region x square equals to y and y equals to 2 minus x. So, we have to calculate the area between these two curves. So, as I always say it is always suggestible to draw the area first. So, let us draw it. So, this is our x; this is our y. So, we have x square equals to y. So, it should be something like this a parabola passing through origin. So, it must touch the origin as well alright and then, we have x plus y equals to 2. So, let us draw this line as well. So, it must look something like this.

So, this is my line x plus y equals to 2 and this is my parabola x square equals to y. So, this is the area bounded between them. Let us call this as point O. Then, this is A; this is my B; this is C; this is D. Now, we since the parabola and the straight line, they are intersecting or the area is bounded between them. So, they must meet at some point.

So, in order to calculate that we have here the parabola x square equals to y and y equals to 2 minus x will meet at the following points, at the following point. So, how do we calculate? We just substitute y equals to x square in this equation. So, it will be 2 minus x equals to y equals to x square and therefore, this will become x square minus plus x minus 2. So, the roots of this equation will be x equals to we can factor it and then, it will be x equals to 1 and another root will be. So, another root will be, I can substitute x equals to 1.

So, one root is 1 and another root will be; if I substitute 2, then this will be 4. So, another root will be minus 2 alright. So, I have minus 2 here and I have 1 let us say 1 here. So, they are meeting at the point x coordinates are minus 2 and 1 and when x is minus 2. Then, y is 4 and when x is 1; then y is 1. So that means, they are intersecting at the point 1, 1.

So, they meet at 1, 1 alright and minus 2 comma 4. So, these are the two points they are meeting. Now, let us calculate the area in this way. So, we first integrate; so we first integrate with respect to y, with respect to y and then, with respect to x and then with respect to x. So, that is how we are maintaining the order. So, if we integrate with respect to x later. So, then in that case x will vary from minus 2. So, we can write simply minus 2 to 1 and y we can write so, y the area bounded between these two curves; so for these two curves x is varying from minus 2 to 1 alright.

So, that is the range for x and for y, y is varying between these two curves; so that means, it is above this parabola but below the straight line. So, it is above the parabola, below the straight line. So, above the parabola; that means, we have x equals to y equals to x square and below that straight line is 2 minus x d A, A denotes the area. So, now, we have x running from minus 2 to 1 and x square running from 2 minus x and we are integrating with respect to y first and then, we are integrating with respect to x. So, that is what we are doing.

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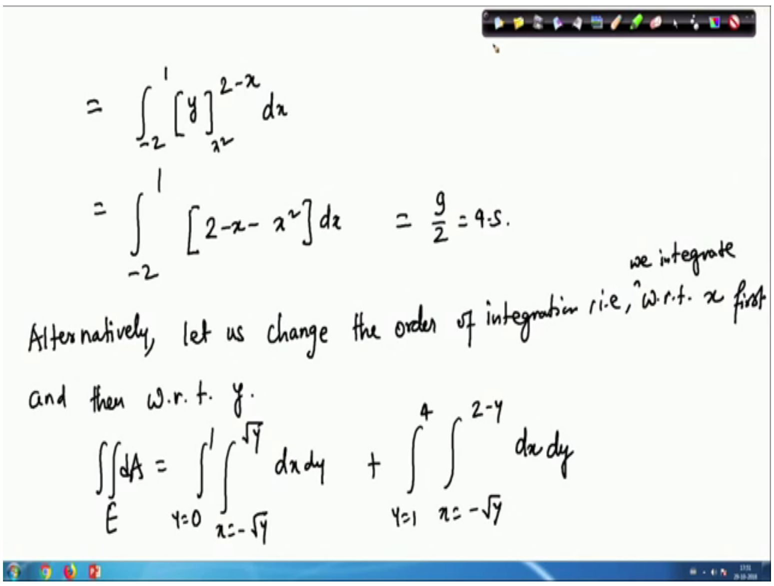

So, if we integrate with respect to y first, then we will end up with integral minus 2 to 1 y. So, integral with respect to dy. So, y and then ranges would be x to x square to 2 minus x dx and then we substitute the value; so, it will be 2 minus x minus x square. So, 2 minus x minus x square dx and after integrating and substituting the value, we will be able to obtain 9 by 2; so, ultimately 4 by 5. So, that is the area bounded between these two curves and not only that we integrate it with respect to y first and then, we integrate it with respect to x. So, that is the order we followed actually.

