Integral and Vector Calculus

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Lecture – 16
Differentiation under Integral Sign (Contd.)

Hello students. So, up until last class we looked into Differentiation under the Integral Sign, where we learnt about if you have an integral of this type let us say, I equals to integral from a to b f x y d x d y then in that case how we can differentiate this integral.

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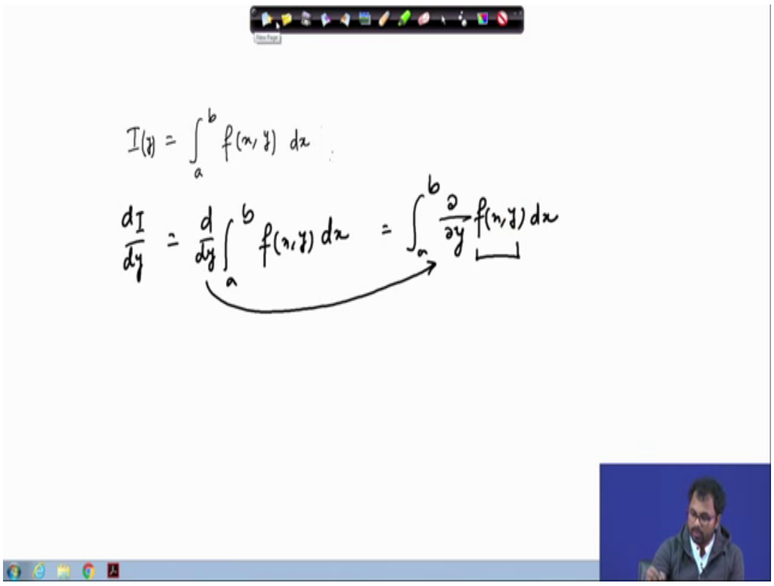

So, here so we do not need to have d y. So, first let us. So, if we integrate this integral then in that case we will obtain a function of y. So, basically this integral is nothing, but a function of y and after the integration, we can talk about the difference ability of this function I y. And if we want to differentiate this integral let us say if we are doing d I d y then in that case, we can write it as integral a to b f x y d x and after the after we perform the differentiation, we will basically bring the differentiation inside the integral and it can be written in this fashion.

Now, bringing this integral bringing this differential inside differentiation inside the integral is not that straightforward, I mean in order to do that we need to have some special properties for this function f. And we saw in the previous class that that this function f x y needs to be continuous on a rectangular domain r, and also the partial derivative of this function f needs to be continuous with respect to y. So, based on that we also solved an example we will continue with the similar topic which is differentiation under the integral sign, and we will now state and an important theorem in that respect. So, the theorem is called as Leibnitz's.

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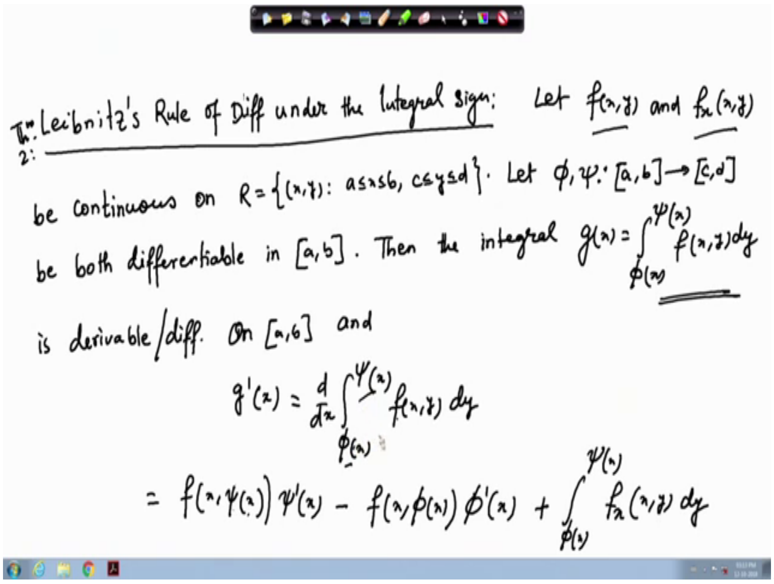

Rule of differentiation under differentiation; Leibniz rule of differentiation under the integral sign so, integral sign Leibniz was a German mathematician.

