Integral and Vector Calculus
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Lecture – 17
Double Integral

Hello, students. So, up until previous class we looked into the first three topics of our integral calculus section.

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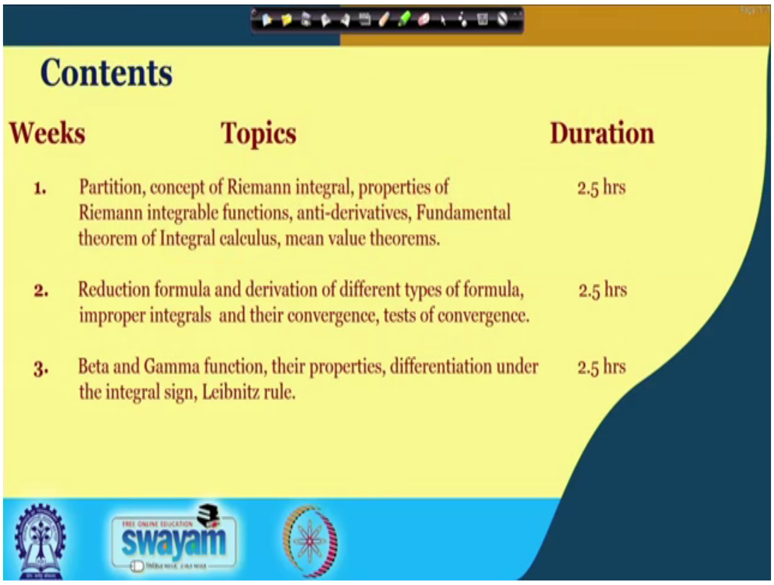

So, they were basically – a partition and concept of Riemann integral, as a fundamental theorem of integral calculus mean value theorems, then we looked into a reduction formula, derivation of different types of reduction formula, then we also looked into improper integral and their convergence and finally, we looked into beta and gamma functions and differentiation under the integral sign.

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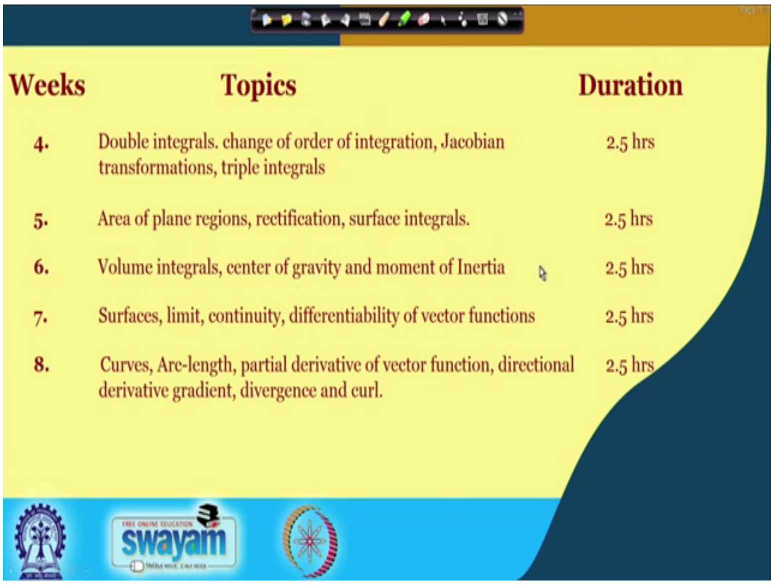

So, the next topic in our agenda is about double integrals change of order of integration and things like that. So, to start with so, today, we will start with double integrals. So, to start with so, what do we mean by double integral and how do we define a double integral and things like that.

