Integral and Vector Calculus

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Lecture – 15
Differentiation under Integral Sign

Hello students. So, in the last class, we worked out few examples on beta and gamma function. Today, I am going to give one more example where you can express an integral in terms of gamma function and then we will move to the next topic which is a Differentiation under the Integral Sign.

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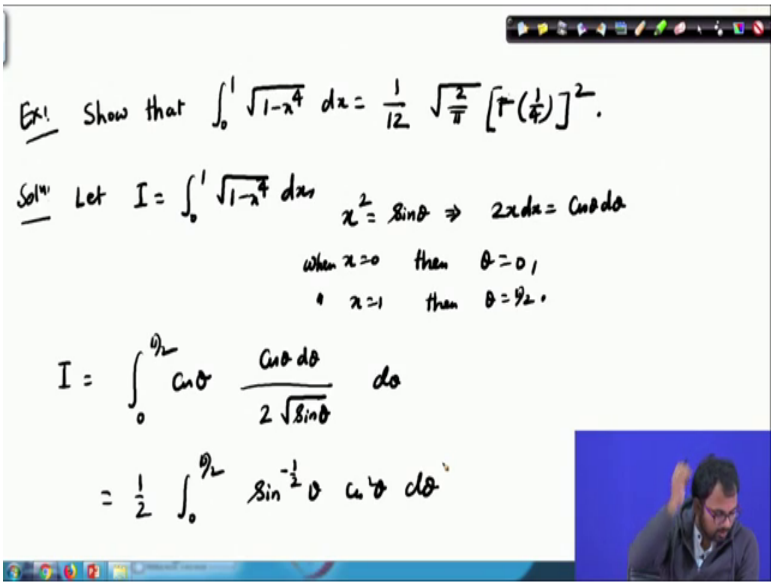

So, to start with, let me write the example. So, the example reads as show that integral from 0 to 1 square root of 1 minus x to the power 4 d x can be written as 1 by 12 square root of 2 by pi times gamma 1 by 4 square, so gamma 1 by 4 square. So, here you can see that the left hand side does not at all resemble with gamma integral; however, the answer can be expressed as gamma function. So, how can we do that? Let us see. So, the solution we will assume that our integral I is equal to 0 to 1, 1 minus x to the power 4 d x.

So, here I will take x equals to x square equals 2 sin theta. So, then in that case this will be 2 x d x equals 2 cos theta d theta and when x is 0, when x is 0, then theta is also 0 and when x is 1, then theta is pi by 2, then theta is pi by 2. So, based on this our integral I would reduce to 0 to pi by 2, 1 minus sin square theta. So, this is basically cos square theta and the square root of it will give cos theta and d x would turn into cos theta d theta divided by 2 square root of sin theta and d theta. So, this is basically 1 by integral from 0 to pi by 2 sin theta or sin to the power minus half theta cos square theta d theta. Now, we will use that beta the sin to the power p theta times cos to the power q theta equals to beta p plus 1.

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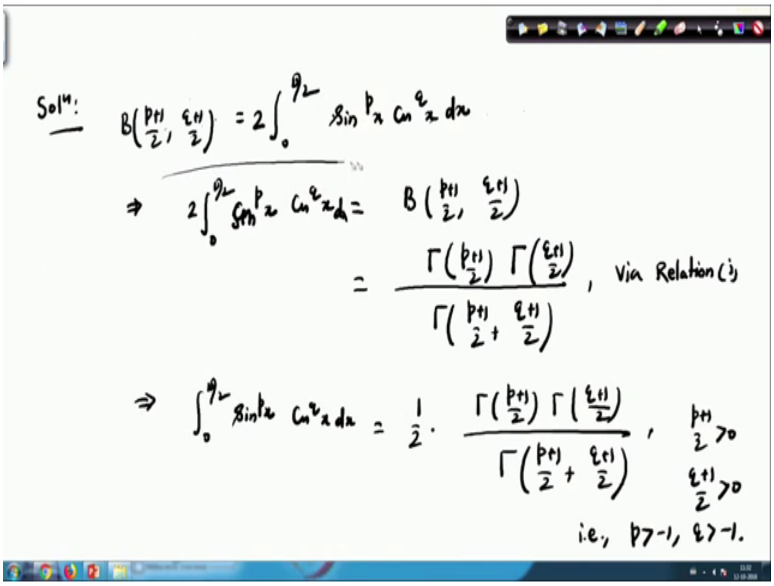

