**Integral and Vector Calculus**
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**Lecture – 19**
**Example of Integral over a Region E**

Hello, students. So, up until last class we looked into double integrals, how do we calculate the double integral on a rectangular domain and we worked out several examples. Now, we also introduced about the concepts of a double integral on a bounded region. So, instead of having a rectangle if we have a bounded region, then how can we calculate the double integral in such cases. So, I have also showed you the idea how do we do the calculation.

Today, we will worked out a few examples just to make those ideas are slightly more clearer.

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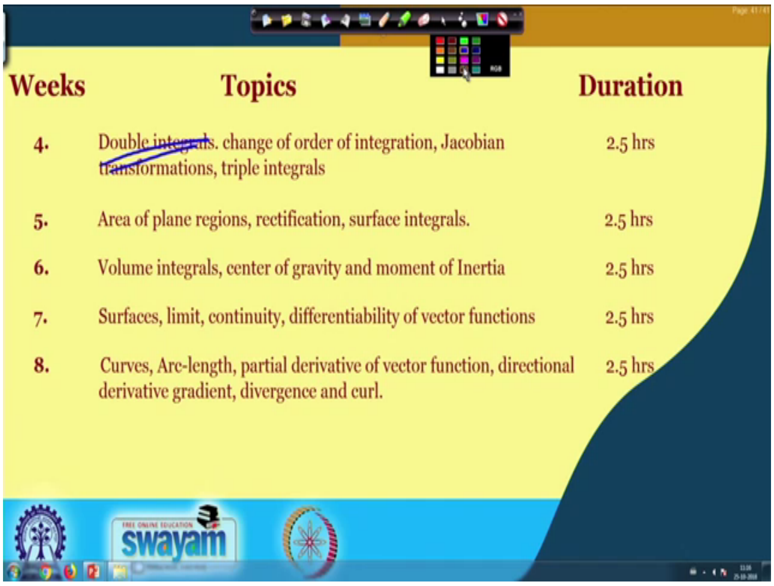


So, today basically we are still continuing with the double integral part. So, we will start with this double integral and if time permits, then we shift to change the order of integration, alright. So, let us worked out a few examples. Let us try to work out few examples, alright.

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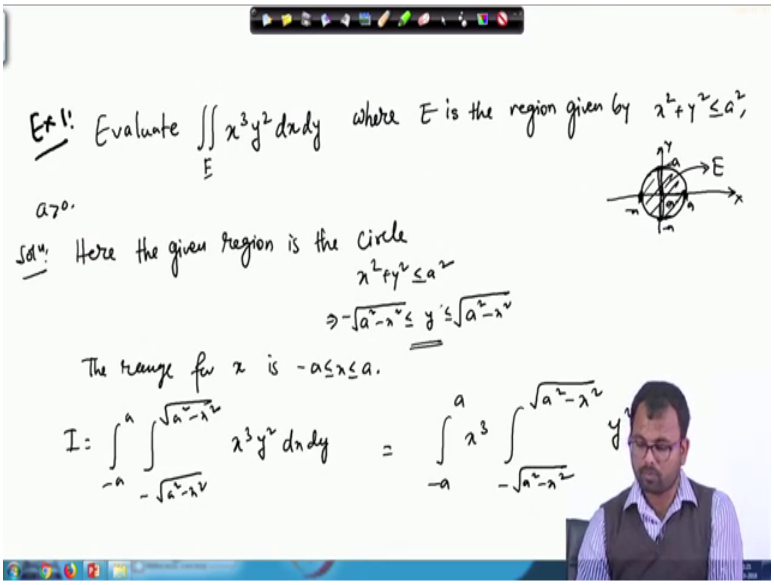


So, the first example is example one. So, the first example is: evaluate the integral on a bounded region; let us say E we are avoiding r for the region because by r be also meant rectangle. So, we will write the region E and the function or the integrand which we have to integrate is x to the power 3 times y square dx dy, where E is the region given by x square plus y square is less or equal to a square, where a is any positive number.

So, here we have to evaluate or we have to calculate this double integral over the circle x square plus y square equals to a square, where a is basically the radius of the circle. So, that means, our region of integration so, if we have x and y, so, our region of integration would be the circle, where a is the radius. So, we have to perform the area in this region alright.