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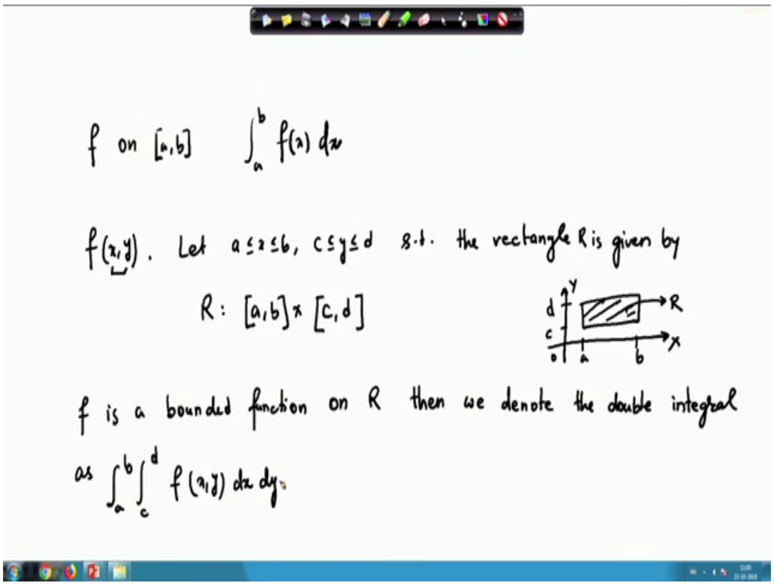

So, we know that a function for the function of one variable; let us say a function f which is defined and bounded on a closed interval a comma b on a closed interval a comma b then our definite integral was given by integral from a to b f x dx. So, this is how we define the definite integral and this actually gives us the area between the point a to b bounded by this curve f x.

Now, an interesting point one could ask in this context is that what happens for the function of two variables. So, instead of having a function instead of having a function let us say for one variable if you have a function of two variable say f x, y then what would happen with the integration of such functions because we know that for the function of one variable we can be able to do, we can be able to check actually its limit, its continuity, differentiability and at the same time we can also perform the integration.

So, we know from differential calculus that for the function of two variables as well we can be able to check its limit continuity, partial derivative things like that. So, what about the integrability of a function of two variables and how do we actually define that?

So, since we have two variables here, x and y we need to find the intervals for these two variables; that means, if you have two variables then you must be having two intervals where these two variables actually belong to. So, let us say x and y. So, I am going to give how to say a formal definition in a way that let a lesser equal to x less or equal to b and c less or equal to y less or equal to d such that the rectangle such that the rectangle R is given by a comma b times c comma d.

So, what does this mean is if you draw this on x and y axis so, here we have the point a, here we have the point b, c here we have the point let us say c and here we have the point d. So, this is basically our rectangle R and we say that f is if f is a bounded function on R then we denote the double integral double integral as a to b, c to d f x, y dx dy.

Now, what do we mean by this integral I am going to write it write it down in probably next couple of minutes, but here we have a range of integration a to b which is for the variable x and range of integration from c to d which is for the variable y. So, do not mix it up. So, when we write a to b that is for x and when we write c to d then that is for the variable y.

So, let us give a formal definition for this double integral and, for the time being we will perform the double integral on the rectangle only.

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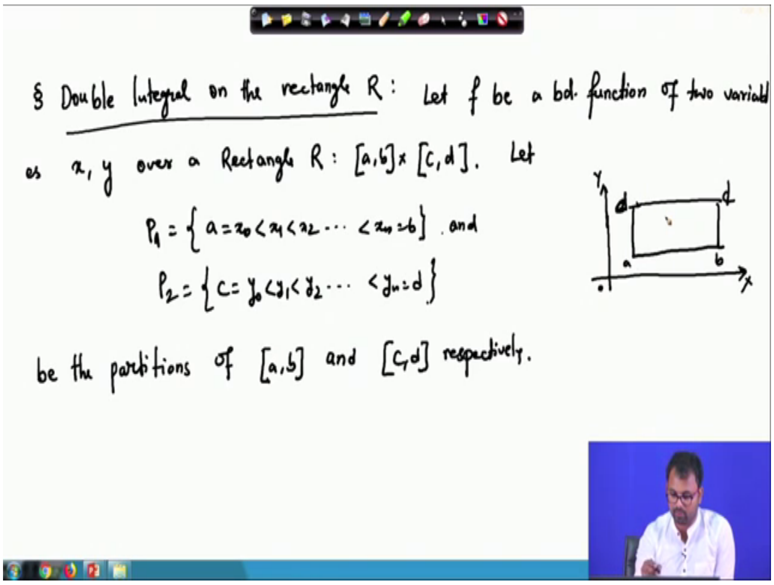

So, double integral on the rectangle R on the rectangle R and R is our a to b times c to d. So, that is pretty much the same.

