Integral and Vector Calculus

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Lecture – 18
Double Integral over a Region E

Hello students. So, in the last class, we started with the introduction of double integral, we also worked out one example. In today's lecture, we will continue with further examples actually just to make the concept a little bit more clear in double integral.

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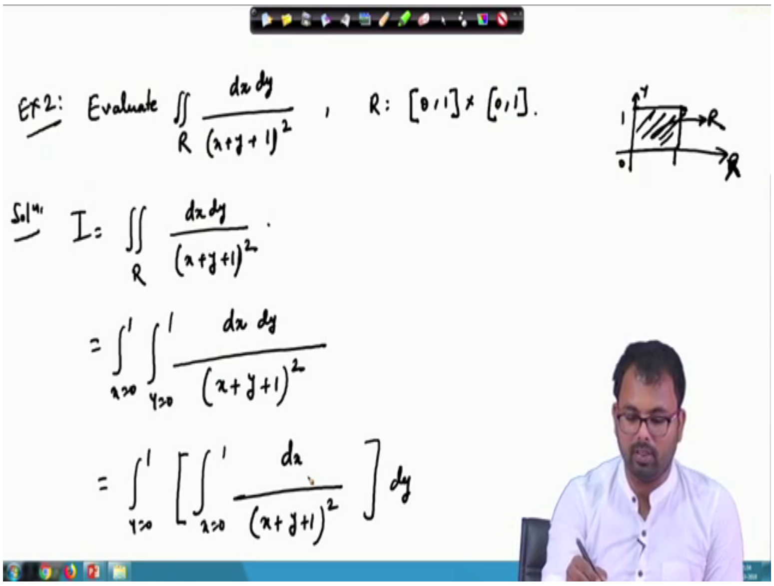

So, let us work our, work out an another example. So, example 2, here it says evaluate, evaluate integral over the rectangle R d x d y divided by x plus y plus 1 whole square where the rectangle R is defined as 0 to 1 times 0 to 1. So, here basically our given domain or how to say sorry range of integration is basically 0 to 1, 0 to 1. So, this is our rectangle. So, this is our rectangle.

Now,, so let us write I equals to integral over the rectangle R d x d y x plus y plus 1 whole square. So, I can write it as integral x running from 0 to 1, y running from 0 to 1 d x d y divided by x plus y plus 1.

So, now I can do that repeated integral calculation here. So, first I will treat y as constant and we will integrate with respect to x first. So, if I integrate with respect to x first, so let us treat y as constant and then I can use that big bracket explaining from 0 to 1 d x y x plus y plus 1 whole square and then, d y.

Now, as we have how to say told several times that when we integrate with respect to x, then we treat the variable y as constant. So, here this y plus 1 is basically a constant and then this integral here integral 0 to 1 d x by x plus constant square is nothing but that our traditional d x by x square integral.

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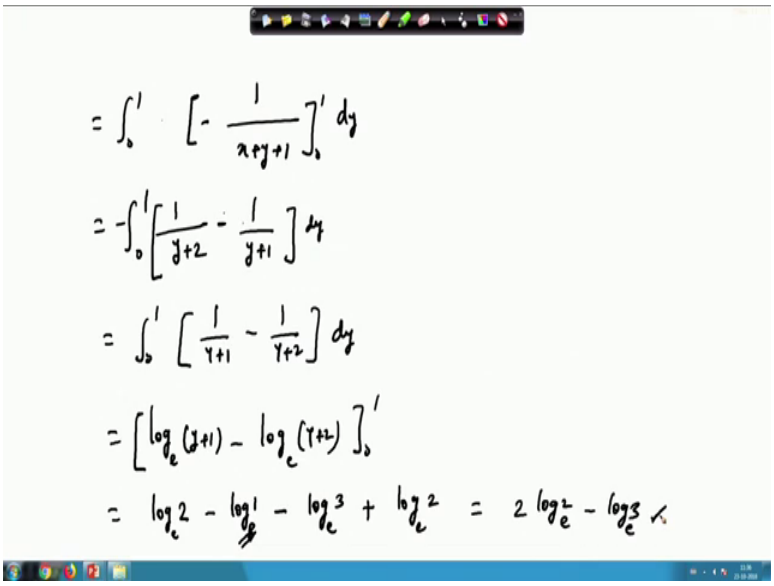

