Integral and Vector Calculus
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Lecture – 02
Partition, Riemann intergrability and One example (Contd.)

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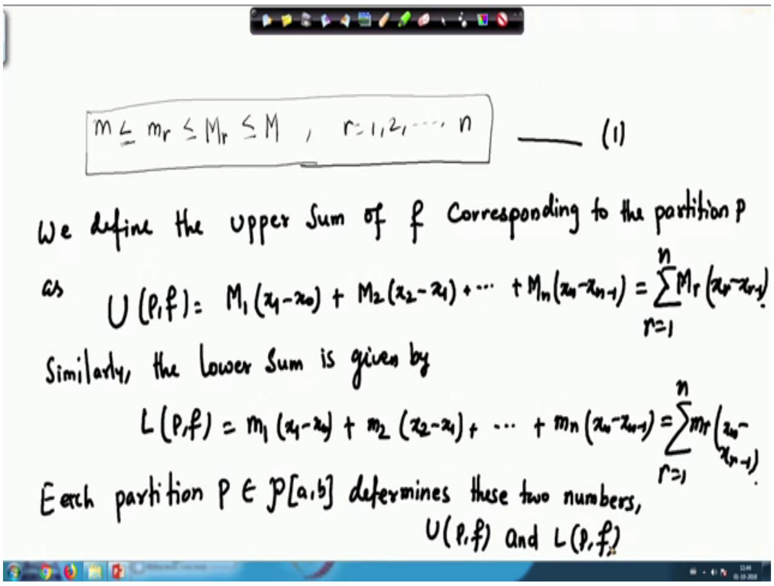

Hello students. So, up until last lecture we lived into an inequality of this type, where we had the lower bound of a function f defined on [a,b] is less or equal to the lower bound on all the sub intervals, which is less than or equal to of course, the upper bound of the function f on all the sub intervals which is less than or equal to the upper bound of the function f on [a,b].

Let us name this relation. So, let us call this relation as equation 1. Now, we define the upper sum of the function f corresponding to the partition P. As we write

$$U(P,f) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + ... + M_n(x_n - x_{n-1})$$

So, you can write it as in the form

$$\sum_{r=1}^{n} M_r(x_r - x_{r-1})$$

So, this is our upper sum, similarly we can define our lower sum. The lower sum is given by L(P,f) and you may have guessed it will be given by

$$m_1(x_1 - x_0) + m_2(x_2 - x_1) + . . . + m_n(x_n - x_{n-1})$$

we can use the summation concept and then it can be written as

$$\sum_{r=1}^{n} m_r(x_r - x_{r-1})$$

So, these two are basically called as the upper sum and the lower sum. Now, you can see that each partition

$$P \in \wp[a, b]$$

determines these two numbers U (P, f) and L (P, f) isn't it. So, for every partition P we will be able to get these points x_0 x_1 x_2 x_3 up to x_n and based on those points we can be able to obtain our sub intervals x_0 to x_1, x_1 to x_2 and so on.

And on each one of these sub intervals, we can be able to find our lower bound for the function f and the upper bound for the function f, mainly because the function f is bounded on [a,b]. And based on that upper bound and lower bound on all of those sub intervals we can be able to calculate the upper sum, which is given by

$$M_1(x_1 - x_0) + M_2(x_2 - x_1) + . . . + M_n(x_n - x_{n-1})$$

and we can be able to calculate our lower sum, which is given by $m_1(x_1 - x_0) + m_2(x_2 - x_1) + . . . + m_n(x_n - x_{n-1})$

So, each partition and this in this family of partitions will determine these two numbers U (P, f) and L (P, f), you can see that they are basically numbers because upper bound and lower bound are numbers and x_1, x_2, x_3 are points on real line. So, basically the difference is also a number. So, ultimately we are obtaining two numbers with the help of this partition P alright.