So, basically we will use this formula here. So, the formula which I am talking about is this one, is this one. So, here p and q are both greater than minus 1. So, here p and q are both greater than minus 1. So, instead of p, I will choose minus half and instead of q, I will choose 2.

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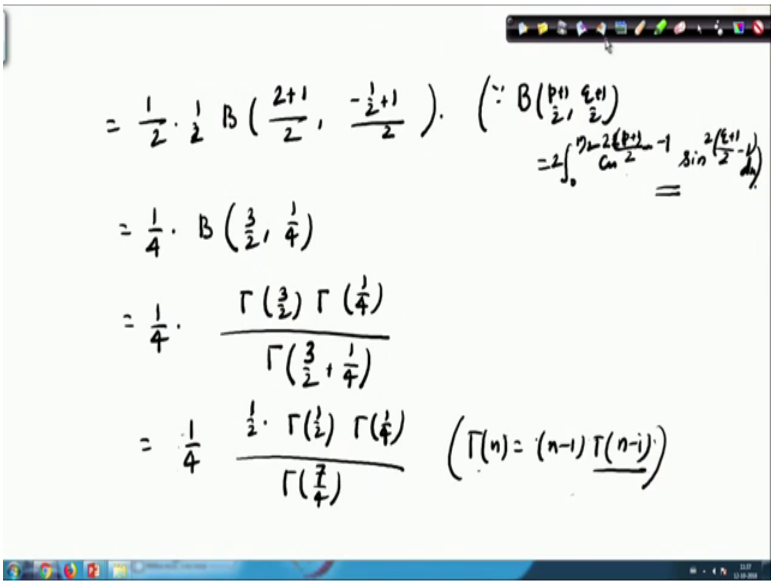

So, then this whole thing will reduce to this will reduce to, so this whole thing will reduce to half times another half beta 2 plus 1 by 2 and minus half plus 1 by 2. So, I can write since beta p plus 1 by 2 and q plus 1 by 2 is equals to the formula is, and beta p plus 1 by 2 q plus 1 by 2 is. So, this formula is p plus 1 by 2. So, this is integral from 0 to pi by 2 cos p plus 1 cos.

So, this will be I can write this formula as I have to do some adjustments here and then, this will be cos of 2 p plus 1 by 2 minus p; so, cos of p plus 1 minus 1 2 times as so in place of m, I have this one minus 1 and sin of 2 q plus 1 by 2 minus 1 d x. So, this is the formula which we are using basically, so 2 times p plus 1 by 2 minus 1 and sin of 2 times q plus 1 by 2 minus 1.

So, ultimately instead of q, I am taking minus half. So, this will reduce to, this will reduce to minus half ok. And instead of p, I am taking 2. So, this will be 3 by 2 and 2, 2 will get cancelled so, ultimately 2, yes. So, I am using this formula here. Now, I can be able to write it as 1 by 4 times beta and this will be beta 3 by 2 times gamma beta 1 by 4. So, this will be our beta 1 by 4 and now based on which, I can use that first relation which says that this will be gamma 3 by 2 times gamma 1 by 4 divided by gamma 3 by 2 plus gamma 3 by 2 plus 1 by 4.