And in order to evaluate that, we simply write integral from 0 to 1 d y and then this will be integral from 0 to 1, this will be one by x plus y plus 1, which we already know from our traditional integral calculus results,

And now, we can substitute the value of x here. So, this will be minus 1 by y plus 2 d y right and, and when x is 1, when x is 1, then this will be minus 1 by y plus 1 d y. So, we can rearrange it a little bit, so this will be 1 by y plus 1 minus 1 by y plus 2, is not it. I just put the minus sign inside the bracket and if we integrate now, then this will be log of y plus 1 to the base e minus log of y plus 2 to the base e integral from 0 to 1. So, this will be log of e log of 2 to the base e minus log of 1 to the base e minus log of 3 to the base e plus log of 2 to the base e.

So, we will basically be having 2 log 2 to the base e minus log 3. You can further simplify, but I am just leaving the result up to here. This value is 1 which we know from the logarithmic results. So, this is our required result. So, see here also we perform the same repeated integral property that we integrate with respect to x first and then we integrate with respect to y like here, all right?

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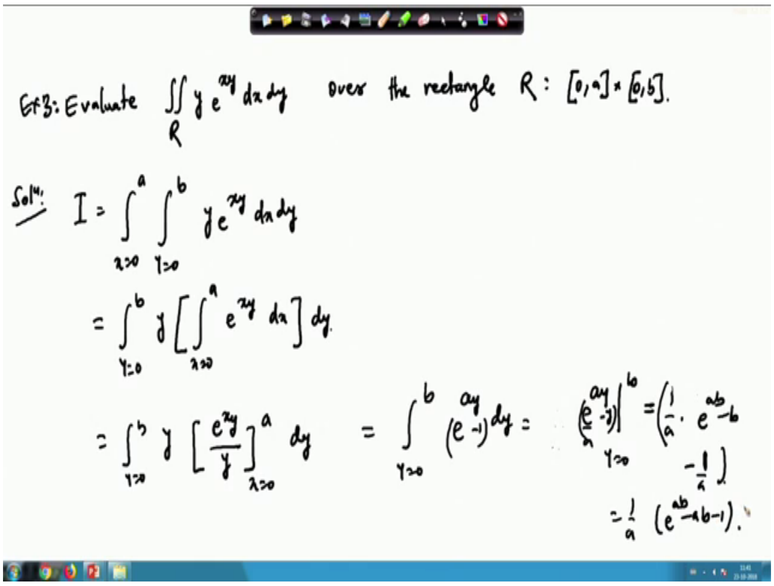

So, now, let us consider another example which is a little bit little bit difficult but not too difficult actually. So, evaluate integral over the rectangle R y e to the power x y d x d y over the rectangle over the rectangle R as 0 to a and then 0 to b. So, this must be example 3 um. I am it is pretty much the similar interval like the previous example. So, I am not drawing that, that area of integration. It is pretty much straight forward here.

So, now I am going to write I as integral over the rectangle R. So, our rectangle R is x running from 0 to a and then y running from 0 to b, y e to the power x y d x d y. This integral also, double integral also exist on this rectangle R. Just looking at this integral integrand, you can be able to tell. And next, we see that we have e to the power x ye to the power x y d x d y. So, if I substitute x y has said let us say by treating y as constant, then this will be basically our usual method of substitution formula.

So, let us see how we can how we can do that or we can separate this. So, there are several ways to do this. So, let us separate this y and then I will integrate with respect to x first and if we integrate with respect to x first, then this will be e to the power x y by y and x running from 0 to a d y. So, this will be integral from y running from 0 to b, then this y, this y will get cancelled and then we will have e to the power a y d y.

So, when we integrate this one this will be 1 by a e to the power a y and y will run from 0 to b. We will have a minus 1 here and then this will be e to the power y minus y and when we substitute for y, then this will be 1 by a e to the power a b minus b minus e to the power 0. So, this is 1.

So, basically our result is one by a e to the power a b minus b minus 1, all right. And we probably, we probably need a 1 by a here, yes. So, we need a 1 by a here. So, this will be, this will be a 1 by a. So, this will be a 1 by a something like this. So, and this will be also 1 by a. So, here I can put up a b. So, yeah now, now it is correct.

