**Integral and Vector Calculus**
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**Lecture – 20**
**Change of variables in a Double Integral**

Hello students. So, in the last class we looked into the examples of double integrals on rectangular as well as unbounded regions. Today we will start with a change of order of integration and Jacobean transformation. So, how you can how to say transform a double integral into a double integral using Jacobean transformations and all.

So, we will start with a basic a basic theory, the proof of which we will avoid, but I will just give you an idea that how you do the change of variables in a double integral or how you change the order of integration. So, let us start change of variables in a double integral.

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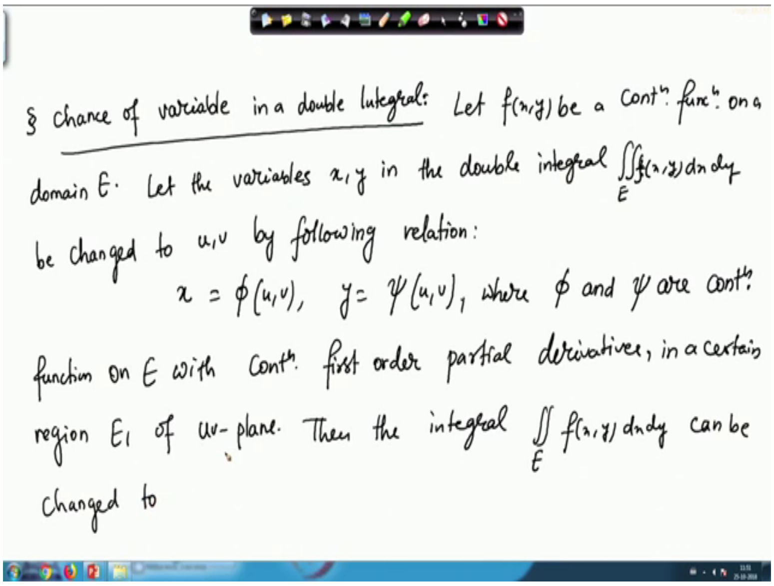


So, the reason why we do the change of variable is that, many double integrals can be evaluated easily if we do the change of variables.

So, sometimes you may have a very complicated double integral given to you and it might seem that it is very difficult to do that, but if you do some certain change of variables, then the overall integral might reduce to a very simple expression and that will be easy to evaluate and of course, after doing the change of variables, your answer will still remain the same. So, it would not happen that the value of that double integral would change. So, it would still remain the same; however, doing that change of variable actually saved us from lot of trouble actually.

So, let us see. Suppose we have a bounded function. So, let f x be f x y be a continuous function on a domain or region on a domain E, and let the integral let the variables x and y let the variables x y in the double integral in the double integral f x y, d x d y be changed to u and v by following relation by following relation. So, what is that relation? So, we are using x equals to phi u v and y equals to psi u v. So, these are some kind of transformation, where phi and psi are continuous functions on e, with continuous first order partial derivative all right; partial derivatives in a certain region E 1 of u v plane.

So, since we are transforming the variable x and y to the variable u and v so; obviously, from the region e we will get transferred to a region E 1 because for the variable u and v after doing the transformation they might belong to a different region. So, they might bound a different region; however, the dimension will not change. So, x and y they signify the two dimensional geometry. So, then in that case once you are using the transformation, we will get away the transformation function as phi u and v, and they will get again transferred to a two dimensional domain. Of course, the region might be two dimensional geometry, but of course, the region might be different and that is why we are saying that these you have phi and psi, they are how to say they are continuous and they have continuous partial derivative in a certain region E 1 so, because they are now in u v plane.

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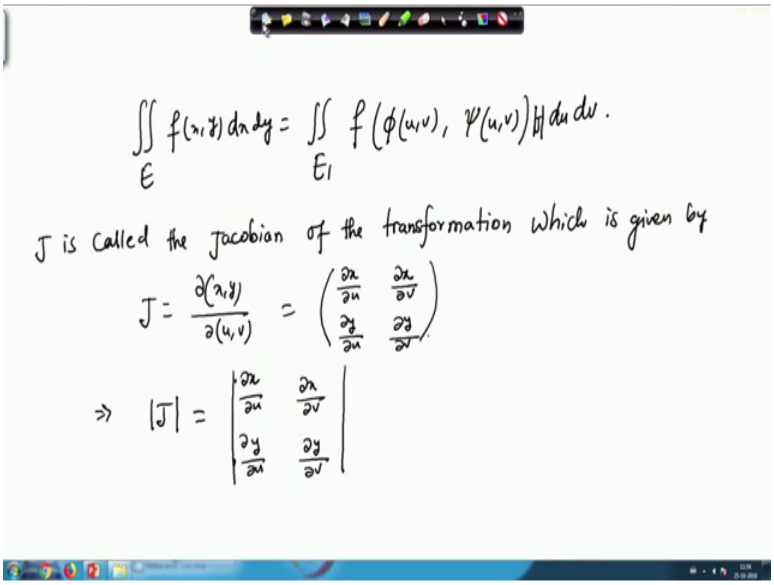