Now, the statement goes like this. Let so, this is this can be basically view as a theorem; we can consider it as theorem 2. So, let f x y and f x y be continuous be continuous on the rectangle R, which is defined as all those x y such that a lesser equal to x less or equal to b and c lesser equal to y lesser equal to d.

And let phi and psi with 2 functions mapping from a b to c d be both differentiable be both differentiable in a comma b, then the integral let us say g x equals to integral from phi x to psi x, f x y d y is derivable or differentiable is derivable or we can also write a differentiable in on the interval a b, and our derivative can be given as g dash x equals to d d x of integral from phi x to psi x, f x y d y is equals to f x y instead of y, I can write psi x times psi dash x.

So, differentiation of psi x with respect to x minus f x instead of y, I can write phi x times phi dash x and plus integral from phi x to psi x f x y d y. So, first of all if we look at this integral; obviously, if we integrate with respect to y and if the limits are both functions of x, then in that case after substitution will basically obtain a function of x, and then we can talk about it its differentiability. So, to talk about its differentiability makes sense, because we ultimately obtain a function of x only.

Now, in order to do the differentiation on this interval a to b, this function f needs to have these 2 important properties that it needs to be continuous and also its a partial derivative with respect to x also needs to be continuous. And then in that case and then in that case we can talk about such difference ability and rule of differentiation is we first substitute f x psi in the place of y we substitute the upper limit which is f of x psi x times psi of x minus f x phi x times phi dash x plus the derivative will come inside.

Now here, whether we would like to see, whether this Leibniz rule of differentiation under the integral sign is actually valid when the limits are constant or not. So, if I substitute phi equals to a and psi equals to b then b is constant. So, side as x will be 0 and phi is constant and in this place of phi we have a. So, phi is constant and therefore, phi dash x equals to 0 as well.

So, the first 2 term will vanish and then in that case we are left with the left with integral from a to a f x, x y d t y so, which is basically our theorem 1 for differentiation under the integral sign. Remember theorem 1 was something which dealt with constant limits so; that means both the limits lower and upper limits where constant in case of theorem 1 and in case of theorem 2, you have both the limits as a function of x. So, this is a general rule of differentiation under the integral sign or sometimes they it is also called as Leibniz rule of differentiation.

So, for the proof we are not concerned with that, because the proof is relatively long and its out of the scope of this syllabus, it is just that knowing the statement is very important because we require this statement quite often in our work. And when we also solve some exercises and some problems, we definitely require this formula to solve those to solve those problems.

So, now we are ready with solving the examples. So, in order to do so, let us let us start with our first problem. So, to start with our first problem let us consider example 1.

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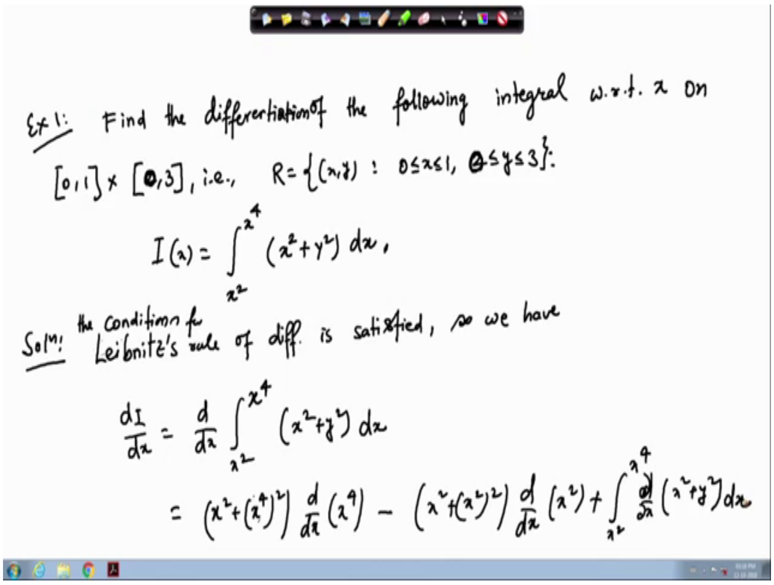