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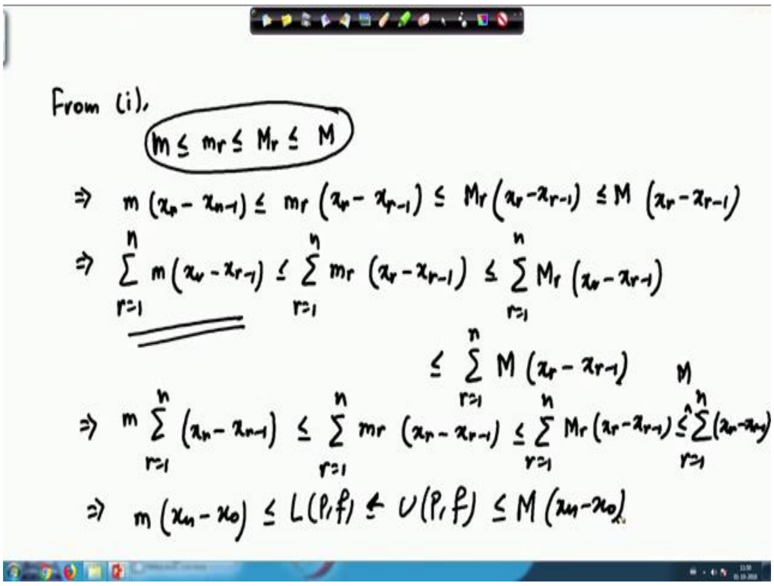

So, now what we have is so from inequality 1 where is that let us go back to the previous slides ok. So, in this slide from inequality one we have

$$m \le m_r \le M_r \le M$$

So, we will write that inequalities so,

$$m \le m_r \le M_r \le M$$

and we will multiply this whole inequality by

$(x_r - x_{r-1})$.

Let us multiply so, as I was saying that $(x_r - x_{r-1})$, they are both points on real line. So, their difference is also a real number basically. Now, even if you have a negative interval; that means [-1,-2] and if you divided into equal subintervals you will still get this difference as a positive. So, $(x_r - x_{r-1})$ is always a positive number and that is why when you multiplied the inequality here, this inequality sign did not change alright.

So, next we will take the summation on both sides. So, this is true for every r running from 1 to n. So, if I take summation on both sides so, this will be

$$\sum_{r=1}^{n} m(x_r - x_{r-1}) \le \sum_{r=1}^{n} m_r(x_r - x_{r-1}) \le \sum_{r=1}^{n} M_r(x_r - x_{r-1}) \le \sum_{r=1}^{n} M(x_r - x_{r-1})$$

So, here in this sum on the left hand side here, in this sum small m is independent of r, basically because small m is the lower bound on the whole interval [a,b]. So, it is not relevant to each one of those sub intervals. So, you can take that m out of the sub interval and then it will be basically

$$m\sum_{r=1}^{n}(x_r - x_{r-1}) \le \sum_{r=1}^{n} m_r(x_r - x_{r-1}) \le \sum_{r=1}^{n} M_r(x_r - x_{r-1}) \le M\sum_{r=1}^{n}(x_r - x_{r-1})$$

also we take capital M out of the interval, because capital M is also independent of r.

Now, here what we are basically doing we are basically summing the difference of all the sub intervals so; that means

$$(x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + . . .$$

.So, if we if we expand this summation then you will be able to notice that except $x_n - x_0$ all other points cancels out. So, if you expand the summation, we will be able to see that and ultimately we will be left with $x_n - x_0$. Similarly, here this term is defined as the lower sum L(P, f) if we look into the previous slide. So, here so L(P, f) is defined by this sum here.

So, we can write it as L(P,f) and this sum is defined as U(P,f) so, U(P,f). And here it is basically $M(x_n - x_0)$