So, you see that you are just looking at the integrand, we see that this integrand is definitely bounded on this interval a b and based on that, I can now treat this in double integral by the method of this repeated it integral. So, I started with treating this integrand as first of all integrating with respect to x keeping y as constant and then, I after I got this function with respect to y, I integrated it and then this is our final result.

So, this is another example where we did repeated integration for a double integral.

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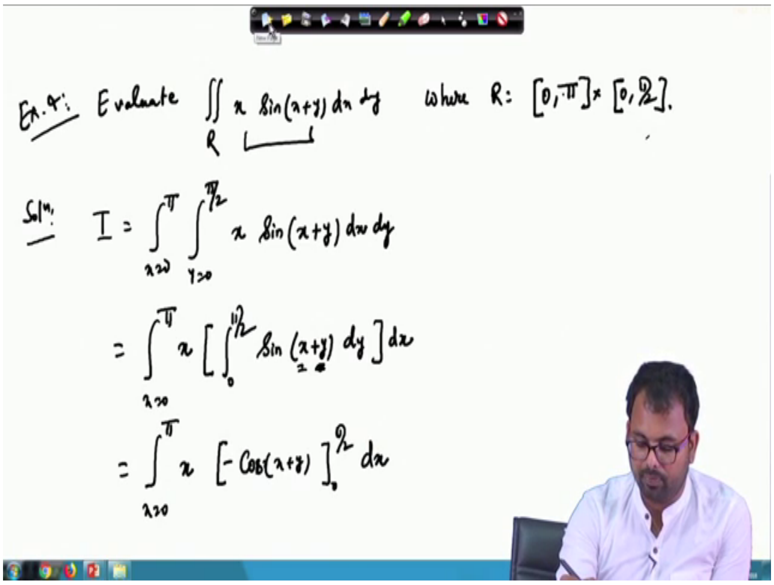

We can consider one more example before we look into the change of order. So, I change the order of integration. So, it is evaluate, evaluate integral over the rectangle R x sin x plus y d x d y where our rectangle R is 0 to pi times 0 to pi by 2. So, again just look at this integrand, it is basically x times R techni metrical function. So, of course, sign actually. So, the integrand is bounded on this on this rectangle R and as a result of which, this double integral will exist and now we will treat this double integral by the by that repeated integral method.

So, let us write it as I equals to x running from 0 to pi, y running from 0 to pi by 2 x sin x plus y d x d y. So, first of all, I will integrate with respect to y. So, let us see what happens. It is totally depend on you whether you would like to integrate with respect to x first or with respect to y. So, here in this case, we choose to integrate with respect to y first.

So, here I can write x and then this big bracket 0 to pi by 2 sin x plus y d y and then d x. So, here y is basically a constant at the moment. So, it does not play any role here and we will just integrate with sorry here x is a constant. So, x does not play any role here. We will basically integrate with respect to y. So, pi and integral of cos is a sin as minus cos x plus y integral value from 0 to pi by 2 d x.

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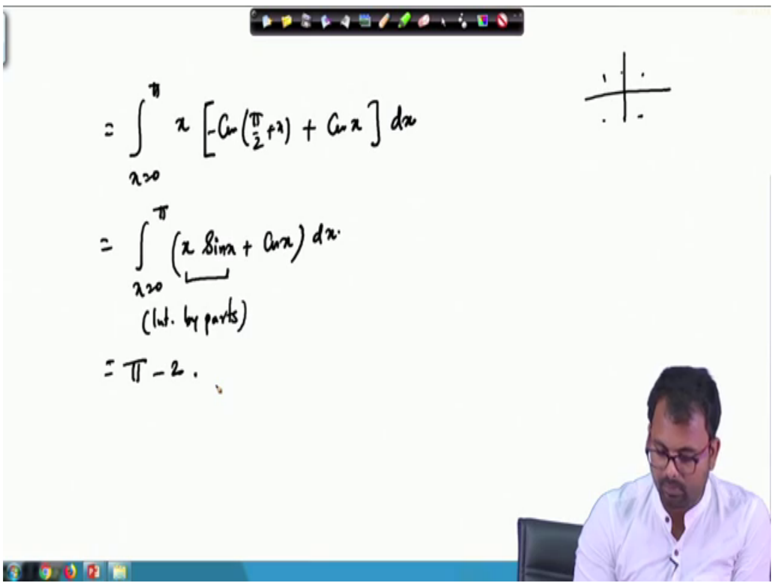

So, now this will be integral from X running from 0 to pi x times cos minus cos of pi by 2 plus x. So, I will have minus cos of pi by 2 plus x minus, minus plus cos of x d x. So, this will be cos of pi by 2 plus x, we know that all sin, tan, cos. So, pi by 2 plus x. So, cos will change to sin and since sin is positive sorry, cos is negative here, so negative and negative will turn into a positive. So, here we will have x sin x plus cos x d x.