So, this can be written as 1 by 4 and this is basically 1 by 2 times gamma half times gamma 1 by 4 divided by gamma 7 by 4. So, this gamma 3 by 2, so we denote that gamma 3 by 2 can be written as gamma 1 plus 1 by 2 and then this can be written as, then this can be written as gamma half. So, this can be written as half times gamma half and, yes. Now, so this can be written as half times gamma half. So, this whole thing can be now converted into, so this whole thing can be now, so this is our 1 by 4 as always; so, this is our 1 by 4 and then, we have half times gamma half gamma 1 by 4 and then gamma 7 by 4.

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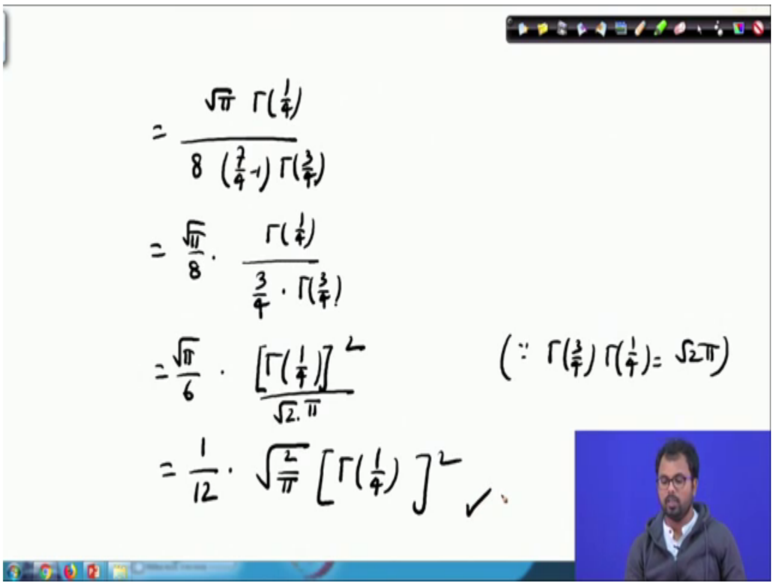

So, this can be written as square root of pi times gamma 1 by 4 because gamma half is a square root of pi and then this is 1 by 8 7 by 4 minus 1 and this will be gamma 3 by 4. So, here what we are doing is n minus 1. So, 3 by 2 minus 1 is half and then, we have gamma half. So, that is the formula which I was using. So, we can either write it or we do not write it. So, the formula is gamma n equals 2 n minus 1 times gamma n minus 1. So, this is what we are using here and then, we will use this formula again here; so, this one.

Now, this is square root of pi by 8. This one will be gamma 1 by 4 and this will be 7 by 4. So, we will obtain basically as 3 by 4 and this will reduce to gamma 3 by 4. Now, we will have square root of pi by 6 and we will have gamma 1 by 4 times. So, we know that gamma 3 by 4 we just derived this formula, gamma 1 by 4 equals 2 square root of 2 by pi square root of 2 times pi. So, gamma 3 by 4 will be square root of 2 times pi and divided by gamma 1 by 4. So, that will go at the numerator and then this whole thing will turn into a square. So, we will basically have square root of 2. So, we will basically have this here and this will turn into 1 by 12 times square root of 2 by pi times gamma 1 by 4 whole square. So, this is what we needed to prove. It is just some algebraic calculation which you can be able to do by yourself.

So, here we see that although this integral here, although this integral here had no connection with gamma function, whatsoever we can be able to reduce into a gamma function first of all into a beta function formula. And, then we can be able to reduce this whole thing into a gamma function formula and with the help of which we can be able to just do some use some gamma function properties and then we can be able to able to derive the required right hand side. So, this is what we needed to prove.