So, find the differentiation of the following integral with respect to x on 0 to 1 times 2 to 3. So, that is our rectangle R s all those x and y such that 0 less or equal to x less or equal to 1 and 2 less or equal to y less or equal to 3.

So, we need to find that differentiation of the integral given as I x equals to integral from x square to x to the power 4, x square plus y square d x all right. So, here our function f is so, let us go to the solution. So, of course, here we will apply the Leibniz rule of differentiation. So, first of all we see that both the functions both the functions let us start this interval by 0 as well. So, that we have a bigger interval. So, 0 to 0 to 1 and 0 to 3. So, of course, these 2 functions are continuous both the upper limits and lower limits in this interval, they also map from 0 1 to 0 3. And this function this x square plus y square since it is an algebraic function in a way, it will be continuous and not only that partial derivative with respect to x is also continuous in this in this in this domain. So, all those conditions for the Leibniz rule of differentiation is satisfied. So, we do not have to worry.

Now, or we can write at least one line we can write at least one line that, Leibniz needs the conditions sorry the conditions for Leibniz rule of differentiation is satisfied. So, we have d I d x equals to d d x of integral from x square to x to the power 4 x square plus y square d x and then this will be x square plus y square then derivative instead of y we substitute the upper limit.

So, let us substitute the upper limit here, which is x to the power 4 and then its square is basically x to the power 4 whole square times the differentiation of the upper limit which is basically d d x of x to the power 4 minus x square plus y square and instead of y we substitute the lower limit. So, x square whole square times d d x of x square plus integral from x square to x to the power 4, derivative of this one with respect to x. So, it is basically d d x.

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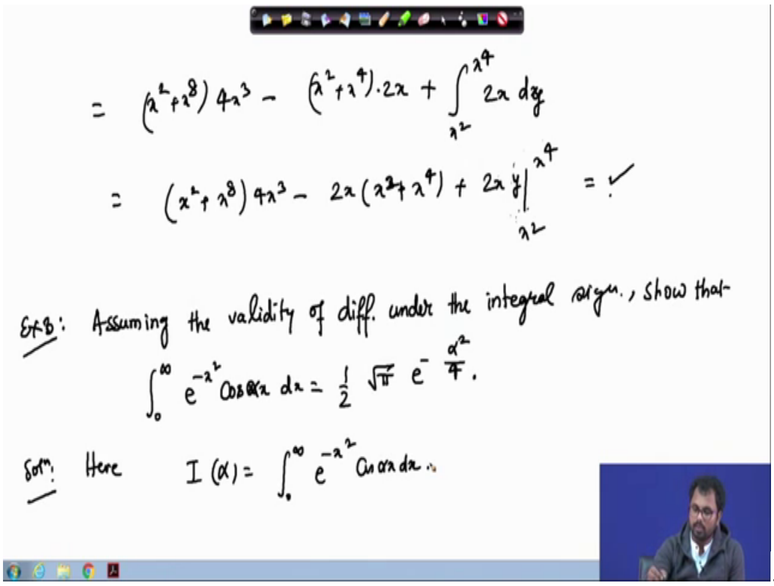

So, as del del x of x square plus y square d x and then this can be written as x 2 x square plus x to the power 8 times differentiation of x to the power 4 would be 4 x cube, minus x square plus x to the power 4; x square plus x to the power 4 times 2 x times 2 x plus integral from x squared to x to the power 4, x square to x to the power 4 the derivative of the second term will be 0, and the first term is 2 x. So, I can write 2 x d x.