And now, we will basically integrate with respect to x here and we will do that by doing that integration by parts for the first one, for the first term and we will basically integrate the second term. So, the second term will result into integral of cos x and integral of cos x is basically sin x and then from 0 to pi, both the values will be 0 and this one can be treated by integration by parts. So, if you do all those things, so I will need I leave these in this part up to the students because it is a basic trigonometric integration which you have done in your plus 2 level.

So, if you do the integration by parts integration by parts, then ultimately you would obtain as pi minus 2. So, that will be your final result. So, here also we started with a given double integral, here we started with treating x as constant and we integrated with respect to y first and afterwards, we integrated with respect to x. And after if you do this integration by parts, then ultimately you will obtain pi minus 2 as the answer.

So, this was one such example where we use this repeated integral method.

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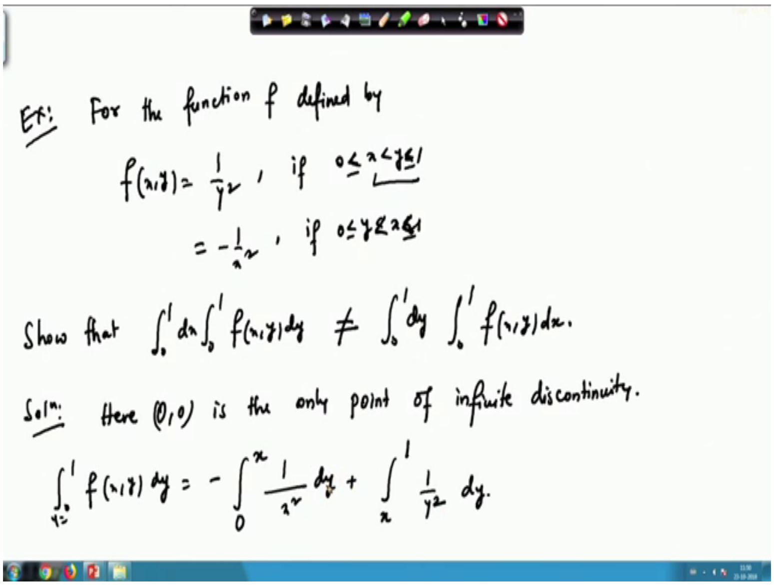

Our next example is for the function f defined by f x y as 1 by y square if 0 less than x less than, y less than 1 and equals to minus 1 by x square if 0 less than y less than x less than 1. So, then show that integral from 0 to 1 d x integral from 0 to 1 f x y d y is not equal to integral from 0 to 1 d y times integral from 0 to 1 f x y d x.

So that means, the function sorry the integral the double integral or to be very precise the repeated integral are not equal. And if the repeated integral are not equal, then that case, the double integral does not exist between the interval between the interval 0 to 1 times 0 to 1. So, let us see how we can evaluate this. So, solution here, it is better to write as 0 and then, let us include the point 0; other basically the endpoints. So, here 0 0 is the only point of discontinuity, point of infinite discontinuity because the function is unbounded at the point 0, 0, infinite discontinuity.

So, the function is unbounded at the point 0, 0. So, then integral from 0 to 1 f x y d y, so integral from 0 to 1 f x y d y, so when y is varying from 0 to 1. So, then in that case, we can write this as minus integral from 0 to x 1 by x square d y plus integral from x to 1 1 by y square d y.

So, we basically divided this interval 0 to 1 by 0 to x. So, y is running from 0 to x minus 1 by x squared d y and then x is running from then, then y 0 to x and then y is again running from x to 1. So, then in that case, the value of the function f would be 1 by y square. So, we have used these two values.