So, alternatively now let us change the order. Alternatively, let us change the order of integration of integration; that means, that is we integrate with respect to that is we integrate; so, we integrate with respect to x first and then with respect to y and then with respect to y. So, what will happen? So, what will happen is the following. So, we still have let us say integral over the region E; I am calling this region. So, I am calling this region as E. Now, if we want to integrate with respect to x first and then with respect to y, we have to see how x is varying. So, first of all they intersect at the point x equals to 1, y equals to 1.

So, let us draw a line parallel to x axis. So, this is my straight line y running from y equals to 1. So, now, here we can see that x is varying from. So, here we can see that y is varying from 0 to 1 right. So, y is varying from 0 to 1 and x will vary from this corner to this corner. So that means, minus square root of y to plus square root of y. So, y is from 0 to 1; x is from minus square root of y to plus a square root of y. So, let us write it. So, we separate the integral into these two sub domain; into these two sub regions.

So, this is my first sub region; let us mark this way and this is my second sub region, this is the second sub region and this is the first sub region alright. So, we will write here d A. So, we will write it as integral from 0 to 1 that is for y and integral x running from minus square root of y to plus square root of y and then, this is dx dy. Now, we are in this region. So, in this region y is varying from 1 to 4 right.

So, y is varying from 1 to 4 and x will vary from y equals to x will vary from y equals to minus is y is equals to minus square root of y to this line right because that is where the x is bounded. So, we will now write, we will now write y is varying from 1 to 4 and x is varying from minus of a square root of y to 2 minus y dx dy and then, we calculate.

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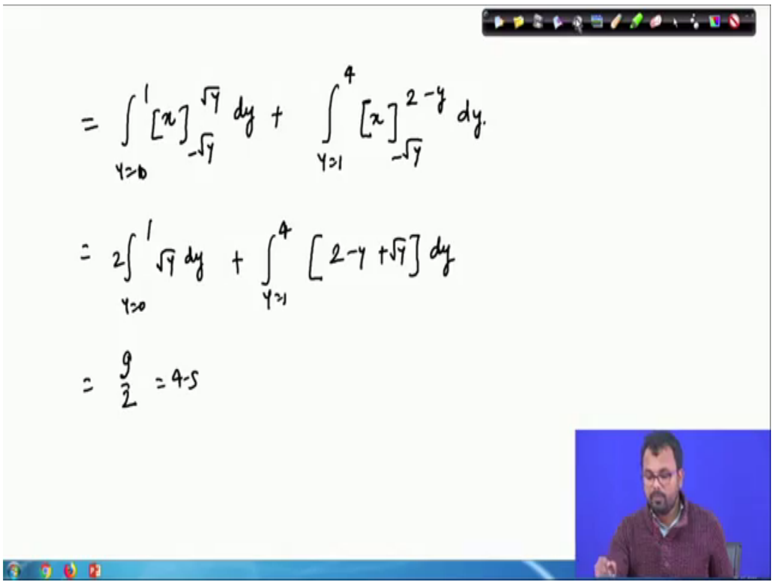

So, if we calculate, then we have basically integral y running from 0 to 1, then we have dx dy. So, we will have x and then this will be minus of a square root of y to plus of a square root of y dy plus integral from y running from 1 to 4; we integrate with respect to x. So, we integrate with respect to x and then, this will be x. Then, minus of a square root of y 2 minus y dy. So, if we substitute, then this will be y running from 0 to 1 this will be square root of y.

So, basically 2 square root of y dy plus integral y running from 1 to 4 2 minus y minus minus plus square root of y dy. So, if we integrate; then, ultimately here also we will obtain 9 by 2; so, 4 by 4.5. So, we saw that just by changing the order of integration, we do not get different answer because the area where we are actually doing the integration is fixed. It is just that we are changing the order of integration; that means, instead of integrating with respect to x first, we are integrating with respect to y. But that will not alter the value of the overall integral.