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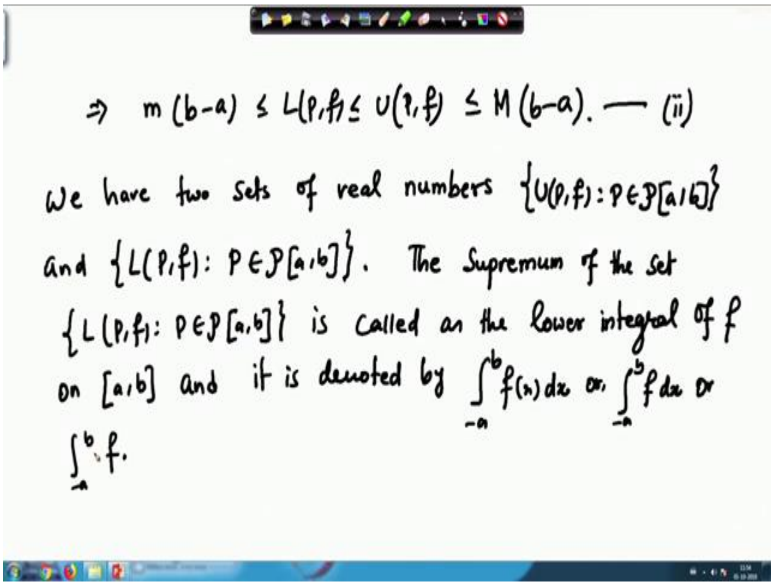

And this $(x_n - x_0)$ we know that x_n is our point b and x_0 is our point a is less or equal to L(P,f) is less or equal to U(P,f), which is less or equal to M(b-a).

So, here what we have is basically an inequality which is also important. So, this inequality says that your lower sum and your upper sum, will be bounded by m(b-a). So, basically the lower bound of the function f on [a,b] times the difference of the endpoints basically and the upper bound is M(b-a) which means that upper bound of the function f times, the difference of the endpoints will be an upper bound for your lower sum as well as the upper sum.

So, this is the second inequality which we needed to establish, before we go to the Riemann integrable functions. Now, based on this we can define two numbers so, the first number is so, here we have two sets of real numbers. So, one is as we said every partition P determines U(P,f) and we have L(P,f) for every partition P in that family of partition [a,b].

So, basically we have two sets of real numbers for every partition P in that family of partitions and the supremum of this set L(P,f) such that P is in $\wp[a,b]$ is called as the lower integral of f on [a,b]. And it is denoted by

$$\int_{-a}^{b} f(x)dx$$

we put a small minus sign here just to signify that it is a lower integral f(x)dx, or you can shorten it by

$$\int_{-a}^{b} f dx$$

or we can even shorten it by

$$\int_{-a}^{b} f$$

So, when I write

$$\int_{-a}^{b} f$$

it basically means that integral from a to b lower integral of course, f(x)dx. This is just one of the notations to save time in a way now that we have the lower integral we can also define the upper integral, for the upper integral lets go to the next page ok.

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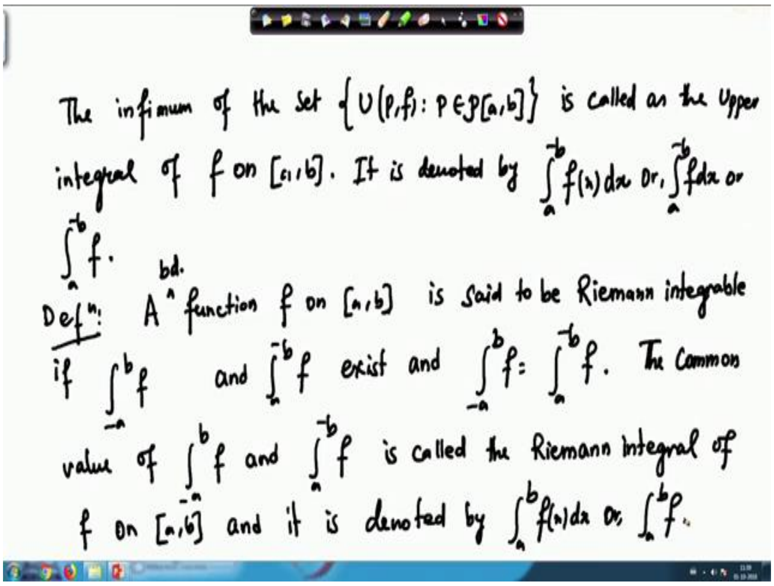

The infimum of the set of all those U(P,f) such that P is in $\wp[a,b]$ is called as the upper integral of f on [a,b]. And we can write this upper integral and it is denoted by

$$\int_a^{-b} f(x)dx$$

we put a dash sign or minus sign in the upper limit f (x)dx, or we can write it as

$$\int_a^{-b} fdx$$

, or to save time we can even write

$$\int_a^{-b} f$$

So, based on the upper sum and lower sum, we can be able to define the upper integral and the lower integral, they are also sometimes called as upper Riemann integral and upper lower Riemann integral.