Now, if we have to perform the integration in this region. So, in order to do that, we have to first of all find out the range for x and the range for y. So, here the given we will write here, the given region is the circle x square plus y square it is less than or equal to a square. So, obviously, a is varying from minus a to plus a; similarly, y is varying from minus a to plus a.

So, we can write the range for y as a square minus x square less or equal to y square less or equal to a square. So, we can write the range for y as minus a square minus x square to a square, square root of a square minus x square. So, this will be the range for y and then we can guess the range for a. So, the range for a will be the range or the variable x, the range for x is so; this is the range for our variable x.

So, now we have the range for the variable x which is minus a to a and range for the variable y which is minus of a square minus x square to a square minus x square, right, x cube y square dx dy. So, it is very easy to see I mean how we are deriving the range for x and y. So, obviously, x is varying from minus a to a.

So, the range for x is already known to us. Now, y will vary if x is known then y will vary from here to here. So, this is the how to say the domain for the variable y. So, y will vary from this point and then it will cover all these points and then it will go up to here. So, that is how we are drawing we are how to say calculating the range for y.

So, now we substitute the range for x and range for y and that is our area or the region of integration. So, that is our region for into a region of integration. So, we substituted or we replaced the replaced the region e by the limits of x and y. And, now from here like we saw in the previous lecture that we separate the variable now we separate the how to say here the integral with respect to x and with respect to y.

So, if we separate, then it will be minus a square minus x square and then a square minus x square y square dy.

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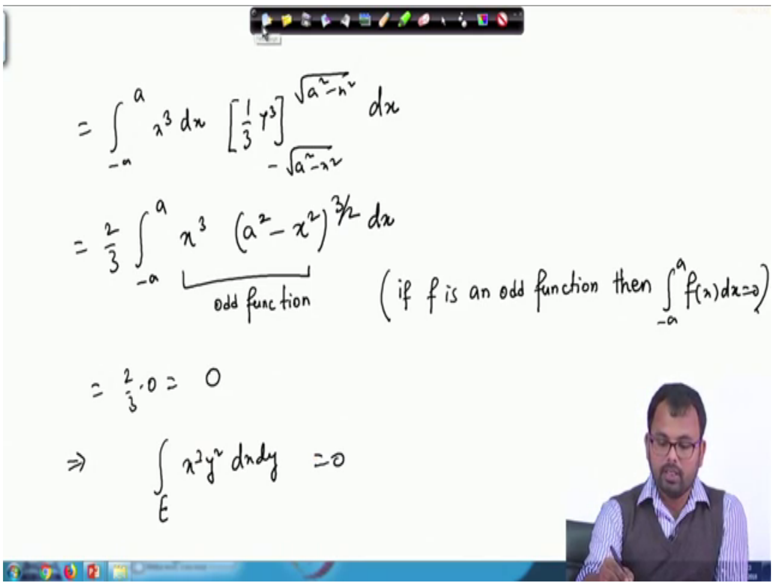

And, this can be evaluated as integral from minus a to plus a x cube dx and then here we will have 1 by 3 y cube a square minus x square times and then up to a square minus x square dx. So, if we substitute the value then this will be 2 by 3 2 by 3 minus a to plus a x cube a square x cube a square minus x square whole to the power 3 by 2 dx, alright. So, this is the function of x and now we have to evaluate this integral.

An another interesting point in this integrand is that if I substitute for x equals to minus x then that because this will behave as an odd function. So, this whole thing is actually an odd function, an odd function and we know a small result from integral calculus which says that if f is a small is a is an odd function is an odd function if f is an odd function then integral minus a to a f x dx will be 0.

So, this results so, this is a very well known result from the integral calculus and since we have an odd function and since the range of integration is from minus a to a the value of this integral will be 0. So, I can directly write 2 by 3 times 0, so, ultimately 0. So, that means, integral over the region E x cube times y squared dx dy will be 0, where E was the given circle basically. So, this is how we evaluate on the double integral over a given region and in this case of course, the answer is 0.