So, then the integral E f x y d x d y can be changed let us write in a new page integral over the region E f x y d x d y. So, we are changing x to phi u v. So, let us write phi u v and we are changing y by psi u v. So, psi u v and d x will be changed to d u and d y will be changed to d v; however, we will have an external factor here which is this J mod and this is not actually J mod this has a meaning. So, this has a meaning which is so, J is called the Jacobean of the transformation and when I say transformation, I basically mean this here.

So, this is my transformation and J is the Jacobean of that transformation which is given by given by J equals to del x y, del u v and its basically a matrix del x del u del x del v del y del u del y del v and the Jacobean determinant, the Jacobean determinant is basically determinant of this matrix here. So, that is del x del u del x del v del y del u del y del v.

So, this is basically our Jacobean matrix. So, first of all when we are doing the how to say change of variable, we have to find out these transformation. So, we are the one who has to calculate these two transformation and once we get these two transformation, then we have to calculate the Jacobean determinant so, that we can do the change of variable. But before that we cannot simply put any transformation here. So, that transformation should also have these two properties. So, it should be continuous a continuous function on E 1 and it should have continuous first order partial derivatives as well.

So, once we have all these ingredients and we can do the change of variable in our integral. Now we might or one might ask that change of how do we change.

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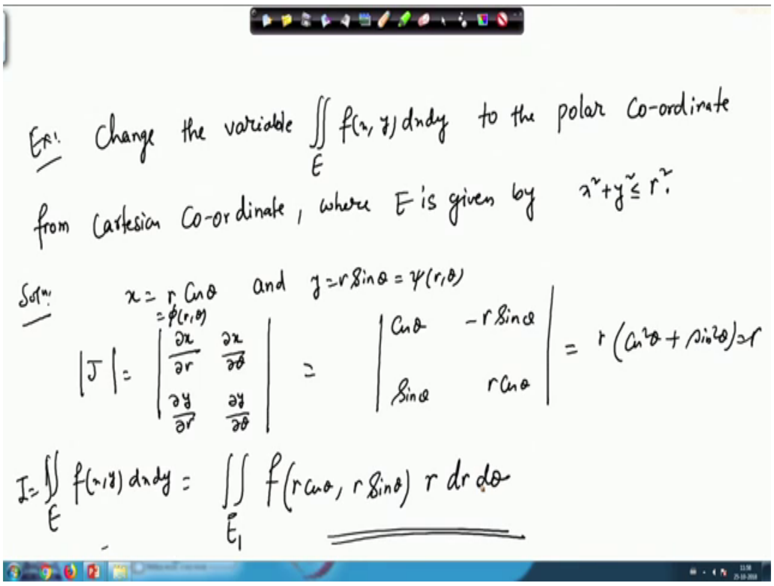


So, an example interesting example could be; change the variable let us say any we have an integral over the region E f x y, change the variable f x y d x d y to the polar coordinate from Cartesian coordinate; t where E is given by x square plus y square is less or equal to let us say r square; r can be any a b c whatever radius we prefer.

So; obviously, for a circle we substitute x equals to r cos theta and we substitute y equals to r sin theta. So, from here we will have. So, we can calculate the Jacobean. So, let us calculate the Jacobean determinant. The Jacobean determinant is determinant of del x. So, our new variables are r and theta. So, I can write this as a function of r and theta. So, phi r theta and this one is basically psi r theta, and what is our phi r theta? Its r cos theta and what is our psi r theta? It is r sin theta.

So, we have del x del r; del x del theta and then here we have del y del r and then del y del theta. So, what is my del x del r? Del x del r would be cos theta simply del x del theta will be minus r sin theta and then del y del r will be sin theta only and del y del theta will be r cos theta. So, if we evaluate this determinant, then this will be r times cos square theta plus sin square theta.

So, ultimately we will obtain r. Therefore, our integral I which is now transferred into a polar coordinate, our new region E 1 then f of x is phi r theta. So, that means r cos theta and y is psi r theta which is r sin theta, and then we have mod then we have Jacobean determinant which is basically r and then we will have d r d theta. This E 1 can be replaced by 0 to r and from theta can be replaced by 0 to 2 pi.