Now, that the definition of the upper integral and lower integral are given and we have also given the notations, we can define the Riemann integrable function. So, definition a function f, or a bounded function f on [a,b] is said to be Riemann integrable if both $\int_{-a}^{b} fdx$

and,

$$\int_{a}^{-b} f dx$$

lets follow one notation. So, I am removing this dx here so, lower integral and upper integral exist and they satisfy this relation

$$\int_{-a}^{b} f dx = \int_{a}^{-b} f dx$$

So, the lower integral and the upper integral has to be equal, then such functions are called as Riemann integrable function and the common value of this is called the Riemann integral of f on [a,b] and it is denoted by

$$\int_{a}^{b} f(x) dx$$

we can write it f (x)dx or we can shorten it like before simply by

$$\int_{a}^{b} f dx$$

or

$$\int_{a}^{b} f$$

So, in order to talk about the Riemann integrability, you see that we had to first look into the concepts of partition that, how you get the partition of a closed interval, then from that partition we formed non overlapping sub intervals based on those sub intervals. We could be able to define those lower bounds and upper bounds not only for the function on the whole interval, but also on there was sub intervals. And for every such partition you will get a different type of non overlapping sub intervals and then you get different types of lower bounds and upper bounds on those sub intervals.

Now, once we have those lower bounds and upper bounds, we were able to obtain the lower sum and the upper sum. And based on those lower sum and upper sum we can be able to obtain this lower integral and the upper integral. And if those lower integral and upper integral are same then in that case we say that the function is Riemann integrable, here the upper sum and the lower sum they depend heavily on what kind of partition you are choosing.

So, for every partition P of this family of partitions, we can be able to obtain this upper sum and lower sum. So, these are basically two sets of real numbers depending on the partition P and based on those two real numbers, which is lower sum and upper sum we can be able to define the lower integral and the upper integral for the function f.

And if those lower integrals and upper integrals are same, then in that case the function is said to be Riemann integrable and their common value; that means, when they are equal then that particular value is called as the Riemann integral of the function f on [a,b]. And we use a how to say a simplified notation in a way simply

$$\int_a^b f dx$$

So, this is just how to say a preparation to define the Riemann integral or Riemann integrable functions on [a,b]. We will look into now an example, where we will show that a function is Riemann integrable. Before I proceed any further I have provided you a list of references, we can where you can look into for the details the things which I am teaching.

Most of a time I will try to follow my own lecture notes and sort of prepared my own lecture notes, which I will follow and I will also try to give you some examples based on the concepts which I am teaching and hopefully that will help. So, now we will look into one example, where we will show that a given function is Riemann integral or Riemann integrable or not.

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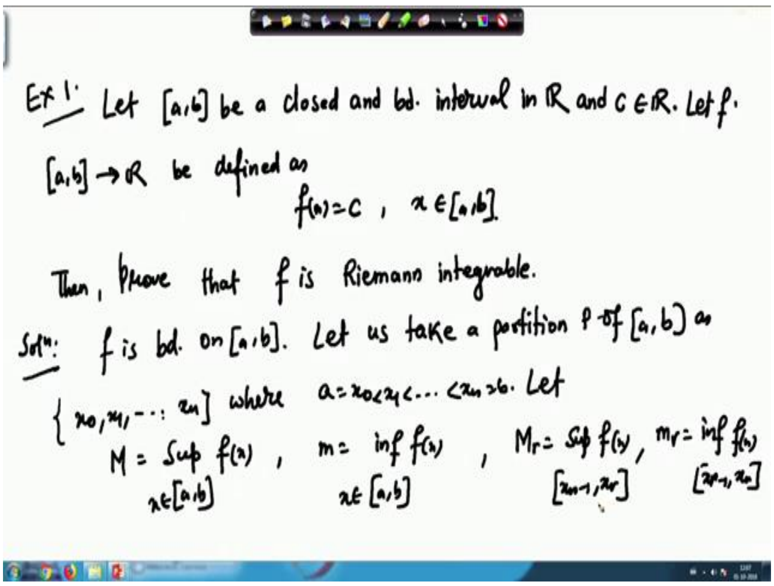

So, let us start with example 1 so, statement let [a,b] be a closed and bounded interval in set of all real numbers $\mathbb{R}$ and c is any arbitrary point or number in $\mathbb{R}$. And let f mapping from [a,b] to R be defined as f(x)=c ∀ x in [a,b], then prove that f is Riemann integrable alright.