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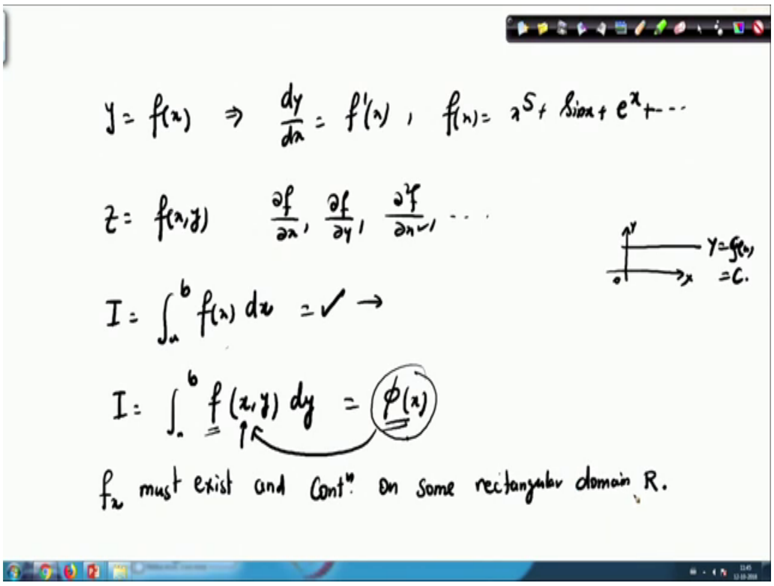

So, similarly you may come across with a different types of integral in gamma function example and based on which you can be able to calculate how to say those integrals using the function of gamma, properties of gamma function. So, I will include some more examples in your assignment sheet just for you to have a look at them and I am pretty sure you will be able to you will be able to solve such problems involving integrals and expressing them in terms of gamma function.

So, the next topic in our third chapter is the differentiation of an and under the integral sign. So, this is also very interesting topic and not only that you will need it here in the integral calculus, but you will be able to see this particular formula has a lot of applications in npde's or also when you are solving how to say, ordinary differential equations where you when. So, for example, you might come across with certain results where you need to differentiate an integral where which is not usual what we do in our differential calculus.

So, in differential calculus, we are usually given a function of this type, y is equals to f x and when we say differentiate, then we basically differentiate like d y d x equals to f dash x and if when f dash x is f x is a function of one variable. Let us say, f x can be our x to the power 5 plus sin x plus e to the power x plus some other function dot dot and so on. So, we can be able to differentiate this function because it is a function of one variable.

Now, now if we have a function of two variable, let us says z equals to f x y, then we usually have partial derivatives. So, we can be able to calculate del f del x and del f del y or del square f by del x square del square by del y squared dot dot and so on. So that means, if you have a function of one variable, function of two variable, then doing the differentiation is quite easy and we have. So, the function needs to be how to say smooth in a way, but differentiation of an integral is not that straightforward.

So, it is not only the function of one variable or two variable, the smooth function you can differentiate. You can also differentiate an integral, but in order to do that, it involves some kind of special properties of that function and we will look into those properties. So, first of all, if I write I equals to integral from a to b f x d x, so here x is a variable and we do not have any parameter. So, f x is a variable when I integrate then it will always result in to constant or we can say that it has a constant function. So, a constant is also a function which assumes the same value. So, if we are on the x and y plane and it will assume always the same value. And so, y equals to f x equals to a constant. So, a constant function is also function, but it is constant at every point.

So, when you integrate a function of one variable, we obtain a constant function and that constant function is always differentiable. So, there is no how to say a special thing to look at. However, if you have an integral of this type, let us say d y, then after you integrate, what do we obtain? We do not obtain a constant, we obtain a function of x mainly because here variable x is involved. So, after we are done with the integration, we will be able to obtain a function of x phi x.

Now, if that phi x is smooth; that means, if the partial derivative of phi r the derivative of the function phi x with respect to x. So, if the differentiation of phi x with respect to x exists, then in that case, we can be able to differentiate this integral I here with respect to x. And this function is differentiable with respect to x depends highly on the differentiability of this function with respect to x because, if this function is not differentiable with respect to x, then this one will also be not differentiable with respect to x.

So, in order to have the derivative, in order to differentiate this function, we need to have the different stability of this function with respect to x. And in other words, the partial derivative of this function x of this function f with respect to x must exist. Not only that, in order to have it to be existing on a given domain of integration, it has to be continuous throughout that interval so that it will exist at the point. So, that means, this function here f x must be, so must exist and continuous on some rectangular domain, on some domain R. So, I can put this whole thing in a in a statement.