So, the definition goes like this; let f be a bounded function bounded function of two variables of course, of two variables of two variables x and y over a rectangle over a rectangle R which is given by a comma b times c comma d and also let P 1 as the partition of the interval for the variable x. So, x 1, x 2, x 3, up to x n which is equals to b and P 2 equals to c equals to y 0, y 1, y 2, dot dot up to y n equals to d.

So, let this and this P 1 and P 2 be the partitions of a comma b and c comma d respectively. So, what we are doing is basically if we have this interval let us say this is x, this is y, so, this is my a, b, c, d. So, what we are doing is we are taking the partition of sorry c and d; d is here. So, a, b, c, d. Now, if I follow that I have to follow the same terminology. So, a b and c d, yes.

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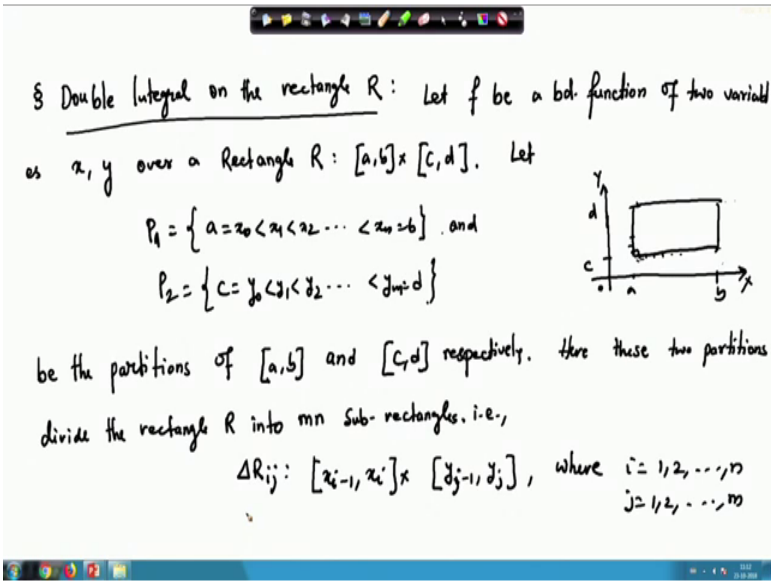

So, a a, b and this one is c, d, yes. So, this one is c and this one is d alright.

So, now what we have is here is ; that means, we are taking the partition of these of these interval for the variable x. So, like x 0, x 1, x 2, x 3 dot dot up to x n and then we are taking the partition for the variable y for the for the interval c d where we have the variable y. So, we take y 0, y 1, y 2 up to y n equals to d. Now, this x 0 to x 1 this is a very tiny rectangle, but here x 0 to x 1 and y 0 to y 1 is one such tiny rectangle, then we have y 1 to x 1 to x 2 and y 1 to y 2 will be another rectangle. So, in a way here we will have if we have. So, here we will keep a rectangle otherwise it will be a square. So, we have y 0, y 2, y 3 up to y m.

So, that means, then here we have n subintervals for the variable x and then we have m sub intervals for the variable y. So, in a way we have m times n. So, m n number of rectangles here so, we in here we will have here these two partitions. These two partitions these two partitions divide the rectangle R divide the rectangle R into m times n sub rectangles, alright because we have n number of intervals in this direction m number of intervals in direction. So, the number of rectangles will be m times n sub rectangles.