Because then in that case we are getting a different area which is not, which will not be the the actual area bounded by the curve that was given to us originally. So, even if you do the change of variables or change of order of integration, the area of the region bounded by these two curves or the integral which you are evaluating has to be same and it is same all the time. Now we see that sometimes we come across with integrals which are really difficult to evaluate and then, just using this tool that is changing the order of integration we might end up getting a little bit in easier integral to evaluate. For example, in this case we had to evaluate the area bounded between these two curves.

So, you see if we are integrating with respect to y first and then, integrating with respect to x then we are calculating a very simple integral and that is the answer. But if we change the order of integration, then in that case we are basically calculating two sub integrals. So, this one and this one and we also have to do some how to say some effort to basically calculate these two domains. So, sometimes it is useful sometimes it is not, it is just that it is from the intuition you have to understand whether the integral can be integrable or not and then, we use the tool. We will work out to one or two more examples just to make an idea clear. So, let us look into an another example.

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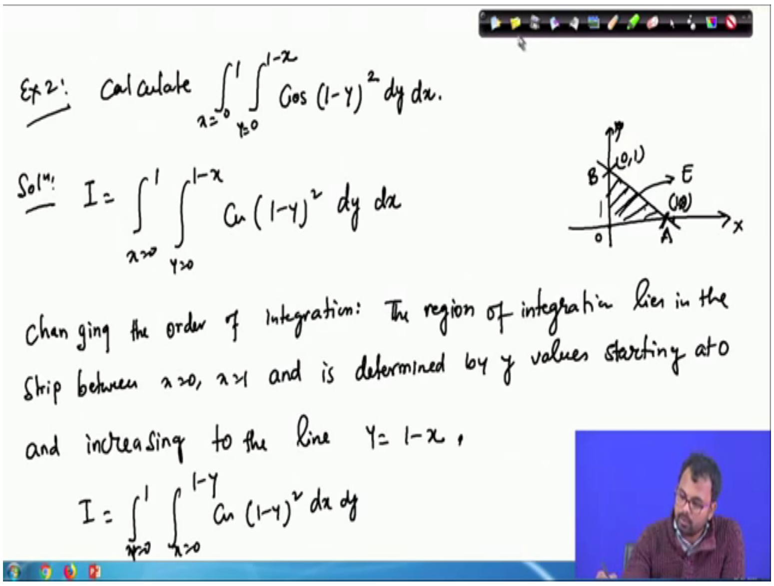

So, example 2, calculate or evaluate integral from 0 to 1; integral from 0 to 1 minus x. So, this is basically for x and this is basically for y cos of 1 minus y whole square dy dx. So, now, solution; first of all, let us draw this region; the y x 0. So, here our x is running from 0 to 1 and y is running from 0 to 1 minus x. So, it is basically a straight line x plus y equals to 1. So, let us draw that line. So, this is that point of intersection 1, 1 sorry as 1, 0 and this is 0, 1 all right.

So, that is basically our given region of integration. So, I am calling it as E. This is my point A and this is my point B. So, here just looking at this integral, let us write it as I. It is very straight forward to see that it is not possible to integrate this because even if you want to substitute 1 minus y square equals to some z or something, then in that case this cos of 1 minus y square dy, it cannot be reduced to a very simpler integral. So, this is quite I have to say a complicated form. Although, it looks simple; but it is not very easy to integrate this. So, then in that case the only way out we have is to change the order of this integration and see whether we can do something about this integral here.

So, first of all if we change the order of integration; so what will happen? So, changing the order of integration; so first of all, we write the area the region of integration; order of integration y, the region of integration lies in the strip between x equals to 0 x equals to 1 and is determined by y values starting at 0 and increasing to the line y equals to 1 minus x. So, so that is the area of integration or region of integration. Now, we see that if we instead of changing instead of integrating with respect to y, if we integrate with respect to x first; then, in that case the same region, in the same region our x will vary from then 0. So, x will start varying from 0 to 1 minus y and y will vary from again 0 to 1.