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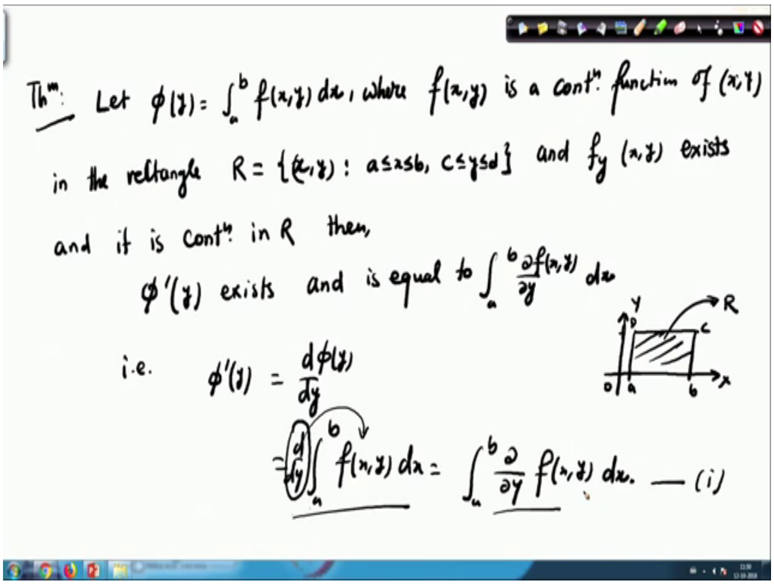

So, let us put this whole thing in a statement or a theorem. So, the theorem reads as, the theorem read says let, phi y equals to integral from a to b. So, now, we are in the function of two variable case, f x y d x where our f x y where f x y is a continuous function, is a continuous function of a continuous function of x y in the rectangle R equals to all those x and y such that a lesser equal to x lesser equal to b and c lesser equal to y lesser equal to d. So that means, we have a rectangular domain sorry, we have a rectangular domain of this type where the function f is defined. So, this is let us say. So, this is our a, b, c and d and this is our rectangle r where the function is defined or where we are performing the differentiation or integration.

Now, and f of y x, y, so like I was saying in the in the previous slide the partial derivative must exist and it is continuous in R, it is continuous in R, then, but then the derivative of phi will exist and is equal to integral from a to b del del y of f x y d x. That is f phi dash y which is basically d y d d y d d y of phi y is equals to integral d d y a to be f x y d x equals to integral from a to b del del y f x y d x; that means, if the function if if the function f is continuous in the rectangle m R which is given in this fashion and if the partial derivative with respect to y for this function f exists and if it is continuous in r, then the derivative of the function for the function phi can be given in this fashion and you can bring the derivative inside the integral. And when we bring the derivative inside the integral, then it will turn into a partial derivative which makes sense because in case of function of two variable we do not have d d y, we always have del del y or del del x.

So, once we bring the integral inside a differential in differentiations inside the integral, it will turn into a partial derivative and this is our first formula for the differentiation under the integral sign. So, we are doing the differentiation under the integral signs. And this is the integral sign which we were talking about and this is our differentiation. So, you see if our integrand has some special properties, we can even differentiate the resulting integral with respect to the parameter. So, here y is our parameter and x is the variable of this integral and at the end, we can differentiate this integral with respect to the parameter. So, here y is basically our parameter, all right.

So, next proof of this theorem is, we are also skipping because it is not in the in the scope of this lecture because it is a little bit how to say extensive in a way and we will mostly focused on working or some examples. If you are interested and you can definitely look into those books where they have used the epsilon delta definition which I am sure you already know about. And by using some inequalities and epsilon delta definition you can be able to prove this theorem. So, we leave the proof up to up to the up to the leader or up to the students. And at first, we will see the application of this theorem which we just stated. Let us name it as theorem 1.