So, we can replace the c 1 by the limits of r and theta and this is the required transformed or changed integral from Cartesian coordinate system to polar coordinate system. And using this kind of formula where. So, here one can be able to transform or change the variable of an integral of a given double integral basically. So, let us work out and another example. So, this one is actually based on polar coordinate system.

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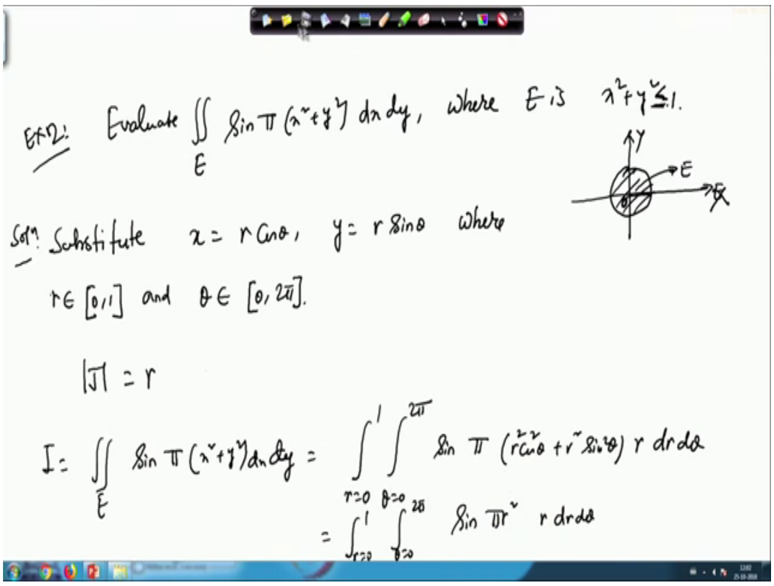


So, its evaluate integral over the region E sin pi x square plus y square d x d y where E is given by x square plus y squared equals to one less or equal to 1, yes. So, here our domain of integration is basically this circle with radius one right. So, this is our region e sorry this is x this is y. So, that is origin.

Now, we can see that here we have a sin pi x square plus y squared d x d y. If we want to evaluate using our traditional method like finding the limit for x which is 0 to 1 and the limit for y is minus of square root of 1 minus x square to square root of 1 minus x square, then it might become a little bit difficult. However, if we do the change of variable from Cartesian to the polar coordinate system it might be beneficial.

So, let us see whether doing change of variable helps or not, that traditional method will always work, but its just that it might get tedious or it might get difficult at some point or the calculation basically. However, doing that the change of variable might make our life a little bit easier. So, let us see. So, substitute x equals to r cos theta and y equals to r sin theta; where our r is running from 0 to 1 because the radius is 1 and theta will run from 0 to 2 pi right. So, where r belongs to 0 to 1 and theta belongs to 0 to 2 pi right ok.

Now, we know from the previous example, that the determinant of Jacobian or Jacobian determinant is basically r. So, here we can do the similar calculation which I am leaving up to the students and I am just going to use that answer, which is Jacobian determinant is r and r is 1. So, let us write just 1 and now we have sin pi yes.

So, now we have integral I over the region E sin pi x square plus y square d x d y. So, now, I am since I am transforming this whole integral into polar coordinate system, I can write the range for r which is 0 to 1, I can write range for pi which is a theta which is 0 to 2 pi and then I have sin pi x square plus y square; x square plus y square is r cos theta r square cos square theta plus r square sin square theta and. So, this value will remain r as it is, because we are taking r as a variable. So, this will become r and d r d theta. So, we take r square common. So, it will be cos square theta plus sin square theta then that value will be o1 and then we are left with r running from 0 to 1 theta running from 0 to 2 pi sin pi r square times r d r d theta. So, here it is a very simple integral to evaluate we can substitute t is equals to r square.