So, this is basically x square plus x to the power 8, 4 x to the power 3 minus 2 x times x square x square plus x to the power 4 and this one will be 2 times x square by 2. So, this is basically just x square and we substitute x square. So, we substitute x square to x to the power 4 and then whatever is the limit we just write the limit here. So, we substitute x square. So, this will be x to the power 8 minus x to the power 4 and we just calculate and that will be the differentiation of the integral.

So, this is how we calculate in this case the differentiation of an integral. So, what we do, we just apply the we just apply the Leibniz rule of differentiation. So, psi x to phi x to psi x and then we integrate with respect to y I am sorry. So, here we are integrating with respect to y. So, this will be this will be 2 x d y. So, this will be 2 x d y and then we can write this as 2 x times y integration from x squared to x to the power 4 sorry. The integration here the integral has to be with respect to y then in that case we get the integral with respect to x yes.

So, if we integrate with respect to y, then we will basically obtain y here and this is of course, 2 x and then in that case this will be 2 x times yth and the limit we can substitute x to the power 4 x square and then we calculate this is basically a calculation of a simple algebraic expression. So, this is how we calculate the differentiation of an integral and of an integral and next we work out few more examples just to get the concept a little bit more clear.

So, example 2 or I would say 3. So, assuming the validity of differentiation under the integral sign, show that integral from 0 to infinity e to the power minus x square cos x, cos alpha x d x equals to 1 by 2 square root of pi, e to the power minus alpha square by 4, where alpha is any positive number.

So, we have to at this integral e to the power minus x over cos alpha x d x is equals to this. So, here it is very clear that alpha is the parameter since we are integrating with respect to x. So, we can write here I alpha equals to integral from 0 to infinity, e to the power minus x square cos alpha x d x. So, this is given. Now, let us differentiate both sides with respect to alpha.

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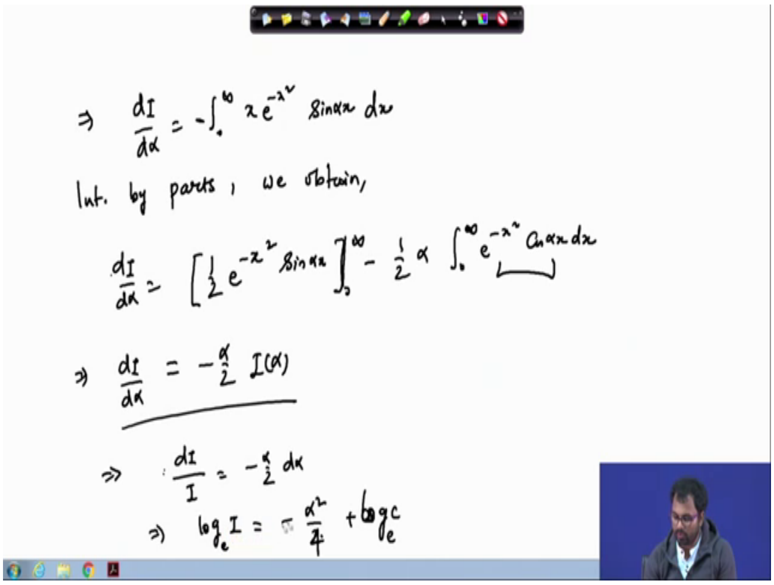

So, this will be d I d alpha and our partial derivative will go inside of the integral. So, this will be 0 to infinity x e to the power minus x squared differentiation of cos x is sine x and differentiation of alpha x is x d x all right. So, this is our d I by d alpha, and now we integrate by parts both sides, then we obtain then we obtain d I d alpha. So, we integrate by parts and then this will this will reduce to 1 by 2 e to the power minus x square sin alpha x, integral from 0 to infinity minus 1 by 2 alpha this will be integral from 0 to infinity e to the power minus x square cos alpha x d x.