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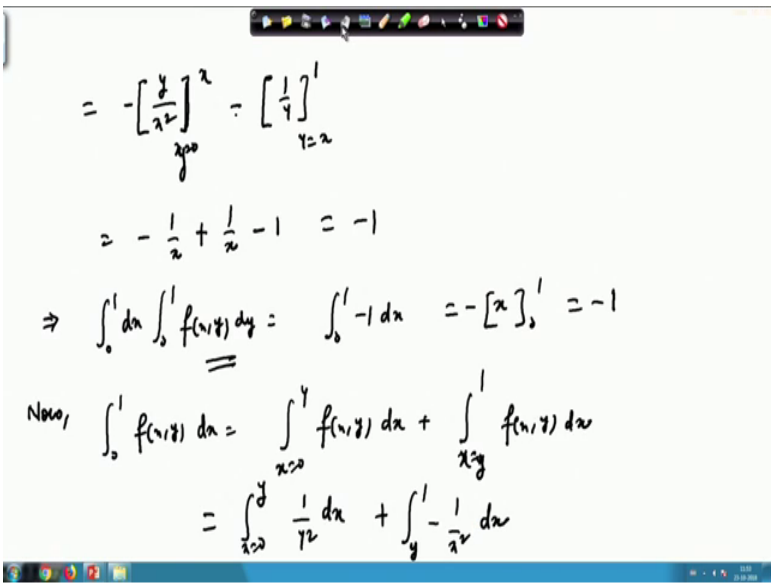

Now, when we integrate when we integrate, then the first term will be minus y by x square integral value will be from x running from 1 from 0 sorry, y running from 0 to x plus we have 1 by y square.

So, this will be minus 1 by y and then y will be running from x to 1. So, if I substitute the value, then this will be minus 1 by x minus minus plus. So, y goes to 0; that means, the whole thing is 0 minus 1 by x and this will be minus 1 by x minus minus plus and then, this will be minus 1.

So, ultimately the value is minus 1 and therefore, from here if I want to evaluate, if I want to evaluate the left hand side. So, integral from 0 to 1 d x integral from 0 to 1 f x y d y is equal to integral from 0 to 1, this value is minus 1 times d x. So, this is minus x integral from 0 to 1. So, this is basically minus 1.

Now, so we have found the value as minus 1. Now, integral from 0 to 1 f x y d x, so what will be the value of this one? So, I can again write integral y x running from 0 to y f x y d x plus integral from y to 1 f x y d x. So, integral x running from 0 to y, f x y, so integral x running from 0 to y, f x y, so we will have 1 by y square right. So, I can write 1 by y square d x and then, x is running from x is running from x is running from y to 1. So, if x is running from y to 1, then this will be minus x square.

So, this will be integral from y to 1 minus 1 by x square d x, all right.

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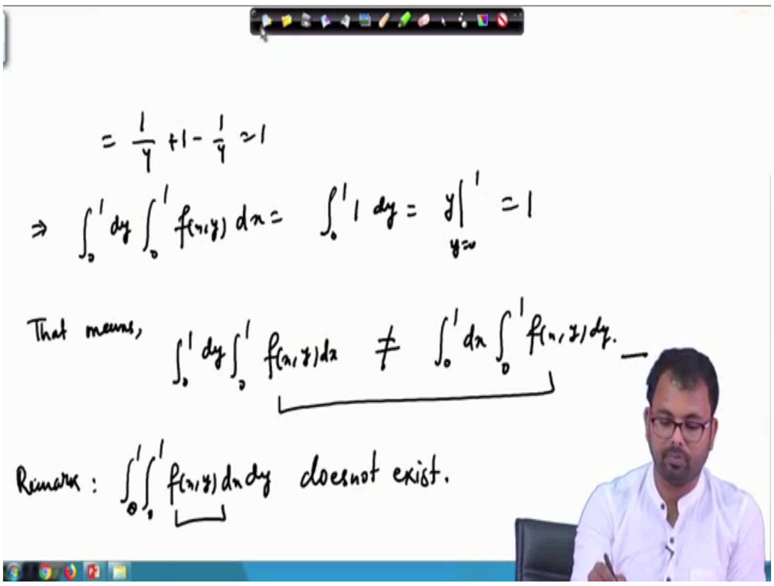

And now, if we integrate like we did before if we integrate like we did before, then we will basically obtain 1 by y plus 1 minus y and then this answer is 1. So, therefore, from here if we have integral from 0 to 1 d y integral from 0 to 1 f x y d x, then this value is basically integral from 0 to 1, 1 d y and then this will be y at y running from 0 to 1. So, this will be 1.