And, the area of this sub and this sub rectangle that is let us say that is our sub rectangle let us write it as delta R as i comma j. So, this one will be given by this one will be given by x i minus 1 to x i times y i minus y j minus 1 to y j where our i is running from 1, 2, 3 up to n and j is running from 1, 2, 3 up to m, alright. So, this is basically our R-th how to say rectangle in a way and it is given by this fashion.

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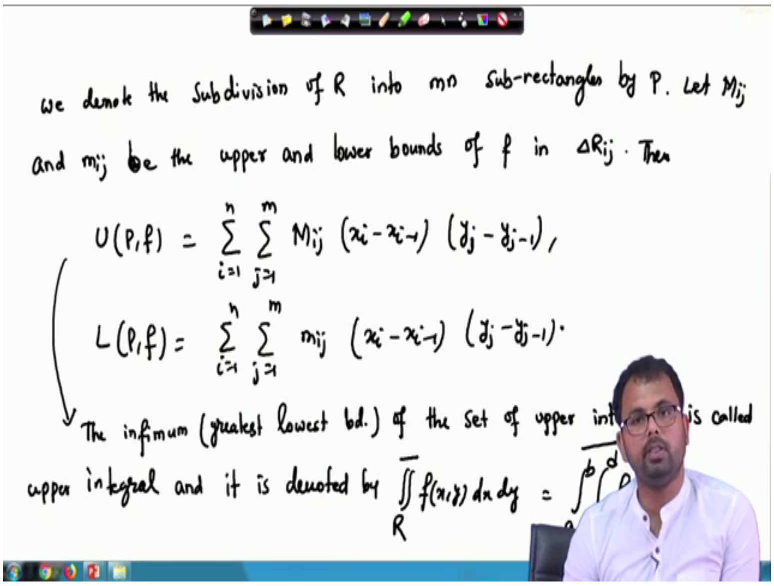

Now, since we have defined the how to say the partition of that rectangle we denote we denote the subdivision we denote the subdivision of R into m n sub rectangles, rectangles by P. So, that means, this one this here we denoted by P. So, P is sort of like how to say a partition of rectangles basically.

So, in case of function of one variable we had P as a partition where we had n number of n plus 1 number of points and that partition P divided the interval into n subintervals. Here in this case the partition P for the function of two variable divides the whole rectangle R into m n number of sub rectangles that is what we have written here. And, let capital M i j and small m i j are the be the be the upper and lower bounds of f in delta R i j.

Then we write upper integral some or U P, f equals to sum over all i running from 1 to m sum over all j running from 1 to m capital M i j times x i minus x i minus 1 and y y j minus y j minus 1. And, we defined L P, f the lower integral sum as i running from 1 to n j running from 1 to m small m i j times x i minus x i minus 1 times y j minus y j minus 1.

Now, let us put a comma here. So, these two are the upper integral sum and lower integral sum. Now, if you remember the function of one variable where we defined the integral we actually took the infimum of all these upper integral sum. So, that gave us the upper integral and then we took the supremum of for all L P f then that gave us actually the lower integral.

So, here from here we will have the infimum the infimum or greatest lower bound greatest lower bound the greatest lower bound of the set of upper integral sum set of upper integral sum integral sums integral sums is called upper integral is called upper is called upper integral and it is denoted by and it is denoted by integral over R upper integral f x, y dx dy you can also write integral a to b, integral c to d then this bar here f x, y dx dy.

So, that is basically our upper integral. Similarly, you can define the lower integral as well by taking the supremum over all L P, f.