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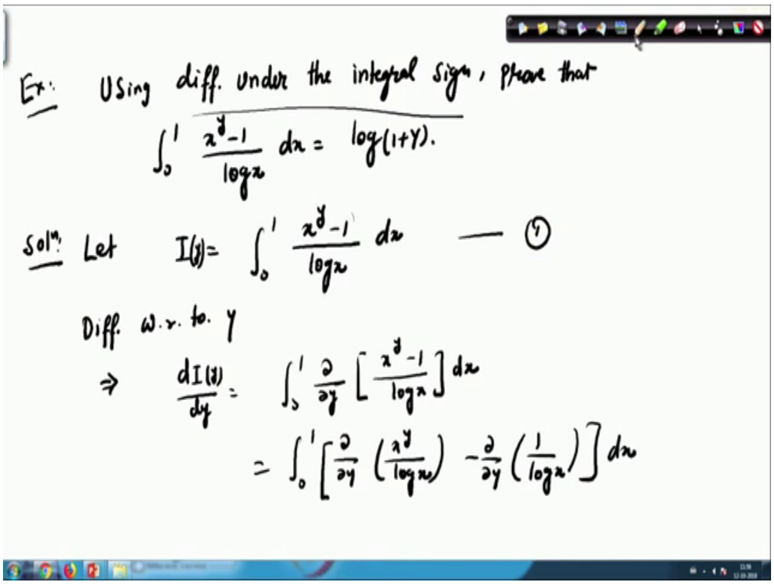

So, first example is, using differentiation under the integral sign, prove that integral from 0 to 1 x to the power y minus 1 divided by log x d x is equals to log of 1 plus y. So, here we have to use the differentiation under the integral sign because it is specifically said in the statement. So, now, let us consider, I equals to integral from 0 to 1 x to the power y minus 1 by log x d x. So, first of all, after integrating this integral, we will obtain a function which is a function of y only because x is our variable. So, we are integrating with respect to x and once the integration is done, we will end up with a function of y only, all right. So, I can write it as I y

Now, what we will do? We will differentiate both sides with respect to y because x to the power y, I is also a differentiable function. So, we can be able to differentiate this f of x with respect to y. So, if we can differentiate with respect to y and also it does not involve any kind of how to say bad behaving functions in a way that partial derivative with respect to y would also be continuous. So, we can differentiate this function now. So, let us differentiate. Differentiate with respect to y. So, what would happen we would obtain, d I y d y equals to integral from 0 to 1 our differential will come differentiation will come inside and this will become x to the power y minus 1 times log x d x.

So, the when we are differentiating, it will be 0 to 1 del del y x to the power y by log x minus del del y 1 by log x. So, now, 1 by log x differentiation of 1 by log x will be 0 because we are differentiating with respect to y and this function does not involve any variable with of y. So that means, this will be treated as a constant and differentiation of constant is always 0.

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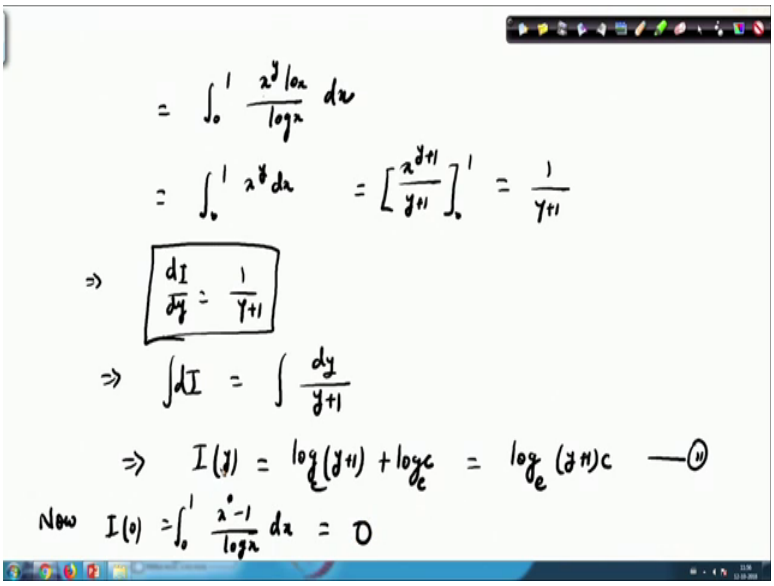