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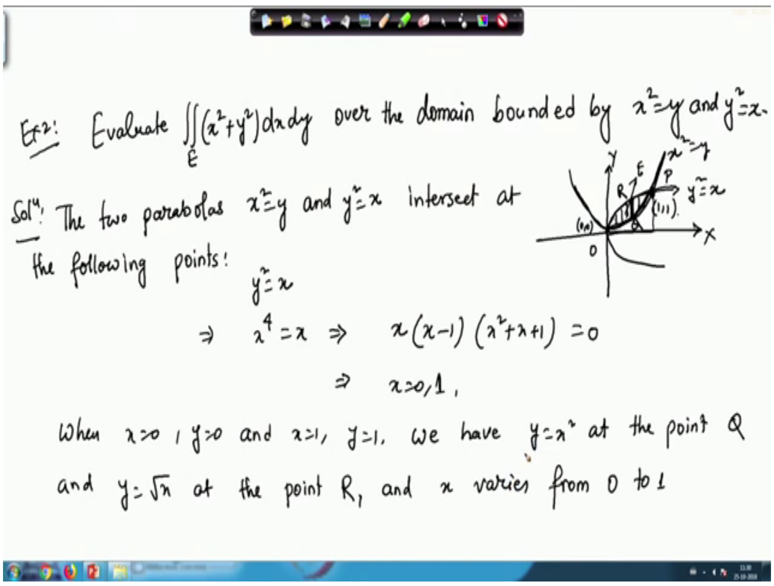


So, let us see an another example where we have to work out a little bit the area of the region of integration in a way. So, let us consider another example. An another example. So, evaluate integral x square plus y square dx dy over the domain over the domain or over the region over the domain bounded by bounded by x square equals to y and y square equals to x. So, this is our how to say area the domain of integration.

So, let us first draw the region. So, this is my x, this is my y that is 0. So, we have a first parabola which is y square equals to x; so, obviously, it will pass through the origin and then we have second parabola x square equals to y which will also pass through the origin. So, this is the parabola y square equals to x and then we have another parabola which is again passing through the origin. So, something like x squared equals to y. So, this is. So, I am not very good at drawing. So, roughly this is what our second parabola will look like.

Now, the domain of integration is bounded by these two parabola. So, it is bounded by these two parabola means the region which is common between I mean between these two parabolas. So, if you can see that this parabola and this parabola enclosed this region here. So, this is our region E or the domain of integration and they intersect these two parabola they intersect at certain point.

So; that means, our domain of integration or the domain or the region E here let us say; region E is basically starting from here to here and of course, it is bounded by these two parabola. So, how do we calculate this area or this point of intersection?

So, here the two parabolas the two parabolas the two parabolas x square equals to y and y square equals to x intersect at the following points. So, if we want to find the point of intersection we basically we basically substitute. So, we have y square equals to x and we substitute for y we substitute x square. So, this will turn out to be x to the power 4 equals to x.

So, from here we can do some factorization. So, this will be x times x minus 1 times x square plus x plus 1, right, equals to 0. So, from here these two will give us the real values of x. So, from here we will get x equals to 0, x equals to 1 and this will this will give us some I would say an imaginary value. So, if we solve this equation if we make this one equals to 0, then in that case we will get imaginary values for x. So, we are not interested in that. So, we will take only x equals to 0 and x equals to 1 as the as the two solutions. So, this equation is a fourth order equation. So, it will definitely have four solutions, but the other two are imaginary. So, we are only interested in real solutions.

So, the real solutions are 0 and 1. And, when x equals to 0 so, from here so, when x equals to 0 we have y equals to 0 and when x equals to 1, then we have y equals to 1. So, that means, the two point of intersections are 0, 0 and 1, 1. So, at these two points these two parabola intersect. So, now we have the point of intersection and we can now calculate the limits. Let us call these two points as O and P. So, they intersect at the point O which is 0, 0 and the point P which is 1, 1.

So, now, we have here so, here we have and let us say this is our point Q. So, this is our point Q anyway. So, this is my point Q which is how to say any point on this lower curve. So, of course, at the point at the point so, we have y equals to x square at the point Q and on the upper point let us say P, Q R. So, and y equals to root x at the point R, right. So, at the on this how to say lower lying parabola we have y equals to x square. So, that is what I have written that it is y cos 2 x is square at the point q. So, q will be any arbitrary point on this parabola and it will always be y equals 2 x square.

However, on the upper how to say parabola any point on this upper parabola which we have y equals to square root of x. So, that is where the y varies. So, y is varying from x square to root x and if I put and x varies from 0 to 1. So, that is x is varying from 0 to 1, alright.