So, here we are integrating this whole thing by parts and considering sin alpha x as the first function, and x e to the power e to the power minus x square as the second function, we will we will obtain this integral here, this is basically a part of method of substitution and on. So, it is pretty straightforward and from here when x is infinity then this whole thing will go to 0 and when x is 0, then again this whole thing will go to 0. So, the first term will not play any role and the second term can be written as I alpha, because this is ourI alpha we started with. Therefore, our d I d alpha is basically this ordinary differential equation.

So, if we integrate both sides then in that case this will be d I d alpha. So, d I by d I is equal to minus alpha by 2 d alpha. So, if I integrate then this will be log of e I and if I integrate on the right hand side then this will be alpha square by 2 times 2 so; that means, this will be 4 plus some constant C.

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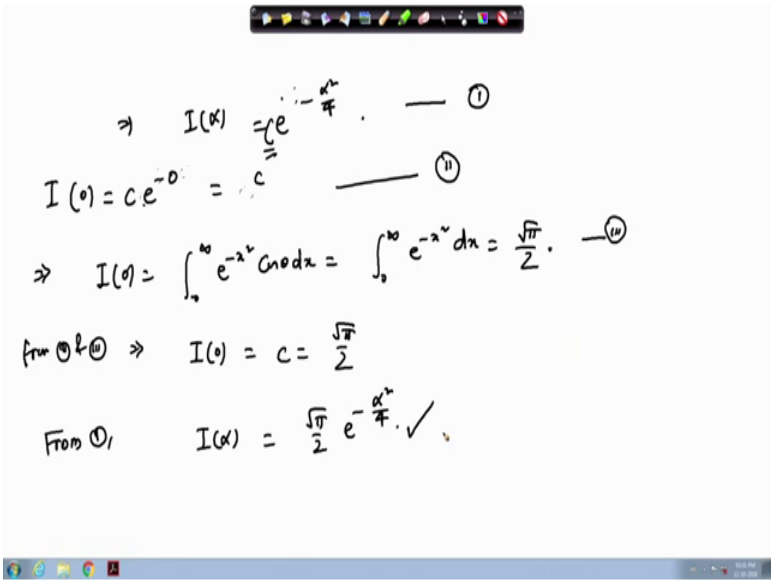

And this can be written as I alpha equals to e to the power some C times e to the power some C plus e to the power some C is it plus its minus alpha square by 4 times this alright.

So, now we find out the value of I 0. So, when alpha is 0 then this will be e to the power C minus 0 so, basically e to the power C. So, let us call this as equation number 1 this as equation number 2. So, we have to evaluate the value of C and I 0 on the left hand side basically I 0 is when I substitute alpha equals to 0, then this is basically cos 0 and cos 0 is 1. So, we have I 0 equals to integral from 0 to infinity e to the power minus x square cos 0 d x so, cos 0 is 1. So, this will be integral from 0 to infinity e to the power minus x squared d x, and in our previous lecture we just we proved that this is nothing, but our gamma half by half.

So; that means, a square root of pi by 2. So, this relation we already proved in previous chapter in previous topic. So, combining these 2 we can see that. So, combining these 2 we can see that I can see that C will be. So, if I choose C as let us say log of C. So, if I choose C as let us say log of C then in that case this will be log of I C. So, this will be e to the power minus e to the power minus alpha square by 4 times C. So, this will be it the C will come here and this will be C times e to the power minus alpha square by 4.

So, we can it is all about playing with the arbitrary constant. So, we can play with this arbitrary constant as we please. So, we can write it as C e to the power minus 0. So, this is basically our C. So, our C is basically I so; that means, from 2 and 3. So, from 2 and 3 we have from 2 and 3 we have I 0 equals to C equals to is square root of pi by 2. So, from here we can write from 1 we can write I alpha equals to square root of pi times 2 e to the power minus alpha square by 4 and this is what we needed to prove.