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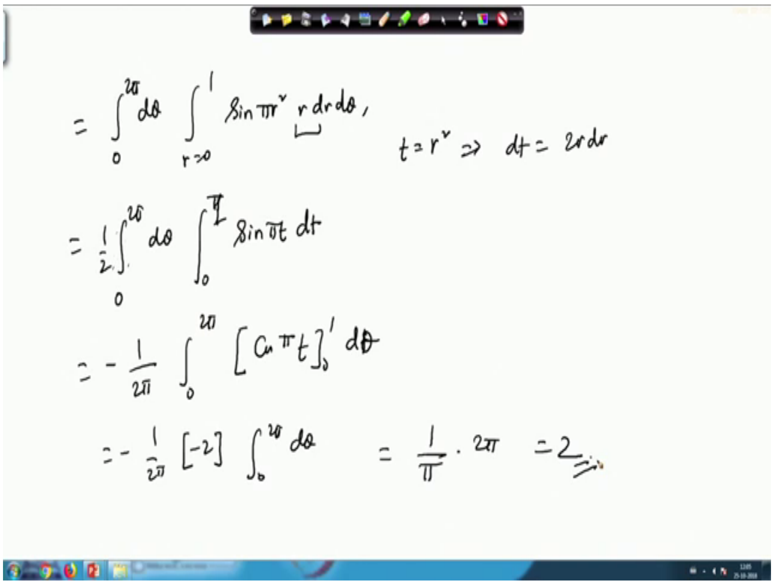

So, putting t equals to r square and we separate these two integrals. So, 0 to 2 pi d theta, and then we have r running from 0 to one sin pi r square r d r d theta. So, let us put t equals to r square, then d t equals to 2 r d r and then this whole thing will reduce to minus 1 by 2 pi or I can write one more step.

So, this can be written as integral from 0 to 2 pi d theta and half integral from 0 to pi sin pi t d t and if I integrate then this will reduce to minus 1 by 2 pi integral from 0 to 2 pi sin pi t is cos pi t and so, we will have cos pi t, and when t is when r is one sorry. So, when t is 0 yeah this is 1.

So, we will have cos pi t and this will be from 0 to 1 d t, now this can be written as minus 1 by 2 pi cos pi t d theta this can be written as cos pi is minus 1 and cos 0 is one. So, basically minus 2 and we will have 0 to 2 pi d theta. So, this will be 1 by pi times 2 pi. So, the answer is ultimately 2. So, here as you can see that initially we were given a very complicated integral to evaluate, but just by changing the variables; that means, changing the variables from Cartesian coordinate system to polar coordinate system, we can how to say calculate this whole integral in a lot more simpler manner.

So, this just in reduced to a sin pi r square times r d r d theta and calculating that one, it was very easy because we substituted t equals to r square which will give you d t equals to 2 r d r, then you can adjust this factor here and the rest of the calculation will become fairly easy. Now not only always polar coordinates, but one can also try to how to say transform them into a spherical coordinate system. If we have a triple integral, then you can also transform them to or the change of variable change the variables to spherical polar coordinate system, which we will learn later on.

So, this is a very handy tool actually to a calculate difficult integrals easily. Next we can we can work out and another example. So, an another example could be let us say prove that.

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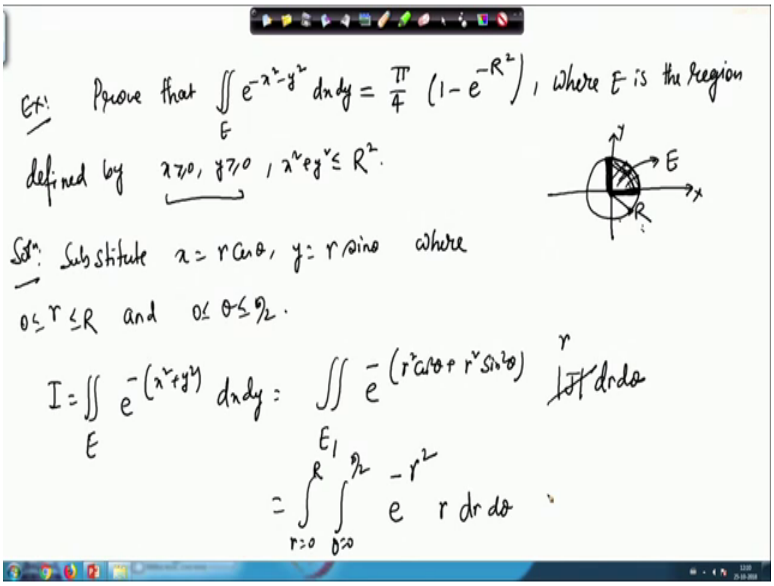


So, we will solve this example also in the similar fashion the way we did before edges that we are practicing one more problem. So, where E is the region defined by x greater or equal to 0, y greater or equal to 0 and x square plus y square is less or equal to R square. So, here there is something new actually in this problem.