So, that I mean; that means; that means, integral from 0 to 1 d y, integral from 0 to 1 f x y d x is not equals to integral from 0 to 1 d x times, sorry d x then integral from 0 to 1 f x y d y.

So that means, these two repeated integrals are not equal which we wanted to prove and from here we can also say that, so remark a small remark ah; the double integral 0 to 1, 0 to 1 f x y d x d y does not exist, does not exist. And the reason for it not existing is that this function is unbounded or it has an infinite point of discontinuity at the point 0, 0 and that is why this integral double integral does not exist or they these two repeated integral they are not same. So, this is a very nice example where we can see that if you have even a little bit a problem in your integrand your function f, then your double integral would not exist.

Although your repeated integral may or may not exist, but they will not be same and at the same time, your double integral how to say would not exist because if it exists, then both the values they are both these repeated integrals these repeated integrals where is that, they also should be same, but it is not happening in this case. So, this is one example which shows that and it is very interesting actually.

So, next, we will look into calculation of double integral. So, not only on the rectangle R, but on a certain region, so up until now, we had a very nice rectangle where we could write integral from a to b and integral from c to d f x y d x d y where the integral was evaluated on a rectangle R. But, what would happen if you have a region or a certain type of domain?

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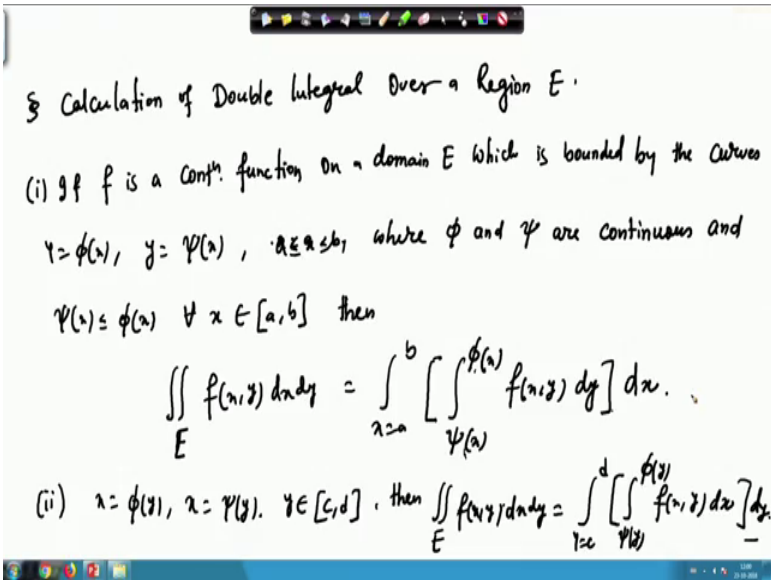

So, calculation, calculation of double integral over a region let us say E over a region E, if I write r, then we might confuse with rectangle.

So, let us write the region E. So, the statement goes like this. If f is a continuous function, is a continuous function, f is of course a function of two variable on a domain E on a domain E which is bounded by. So, the domain is now given by which is bounded by the curves y equals to f phi x and y equals to psi x.

So, these two are the curves that are defining the region or the domain e where x is equals to where a less or equal to x less or equal to b. Then, where phi and psi are continuous are continuous and psi is less or equal to phi for all x in the interval a comma b, then the double integral over the region E or over the domain E can be given as, integral x running from a to b then integral psi x to phi x f x y d y d x. So, this is the way we define the double integral on a region E.

Similarly, if our function f is continuous on a domain e which is bounded by the curve x equals to phi y and x equals to psi y. So, if we have the curves x equals to phi y and x equals to psi y where y is running between c to d, where y is running between c to d and both phi and psi are continuous for all y in c to d, then our double integral can be written as integral over the domain E f x y d x d y. Let us write it instead of dot dot f x y d x d y can be given by integral y running from c to d, then integral psi y phi y f x y we integrate with respect to x, substitute the value and then integrate with respect to y.

So, either way; either you can integrate with respect to x first by putting the values of these two curves, but for the variable x and then, we integrate with respect to y at then because we will have only functions of y or we can first integrate with respect to y by putting, by putting the values for y as psi x and phi x and then we integrate with respect to x. So, either way would do. We will stop here for today. And in the next lecture, we will basically start with the examples where instead of rectangle; we will have regions to perform the double integral.

Thank you.