So, you see by considering this integral. So, this integral here it is also very important to notice whether this falls under that under the category of differentiation under the integral sign or not. So, if it does falls into that category, then in that case we have to identify the parameter it is usually the variable with respect to which we are not differentiating not integrating. So, here we are integrating with respect to x. So, alpha is the parameter and then after the integration we will obtain a function which is a function of alpha only, and therefore, we can write it as I alpha.

Now, we differentiate this I alpha with respect to alpha, and we obtain this integral here then we this we different we integrate this integral by parts and then we obtain this here, and we can evaluate the limit and we realize that the limit in the first term vanishes therefore, the second term will reduce to minus alpha by 2 times 4 I of alpha, and next we integrate and in after integrating choosing a constant we know that from our plus 2 level that we choose this constant in such a way that you can use those logarithmic formulas and then you can write it here as e to the power that whatever you have on the right hand side.

So, instead of choosing C, I chose a logarithmic to the base e and then I can bring this log here and then this will be log of 5 minus log of c. So, it will be log of I by c and then a on the right hand side we will obtain C times e to the power minus alpha square by 4. Now we need to evaluate the value of C. So, what we do? We substitute alpha equals to 0 and therefore, this will reduce to a square root of pi by 2 and by substituting the value of C here we will obtain require I alpha, and this is what we needed to prove in our example. So, in this problem this is what we needed to prove. So, it is not that how to say complicated it might get sometimes lengthy, but solving this type of problem would not be that much complicated.

Next so, let me give you 1 or 2 more examples, but I would say I would assume that solving those problems can also be fairly easy because they do not involve such a complicated calculation or something.

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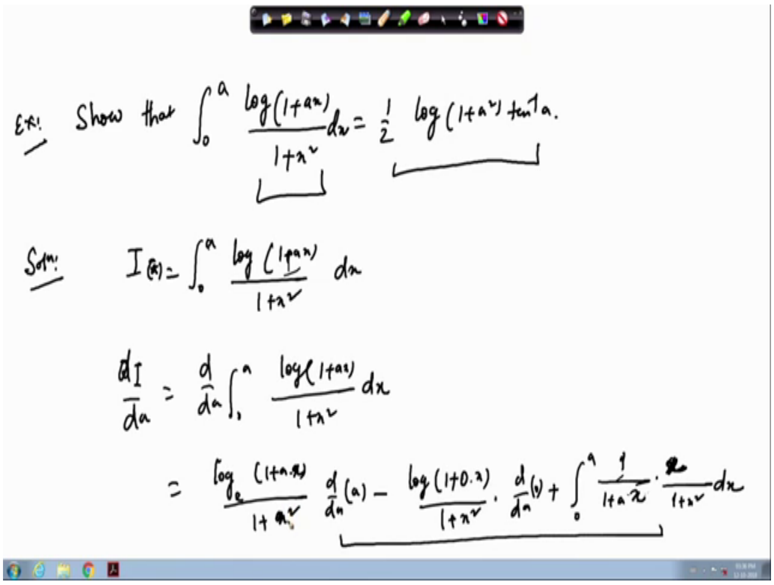

So, another example could be show that integral from 0 to a log of 1 plus a x divided by 1 plus x square equals to 1 by 2 log of 1 plus a square tan inverse a. So, again if you look at this integral it might look a little bit complicated, but and on the right hand side we have of we have an integral we have a value of the integral, which sort of involves log and tan inverse.

So, I can write I equals to integral from 0 to a log of 1 plus a x divided by 1 plus x square. Now again here we are integrating d x. So, here we are integrating with respect to x so; that means, a here will be considered as the parameter and we have an a in the integrand and as well as in the limit upper limit as well. So, this I of a is not that simple to differentiate here we have to use the Leibniz rule of differentiation, before we use the simple our function 1 sorry theorem 1 of integral under the differentiation under the integral sign, but in this case we have to use the Leibniz rule.