So, we will look into the solution, before we start solving this problem. We first have to just gather the ingredient what do we need to show that Riemann integrability. So, we have to show that the lower integral is equal to the upper integral. Now, in order to obtain the lower integral or upper integral, we first have to obtain the lower sum and the upper sum.

And before we can obtain the lower sum and upper sum, we first have to find out a partition for [a,b] based on that we have to find out the lower bound and the upper bound for this function f, on those sub intervals as well as on [a,b]. Here we are a little bit in luck because the given function is constant. So, for the constant function it does not matter what kind of partition, or what kind of points you were choosing the value would remain always the same so; that means, the lower bound and the upper bound would always be same not only on that closed interval, but also on those sub intervals as well. However, just to make things clear or the concept clear, we are going to solve this example.

So, let us see first of all f is bounded on [a,b] and let us take a partition P of [a,b] as

$$[x_0, x_1]. \; .x$$

sorry so, $\{x_0, x_1, . \; . \; . x_n\}$

where

$$a = x_0 < x_1 <. \; . \; . < x_n = b$$

and up; obviously, the sub intervals would be

$$[x_0, x_1], [x_1, x_2],. \; . \; ., [x_{n-1}, x_n]$$

And let capital M be the supremum of the function f(x) on [a,b] and small m is the infimum of the function f(x) on [a,b]. Similarly we define capital M_r as the supremum of f(x) on $[x_{r-1}, x_r]$. And a small m_r as the infimum of the function f(x) on the interval $[x_{r-1}, x_r]$, by the way if I do not write

$$x \in [x_{r-1}, x_r]$$

like the way, I have written here it always means that x is in $[x_{r-1}, x_r]$ where r is running from 1 2 3. . . n.

So, since f is a constant function its upper bound and the lower bound would always be same, not only that it will be same on that I mean on those sub intervals.

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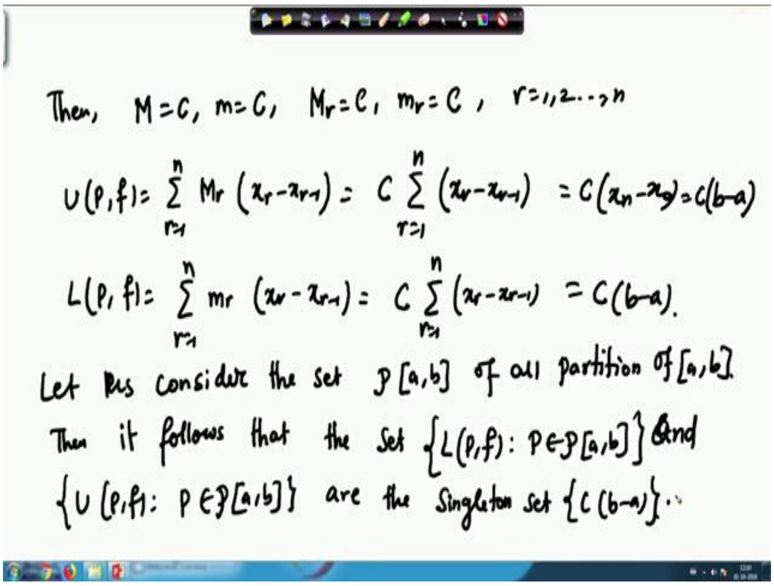

So, since f is a constant function our capital M, the upper bound is basically the constant c our lower bound on the whole interval [a,b] is again c our upper bound on each one of those sub intervals is c and our lower bound on each one of those sub intervals is again c, where r is running from 1 2 3 up to n.