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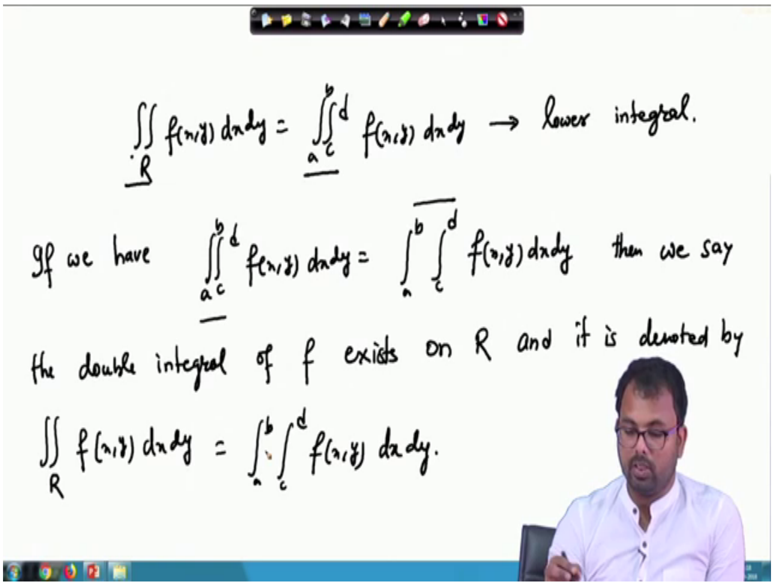

And, it will be denoted by it will be denoted by rectangle R we put a bar here or maybe here f x, y dx dy or one can write integral a to b c to d and then let us put a bar here then we have f x, y dx dy and this is our basically lower integral lower integral.

And, like function of one variable when we have so, if we have if we have lower integral a to b c to d lower integral f x, y dx dy equals to integral from a to b integral from c to d which is upper bar which is basically upper integral dx dy, if they are equal then we say the double integral the double integral of f exists on the rectangle R and it is denoted by and it is denoted by integral over the rectangle R f x, y dx dy or you can write integral from a to b integral from c to d f x, y dx dy.

So, that is how we define formally the double integral of a function f which is bounded on the rectangle a to b times c to d. So, it is how to say a nice way to define, but there is another question here that how do we evaluate the double integral. So, it is not only just the definition, but at the same time we have to address that how do we define that how do we calculate the double integral.

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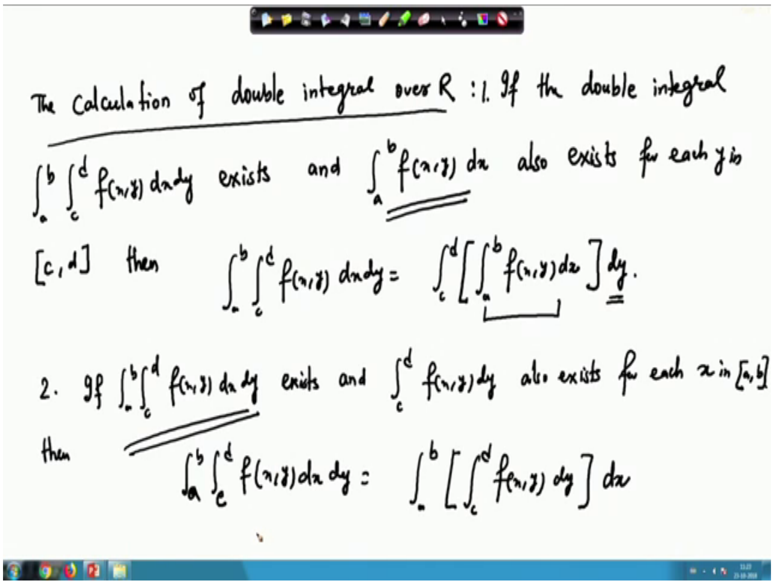

So, in order to calculate the double integral the calculation of double integral basically can be done. The calculation of double integral over R over R; so, if the double integral how do we calculate. So, if the double integral a to b, c to d f x dx dy exists and integral from a to b f x, y dx also exists for each y in c, d then integral from a to b integral from c to d f x, y dx dy can be written as integral from c to d then integral from a to b f x, y dx and then dy which means that if this integral exists; that means, if for every y if the integration with respect to x exists then in that case which we can evaluate this double integral.