So, it is not really changing the limits here. So, if we change the order. So, changing the order of integration our I will look integral x running from 0 to 1 and sorry y running from 0 to 1 and x will run from 0 to 1 minus y because the integrand would remain same. It is just that we have changed the order. So, if we change the order, then in that case since here we do not have two sub regions like parabola and straight line and all that. Here, we just have this straight line and the usual coordinate x axis.

So, it is very easy to calculate the range for x and range for y. So, now, we have cos of 1 minus y square dx dy. So, when we integrate; now when we integrate, then it will be integral 0 to 1.

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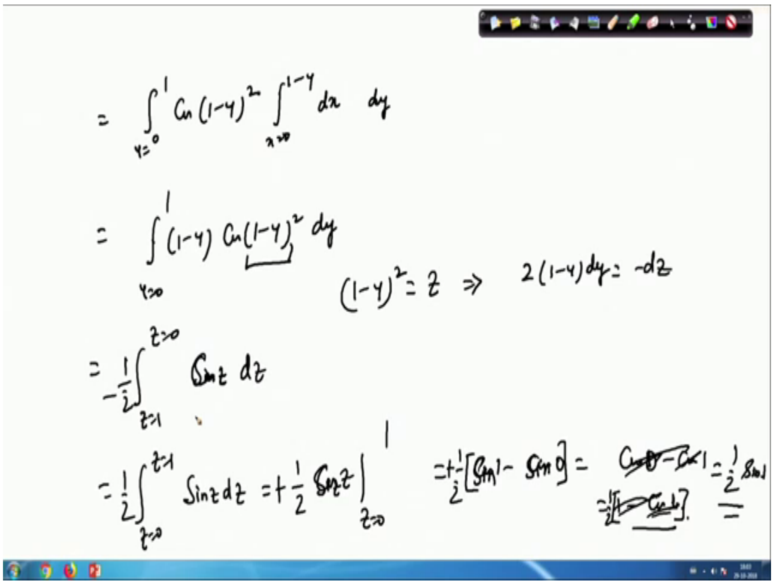

So, y running from 0 to 1 cos of 1 minus y square and then here we will have integral x running from 0 to 1 minus y dx and then, dy. So, this will be 1 minus y times cos of 1 minus y whole square dy y running from 0 to 1. Now, we can substitute this 1 minus y equals to z. So, now, we can substitute 1 minus y square equals to z. So, then this will be two times 1 minus y dy equals to minus of dz and if I substitute the value.

So, when y is 0. So, when y is 0, z is 1 and when y is 1, z is 0 and here I have minus and this will be sin z dz and since we have minus here. So, we can change the limit. So, sin z to z equals to 1 sin z dz and then this will be cos z z running from 0 to 1 it will cos 1 minus cos 0. So, basically cos of 1 minus minus. So, it will be minus cos 0 minus cos 1 basically 1 minus cos of 1. So, that will be the there is a half here. So, of course, there will be a half here. So, half and there we will have minus 1 by half.

So, 1 by half minus and then this will be 1 by 2. So, this is basically cos z and cos z is sin z. So, we will not get any minus here and then, this will be sin z sorry this is plus. So, this will be sin 1. So, this is sin 1 minus sin 0. So, ultimately we will get half of sin 1 right. So, we will not get these. So, half of sin 1 is the required answer.

So, just in this example also we saw that. So, this is basically how to say calculation error, I know I am pretty sure you can be able to fix that. So, here in this example also initially if we look at the; if you look at the problem, it is very clear that we cannot be able to integrate this function by with our traditional method.

Because there is no dz; there is no how to say that anti derivative form here. So, we cannot substitute 1 minus y square and get something and get something compensated from here or something. So, then in that case, we have to the only option have is to change the order of the integration. So, if we are changing the order of integration then, we have to find the range for the variable x and y. So, that is what we did and another good thing about this example was x was running from 0 to 1 minus y as y was running from 0 to 1 minus x. So, it was very fairly easy to calculate the range for the variable x and for the variable y it is just interchanging.