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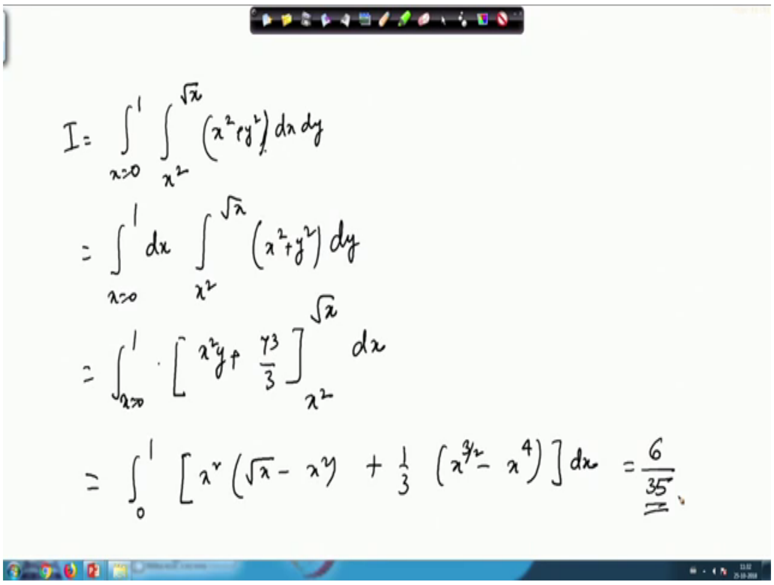


So, let us see how we calculate the double integral. So, instead of writing the region, now I am writing the range for x. So, x is running from 0 to 1 and y is running from x is square to square root of x and then we have x is square plus y square dx dy. So, now, we can again do that separation thing. So, we will first integrate with respect to y.

So, let us take out dx here and then we will have x square plus y square dy and if we integrate so, when we integrate this will yield x square y plus y cube by 3 and then range of integration is x square 2 root x dx and after integrating and after substituting we will obtain 0 to 1 x square times root x minus x square plus 1 by 3 x to the power 3 by 2 minus x to the power 4 dx and this is just how to say an algebraic expression to integrate. I am pretty sure you have done such kind of integration in your plus 2 level. So, I will leave this part up to the up to the students and once you integrate you will ultimately obtain 6 by 35. So, that is the required answer.

So, here let us go to the previous slide. So, here the how to say region or the domain was bounded by these two parabola. So, drawing this figure always helps them and it is a suggestion that whenever you come across a problem like that it is always advisable to first draw the domain of integration.

Because from there you can be able to draw the conclusion that what will be my range of x and what will be my range of y, like we did here. And, then just substitute the values of replace the region E by the values of x and y like we did here. So, we replace the region E with the limits for x and with limits for y and then we just do the normal integration and that will give you the answer.

So, this was another example where we where we calculated the double integral bounded by a region. We will see one or two more examples like that before we jump to our new topic which is change of order of integration.

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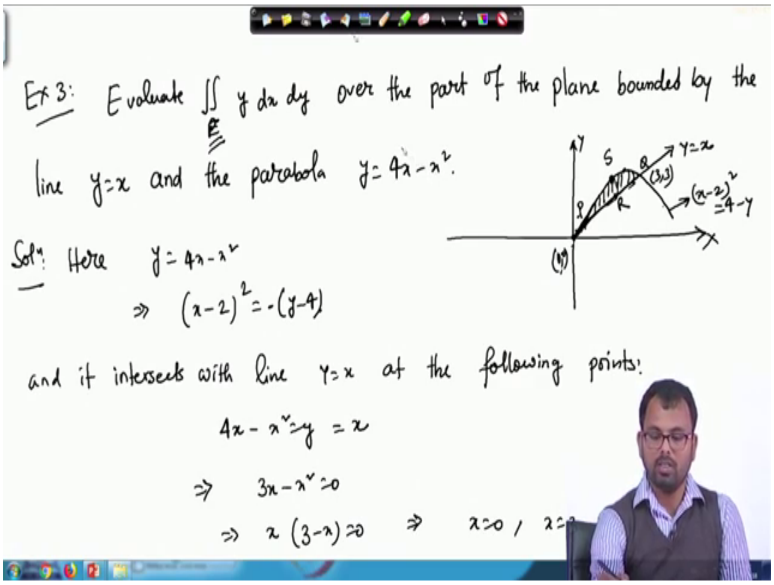


So, let us let us consider another example. Evaluate integral over the region E y dx dy over the part of the plane over the part of the plane bounded by the line y equals to x and the parabola y equals to 4x minus x square.