So, first we evaluate this integral with respect to x only where y will be treated as constant and then we after the integration then we substitute the values of x and then we will basically obtain a function of y only and afterwards we integrate with respect to the function y only because other than that you have rest of the things as constant and then that will be actually the value of that double integral. So, here we first integrated with respect to x and then we integrated with respect to y.

Similarly, so, let us call it as a point – 1. Similarly, if the double integral so, all these language would be same. So, if integral from a to b, c to d exist so, if this double integral exists and integral from c to d f x, y dy also exists for each x in the closed interval a comma b, then we can write integral from a to b integral from c to d a to b, c to d f x, y dx dy as integral from a to b, then integral from c to d f x, y first dy and then dx.

So, that means, if we have this double integral exist here and if the integration with respect to y also exists, where x is treated as a constant then we can be able to evaluate this double integral in this fashion. So, first we integrate with respect to y treating x as constant and then we integrate the rest of the result with respect to x because then we will have only a function of x. So, either we integrate with respect to y or we integrate with respect to x first and then we integrate whatever the variable is left afterwards and if the double integral exists then these two repeated integral would also be same.

So, if because you see if the double integral exists; that means, this side has a value. So, then in that case you will have the value for this side as well and the same value should be equal to this side as well so; that means, integral from c to d integral from a to b f x, y dx then dy is equals to integral from a to b, integral from c to d f x, y dy dx. So, these two repeated integral will be equal if this double integral exist. The proof is a little bit lengthy and it is actually out of the scope of this lecture.

So, instead of going into the theoretical proof of this of this result we will emerge that and will now look into few examples where we can actually apply this result where we can calculate the double integral of a given function defined on a rectangle.

So, let us start with our first example.

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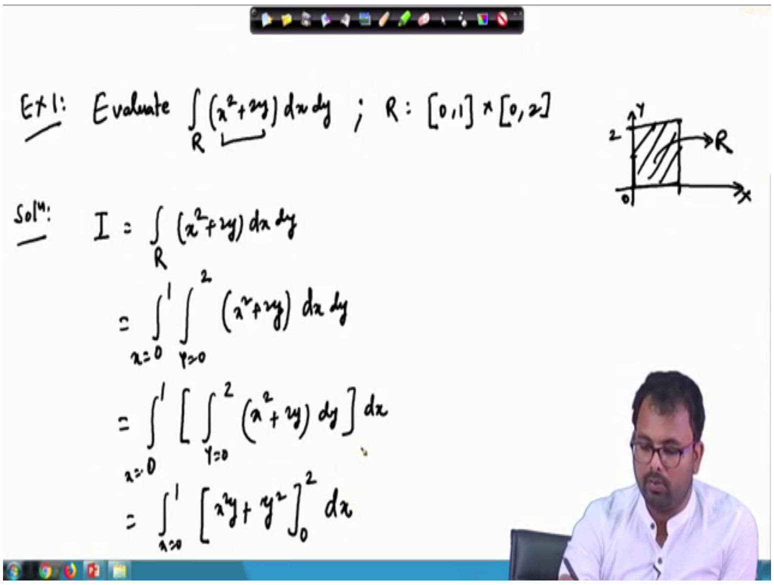

Example 1. So, evaluate integral over the left angle R; you can also use only single integral symbol and if you write R then that is also how to say a way to say that it is basically a double integral defined on the rectangle R. So, we do not necessarily have to write a double integral or double integral R all the time, alright.

So, integral over the rectangle R x square plus 2y dx dy where our R, so, where our R is basically 0 comma 1 times 0 comma 2. So, that means, our rectangle looks like so, 0 comma 1, 1, 2. So, this is our how to say range of integration. So, that is our rectangle R and this is where we have to perform the integration, alright.

So, solution: so, let us write I as integral over the rectangle R x square plus 2y dx dy. Now, I can write R as 0 to 1 or I can write x running from 0 to 1 and then I will write integral from y running from 0 to 2 x square plus 2y dx dy. So, now, what we would do? We would integrate with respect to y first, because here this is obviously, a very nice behaving function. There is a polynomial function in a way and it is definitely defined and bounded on this rectangle R. So, this double integral will exist this double integral will exist. So, we can actually do that repeated integral property here.