So, first of all let us draw our region. So, if we draw our region, then we have a circle. So, let us draw this circle and that is our radius R; however, we are also given two separate conditions which is x is positive and y is positive; that means x is positive. So, we are not interested in this half and then y is positive. So, we are also not interested. So, x is positive so; that means, we are not interested in this half and then we have y is positive, then we are also not interested in this half. So, then our region is bounded by this circle and then these two lines x equals to 0 and y equals to 0. So, this is our region E right.

So, in this region of course, we have r is running from this small r is running from 0 to capital R up to the radius, but then theta is running from 0 to pi by 0 only because we are not completing the whole circle we are just sticking to this half only. So, in that case we have theta running between 0 to pi by 2. So, let us provide this transformation. So, substitute or put substitute or put x equals to r cos theta and y equals to r sin theta, where r is running between 0 to capital R and theta is running between 0 to pi by 2 right.

So, now we write this integral here, I equals to integral over the region E to the power minus x square plus y square d x d y. So, when we transform it to the region E 1, we will write this region in E 1 in a minute we can write minus r square cos square theta plus r square sin square theta and then d x d y will be converted into Jacobian Jacobian determinant, and then d r d theta. So, this Jacobian is basically our r. So, we can write instead of this Jacobian we can simply write r.

So, now here we will replace this region E 1 with the limits for r and theta. So, r is running from 0 to capital R and theta is running from 0 to pi by 2. In the integrand, we will take r square common and then we have cos square theta plus sin square theta. So, this will remain as it is and we will have r d r d theta right.

So, now I can substitute instead of r square, I can substitute t again and like what we did in the in the previous example we can also do the same thing here and ultimately we will end up with.

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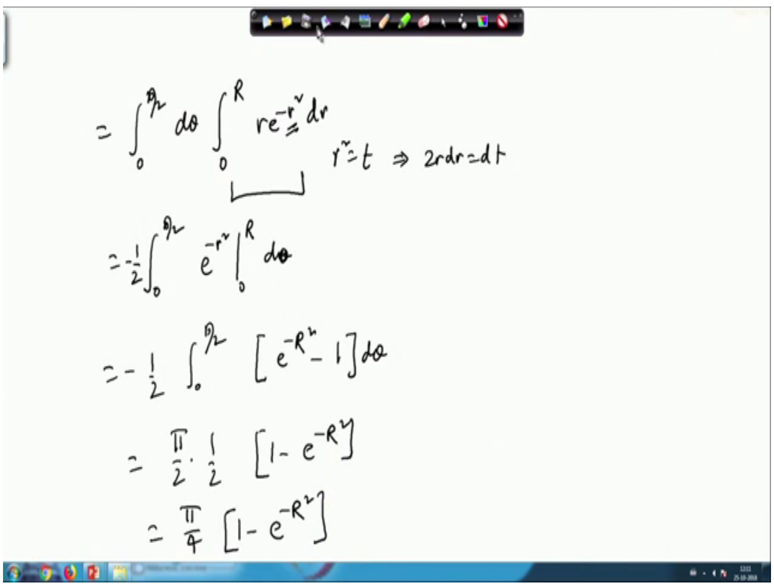

So, we will first write 0 to pi by 2 d theta then integral from 0 to capital R e to the power minus r squared d r and then we substitute for r square equals to t and then we transform this whole thing into function of e to the power minus t and then this will ultimately give us e to the power minus r square integral from 0 to capital R d r and we will have sorry d theta and we will have minus half at the front yeah.

So, this will be 21 d r it was to d t and this can be written as minus half integral from 0 to pi by 2 this will be e to the power minus r capital R square minus e to the power 0 so, basically 1 and then d theta. So, this will be pi by 2 times half times 1 minus e to the power minus R square. So, ultimately we will have pi by 4 times one minus e to the power minus r square and I believe this is what we needed to prove yes.

So, see initially it was a little bit how to say complicated in a way to evaluate this integral, but just using some transformation which was how to say polar coordinate system which transferred this Cartesian coordinate system into a polar coordinate system with we were able to reduce this whole e to the power minus x square plus y square into a fairly simple method of substitution integral and by doing that metal substitution our required result will be this.

And of course, using these transformation of course, makes life quite easy and there will be a lot of situations where we come where we come across a very complicated integral double integral, but just using some simpler transformation we can be able to reduce that complicated integral into a fairly simple one and this is one such example. We will we will restrict ourselves to practicing examples on this topic up to here, in the next class we will start with change of order of integration. So, today we saw a change of variables using Jacobian transformation, in the next class we will see change of order of integration and triple integral and that we conclude this section as well. So, I look forward to your next class.

Thank you.