So, here the given curves are slightly tricky. So, this is not our traditional parabola, this is a slightly different form of parabola, but it is still a parabola and of course, the region is bounded between or the domain of integration is bounded between this parabola and the straight line. So, as I was saying that it is always helpful to draw the domain of integration first.

So, let us draw. So, that is our x-axis and this is our y-axis that is the origin. Here we can write this sentence here y equals to 4x minus x square. So, we can be able to write it as x minus 2 whole square and y minus 4. So, the vertex of this parabola is at the point 2 comma 4.

So, let us let us let us draw the parabola first. So, let us say this is my vertex and since we have x square minus 2. So, that means, it will be. So, it will be as minus. So, it will be an inverted parabola. So, it will go something like this x and it the region is bounded between the points this as straight line and the parabola.

So, let us draw a straight line passing through the origin. So, this is the region of integration and of course, they intersect at two points the first one is here; I can call it as let us say P and this region is Q, any point on this straight line is R and any point on this parabola is S. So, this is our y equals to x and this is our, so, this is our x minus 2 whole square equals to 4 minus y, alright.

So, now we have to guess the values for these we have to we have to calculate not guess, but we have to calculate these two point of intersection. So, here this is our parabola and it intersects with the line y equals to x at the following points at the following points. So, we have 4x minus x square equals 2 y and we substitute for y equals 2x here. So, this is equals to x now if I bring this x here. So, this is 3x minus x square and this can be written as x 3 minus x.

So, from here we will basically obtain x equals to 0 and x equals to 3 and when x is 0, when x is 0, y is 0 and when x is 3, then y is 3.

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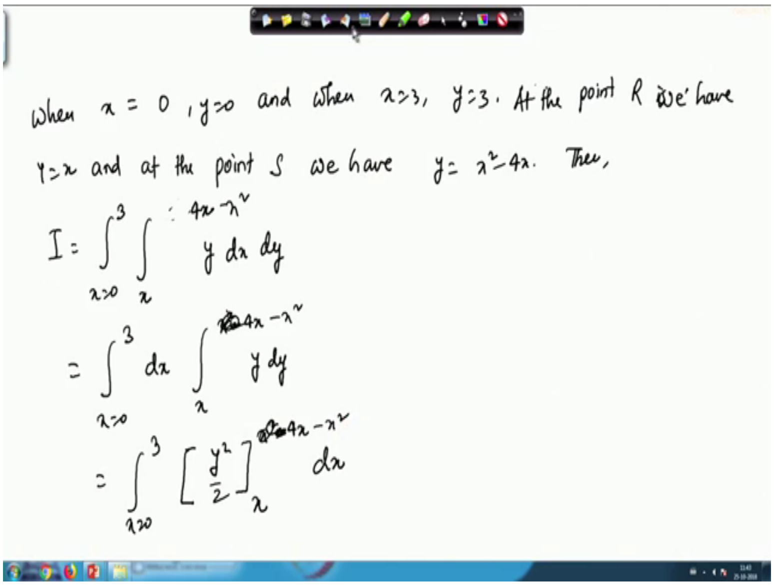


So, that means, this point of intersection was actually origin. So, they are intersection are intersecting at the origin. So, this origin O we can rename it as P, so, we do not no need to rename twice. So, they are basically intersecting at the point 0, 0; we just saw here and then we they are intersecting at the point 3, 3 which we also saw.

So, they are intersecting at the point 0, 0 and they are intersecting at the point 3, 3 and this is 0, 0 point is named as P the point this 3, 3 point is named as q now we have the area or sorry we have the point of intersection, we need to find out the limits for the variable x and y so, that we can replace this region E with those limits, alright.

So, here any point on this on this straight line is given by y equals to x. So, we can write the point like we did in the previous example, so, we can write the point we have what we have named it we have named it R. So, the point R is R at the point R we can write at the point R at the point R we have y equals to x and at the point S, right? So, at the point S we have y square equals to we have y equals to y equals to x square minus 4 x. So, we have y equals to x squared minus 4x.