So, now y is varying from 0 to 1. So, that is what we have written here alright and if we have that; so then, in that case we integrate with respect to x first and then, we can have now we have that z how to say method of substitution formula working. So, we substitute 1 minus y square equals to z and then we just do some simple calculation. So, that is what your answer is. So, this is one such example where how to say changing the order of integration made our life a lot easier actually. Now, it may be possible that sometimes, it may be possible that sometimes changing the order of integration may not be that useful.

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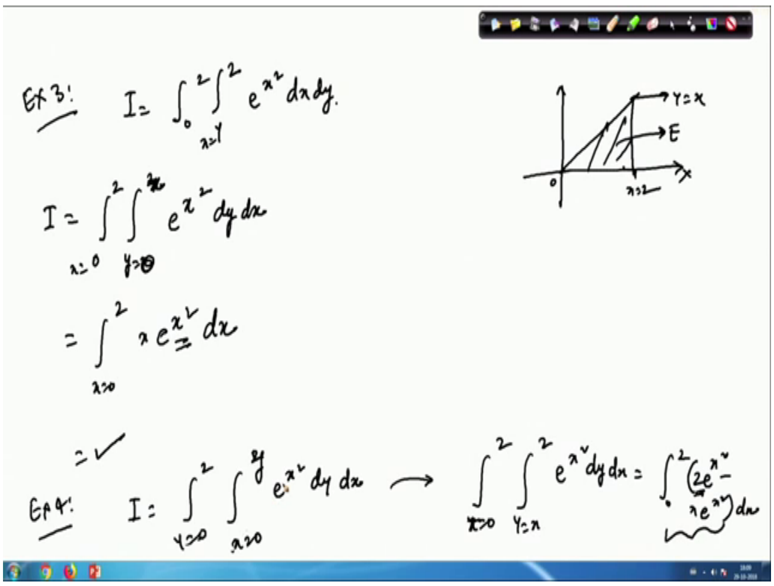

So, what I mean by that is let us consider an another example. Let us consider an another example. So, here we have our area of integration as 0 to so x is 2 and y equals to x. So, this is this is now y equals to x and this is the area of integration right so E. So, this is my area of integration and. So, after changing the area of integration, we can write I equals to; we can write I equals to 0 to 2. So, y would still be varying from 0 to 3 right? Yes and then x will vary from x will vary from 0 to 2, yeah and then y will vary from x to 2 and then we have e to the power x square dy dx.

So, if we have done that, then in that case it will be, no when we are changing the order of integration; then y is varying from 0 to 2 and x is varying from 0 to 2 and y will vary from; so y will vary from; y will vary from 0 to x. Yes, so y will vary from 0 to x. So, now, we can be able to integrate this integral with respect to y first. So, if we want to integrate then this will be integral x running from 0 to y and then this will be dy. So, substituting x, then it will be e xxx e to the power x square dx. Now, we can substitute this and then we get the value out of it.

However, if we had started with let us say; if we have started with let us say instead of this, we can have a problem where we can start with let us say this example. Let us say this example if we had integral I equals to y running from 0 to 2 and x running from 0 to 2 and y running from x to 2 e to the power x square dy dx and x running from y running from 0 to 2, where is that?

So, if we have started with y running from 0 to 2 and x running from let us change this x running from 0 to y. Then, in that case if we change the order of this in order of this integral, it would converge to integral y running from 0 x running from 0 to 2 and y would run from x to 2 e to the power x square dy dx and then, this will reduce to integral from 0 to 2 2 e to the power x square minus x e to the power x square dx which is a rather complicated integral to evaluate than evaluating this integral.

So, just starting with a different form of integral, we can be able to see that always changing the order of integration may not help. So, sometimes it is useful sometimes it is not, but most of the time it is useful. So, as in this example, it becomes a rather complicated to integrate especially the first term. So, sometimes we do not have to change the integral, but most of the time we do.

So, we will conclude this section here today and in the next class we will start with area and how do we calculate the area of a under a curve and several other things related to area volume and surface integral and we will work out more examples based on this topic in our assignment section, assignment sheets.

So, thank you for attention today and I will see you in the next class.