So, let us first treat x as constant. So, I am going to integrate with respect to y. So, if we integrate with respect to y then this will be x square y plus y square, right because it will be y square by 2 and then 2 will cancel out integral value from 0 to 2.

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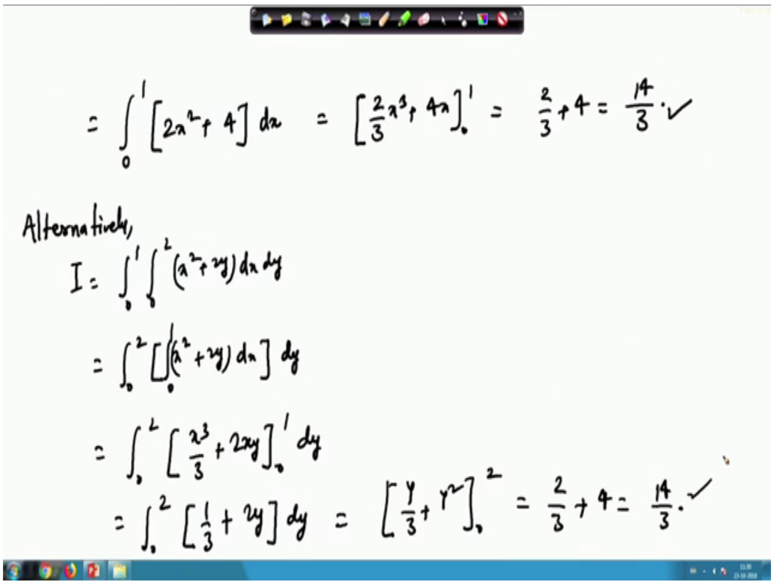

And, if I substitute the value then this will be integral from 0 to 1 then we would have 2x square plus 4. So, we would have 2x square plus 4 dx and after if we integrate this here then it will be 2 by 3 x cubed plus 4x integral from 0 to 1 and this will be 2 by 3 plus 4, right. So, ultimately 14 by 3. So, if we so, if we integrate with respect to y first; so, here we see that we integrated with respect to y first and then we integrated with respect to x and our answer is 14 by 3.

So, now let us verify what happens if we integrate with respect to x first and then we integrate with respect to y. Because, our conclusion is if the double integral exists then both the repetitive integral must be equal. So, let us see what happens? Alternatively, so, the our result is done here. So, basically we have a fine we have found the answer it is just that we want to verify whether we will obtain the same result or not. So, alternatively we can do I equals to integral from 0 to 1, integral from 0 to 2x square plus 2y dx dy.

So, here if I integrate with respect to x first then I can write it as x square integral 0 to 1 plus 2y dx and then dy and if I integrate with respect to x then this will be x cube by 3 plus 2xy integral from 0 to 1 dy and this will be integral from 0 to 2, 1 by 3 plus 2y dy and if I integrate this then this will be y by 3 plus y square integral from 0 to 2. So, this will be 2 by 3 plus 4; that means, 14 by 3.

So, even if we integrate with respect to x first by treating y as constant we would obtain the similar result and this is basically happening because this double integral exists on this rectangle R. Sometimes it is also a intuitive just looking at the integral you can be able to make out whether that integral would exist on that interval or not. So, the first criteria is it has to be bounded on that interval and then we can at least talk about that the double integral exists.

And, then afterwards it does not matter whether you integrate with respect to x first treating y as constant and then integrating with respect to y or integrating with respect to y first treating x as constant and then you integrate with respect to x at the end. Both the repetitive integral would give you the same result like here and you do not have to do both of them you just have to calculate at least just one of the repetitive integral and that will be a required answer.

So, we saw an introduction to the double integral in this lecture and we also worked out an example, where we showed that about the repeated integral will be equal if the integral double integral exist we. We will stop today lecture here and then in the next class we will continue with some further examples.

Thank you.