So, based on that I can calculate my upper sum U(P,f), which is basically

$$\sum_{r=1}^{n} M_r(x_r - x_{r-1})$$

So,
M_r is basically c so, c will come outside of the summation because it is independent of r. And here we will have (x_{r-1}, x_r) . Now, again this is similar to what we did earlier. So, if you expand the summation, then you will basically have $(x_n - x_0)$ left. So, we will have

$$(x_n - x_0)$$

left x_n is basically our b and x_0 is basically our a, similarly we can calculate our lower sum which is

$$\sum_{r=1}^{n} m_r(x_r - x_{r-1})$$

this is

$$c\sum_{r=1}^{n}(x_r - x_{r-1})$$

Similarly this will also give

$$c(x_n - x_0)$$

And therefore, we will obtain

$c(b-a)$ alright.

So, now let P be let us so,let us consider the set of all partition the set $\wp[a,b]$ of all partitions of [a,b], then basically it follows that, the set L(P,f), where all such P is in $\wp[a,b]$. And the set U(P,f) where P is in $\wp[a,b]$, I mean they are both basically how to say the singleton set $c(b-a)$.

Because it does not matter what kind of partition, you will choose since it is a constant function, we will always obtain $c(b-a)$. And therefore, the least the how to say the supremum or the infimum that is the least upper bound or the least or the how to say the greatest lower bound, will be always $c(b-a)$.

So, like we are talking about the supremum of L(P,f) and infimum of U(P,f)

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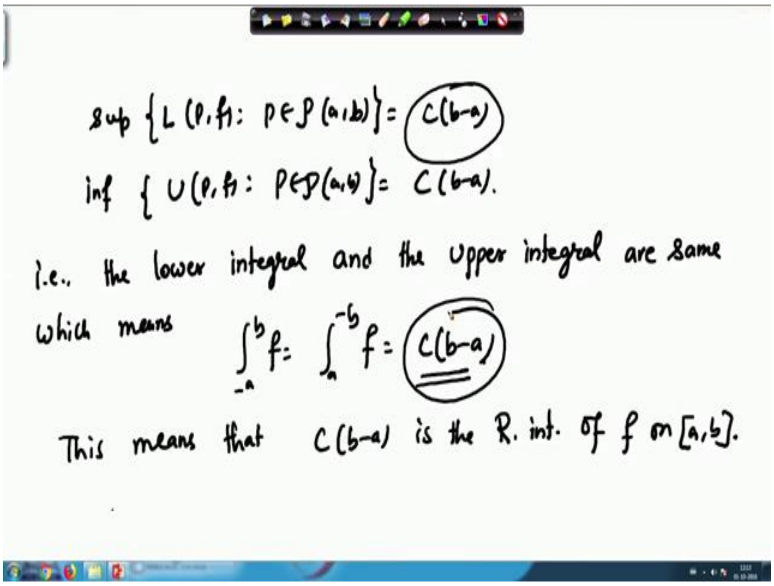

If we take for this these two sets actually, it will always be so, what I am trying to say supremum of all such L(P,f) such that P is in $\wp[a,b]$,will be $c(b-a)$ and infimum of all such U(P,f) such that P is in $\wp[a,b]$ is again $c(b-a)$ so; that means the lower integral.

So, that is the lower integral and the upper integral are same which means

$$\int_{-a}^{b} f = \int_{a}^{-b} f = c(b-a)$$

And since they have a common value which is basically $c(b-a)$ this is the Riemann integral of the function f on [a,b].

So, this means that $c(b-a)$ is the Riemann integral of f on [a,b] where f is the constant function and it is denoted or we can just leave it like that. So, for this constant function f is equals to c we were able to show that this constant function is Riemann integrable on [a,b] for that of course, we needed to calculate the upper sum and lower sum which was basically $c(b-a)$. And if we take the supremum of all such upper sum as sorry of all such lower sum and the infimum of all such upper sums.

For all I mean how to say set of all these partitions, then you always get it has a $c(b-a)$ We will always get it as c times b minus a. And since the lower integral and the upper integral is same, this common value is basically the Riemann integral of this function f on [a,b], we will look into one more example in the next lecture. And we will conclude today's lecture on this example.

So, thank you for your attention.