So, we will obtain 0 to 1, x to the power y times log x to the power y times log x times log x after differentiation. So, this is what we will obtain. Now, we can integrate both sides. So, if I, so we can integrate this one not both sides, we will integrate this one and this will reduce to x to the power y plus 1 divided by y plus 1 integral from 0 to 1. So, this will be 1 by y plus 1. So; that means, d I d y equals to 1 by y plus 1. So, we will have I equals to, d I equals to d y by y plus 1. This is basically our ordinary differential equation, alright. So, at the end, we are obtaining an ordinary differential equation.

So, if I integrate this will turn into I y equals to log of y plus 1 plus log c to the base e. So, this can be written as log of y plus 1 times c to the base e. Now, I need to obtain the value of c. So, what is y 0? So, let us call it as equation 2 and I will call the original problem by equation 1. So, our I will call this relation, not the original problem but I will call this relation as 1. So, what is I 0? So, I 0 now I 0 is nothing but integral from 0 to 1 x to the power 0 minus 1 by log x; now, x to the power 0 is 1 and 1 minus 1 is 0.

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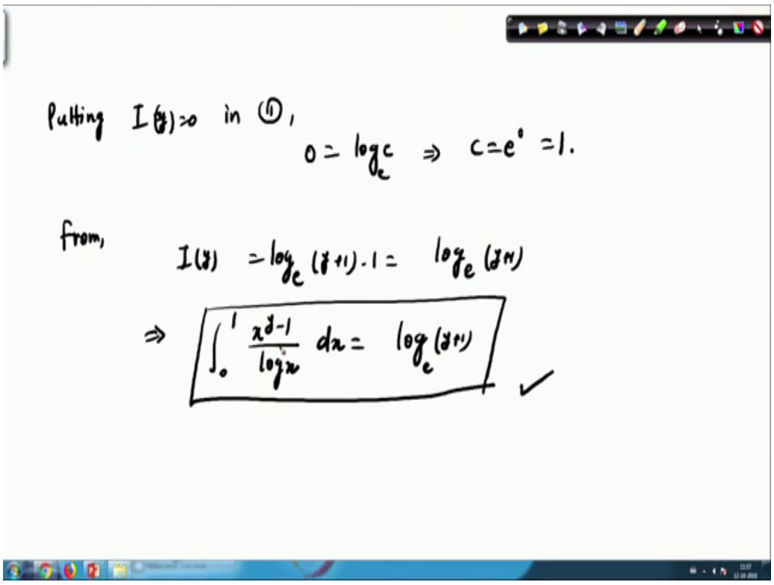

So, the value of I 0 is 0 and using substituting I y equals to 0 in 2, putting I equals to putting I equals to, I y equals to 0 in 2, we will obtain we will obtain 0 equals to log of c to the base e. So, therefore, c equals to e to the power 0 which is basically one and we will use this value of c in 2. So, from 2, we will have I y equals to log of y to the base e plus 1 times c c is 1. So, this is basically log of e y plus 1 and I y is nothing but our given integral, right. So, this is what we started with equal to log of y plus 1 to the base e and this is what we needed to prove.

So, you see just using the formula of differentiation under the integral sign, we can be able to calculate the value of this integral without doing any complicated method of substitution or anything. So, this is a very nice tool which helps us calculate the value of an integral by solving some kind of ordinary differential equation like here. And, in the next class we will look into a very important theorem of differential under integral sign which is Leibniz rule of differentiation under the integral sign. So, I will stop here for today and I will look forward to your next class.