So, the Leibniz rule the Leibniz rule can be applicable here. So, d I d a which is which can be given as d d a integral from 0 to a, log of 1 plus a x divided by 1 plus x d x and this can be written as log of x y. So, instead of y I substitute a. So, this will be 1 plus a times a divided by 1 plus x square and this instead of 4 a, we as substituting again a times aside as x side as x is this one here. So, this will be d d a of a minus this second one would be log of 1 plus instead of a, I will substitute 0 times x divided by 1 plus x square and then this will be d d a of 0 plus integral from 0 to a, we have d d a.

So, this will be 1 by a square x square times a 1 plus x square d x right. And differentiation of a x would be a times 1 by. So, differentiation of this function would be 1 by a plus x square plus 1 by x square. So, it will be slightly different. So, the value of this integral the value of the difference the differentiation would be x 1 plus x square 1 by 1 plus a x. So, here this will be 1 by 1 plus a x time and in the numerator we will have and x actually yes.

So, this will be the value of the differentiation of this integral with respect to x here and now we have to again calculate like we did before. So, substitute. So, this will reduce to this will reduce to a function with respect to with respect to a. So, here this will reduce to a yeah and then we basically formulate the o d. So, let me go to the next page. So, we will basically obtain an o d e of following type.

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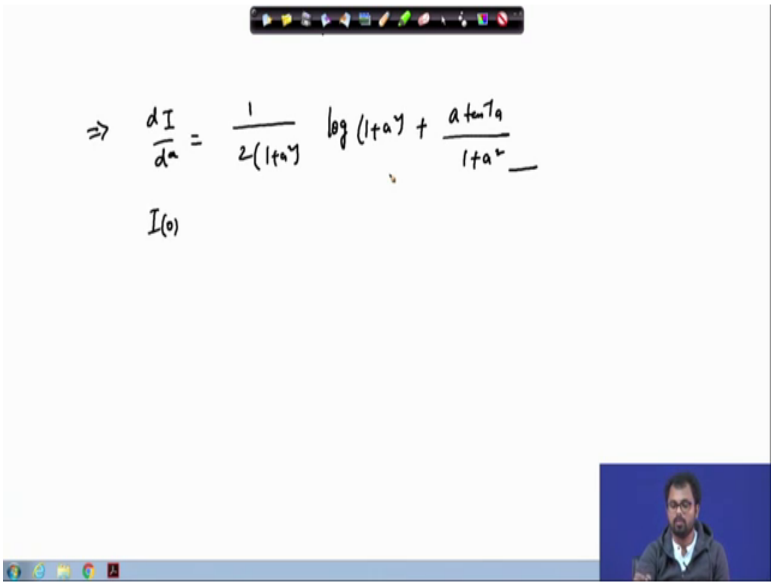

So, the o d will look like d I d a equals to 1 by 2 1 plus a square log of 1 plus a square plus a and inverse a by 1 plus a square. So, this is the required ordinary differential equation that we will obtain.

It involves a slight simplification or calculation here, but since it is relatively big I leave this to the students and ultimately we will obtain an ordinary differential equation of this type, and from here we will basically obtain I 0 like we did before, because we will obtain a constant and then we substitute the value of the constant to solve this ordinary differential equation, and that will give us the value of this integral I which we started with.

So, based on these the calculation, we see that we can be able to evaluate this type of integral, which is basically involves another parameter and this type of integral do come up quite often in engineering sciences also, and there we might need to do the differentiation under the integral sign and this theorem 1 and theorem 2 which is basically Leibniz rule of differentiation will prove to be very handy, and I will include some more examples in your assignment sheet and I look forward to your next class.

Thank you.