So, then our double integral will be first we replace the region E by the limits of x and y, so, x is varying from 0 to 3 and y is varying from x to x square minus 4x and the curve which are the function which we needed to evaluate was y dx dy alright.

So, again we do the separation. So, first we write this as integral from x running from 0 to 3 dx and then integral x to x minus 4x y dy. Now, if I integrate then this will be y square by 2 integral from x or sorry the limit from x to x square minus 4x dx.

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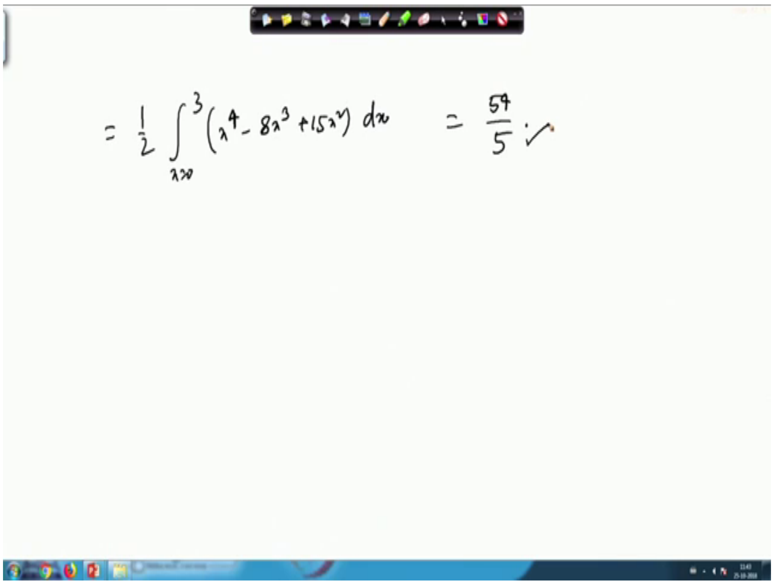

And, this can be written as 1 by 2 integral from expanding from 0 to 3, we will have x to the power 4 minus 8x cube plus 15 x squared dx right. And, then we just use the integration. So, it is the basic how to say integration of an algebraic function. So, I leave this integral to evaluate up to the students and if you evaluate then at the end you will ultimately get 54 by 5. So, this is how we calculate the area bounded by this parabola and this is straight line.

So, remember, the rule of thumb is that you always need to draw the figure first. So, once you draw the figure first it will become very clear that what would be our domain of integration because you do need that domain of integration in order to replace this E with those limits for x and y. So, here in this case we draw the domain first and we can see that for this inverted parabola and this is straight line they intersect. So, what are the two points there are intersecting we can solve actually and after solving we obtained that they intersect at the point 0, 0 and the point 3, 3.

So, now that we have the point of intersection we know the range for the variable x and y and then we basically look at the variable y it is always suggestible to integrate with respect to y first and then do the integration with respect to x and in order to do the integration with respect to y first you need to find out the range for y here the range for y is basically y is varying in this direction, right.

So, y is varying in this direction and the domain of integration is bounded between these two curves. So, first if y is running between these two how to say these two curves in a way, so, we can take any arbitrary point on this straight line and at those points we will have y equals to x and if we take any arbitrary point on that parabola then at those points we will have y equals to 4x minus x square. So, that is what we have done, ok.

So, here there is a small error, we can correct it. So, 4x minus x square, sorry. So, this is basically 4x minus x square this is also 4x minus x square. And then just calculating this algebraic expression will give you the required answer. So, this is the way we evaluate the integral over a certain region or over a bounded region in a way. So, it is not only that you always have a rectangles for the double integral, you may also have regions bounded regions basically and you that is how you basically evaluate those integrals you just have to draw the domain and then guess the range for the variable x and y and there are several examples which we can work out, but there are more or less of the similar type.

So, it is the nothing that I can teach is just that you have to practice a lot of examples like that, but the basic idea is same. I will include some problems like that in your assignments for you to practice and for you to clear your doubts and we will also look how do you say provide the solutions of them as well and hopefully it will help you clear out the doubts if you have at all, in this double integral over a region or over a rectangular domain.

So, we will stop today's lecture here and in the next class we will begin with change of order of integration.

Thank you.