Integral and Vector Calculus
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Lecture – 14
Beta and Gamma Function

Hello, students. So, in the last class we looked into the definition of Beta and Gamma Functions and we also tried to derive some properties related to these two special type of integrals. And, in the last in this lecture we will actually work out few examples and we will also derive some properties of these two integrals and we will see how we can apply those in those properties to solve some problems.

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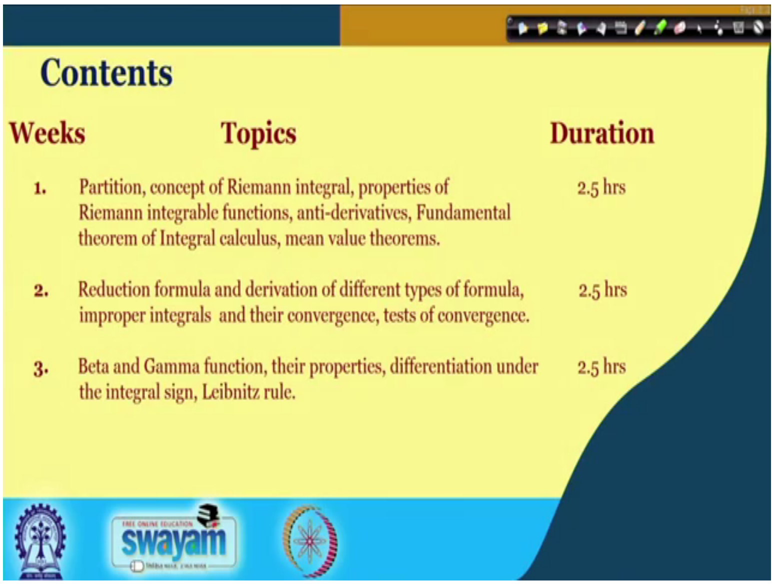

So, in order to start with let us. So, this is the first this is the topic which we are going to cover today; beta and gamma function, their properties and if time permits then we will start with differentiation under the integral sign. So, let us begin.

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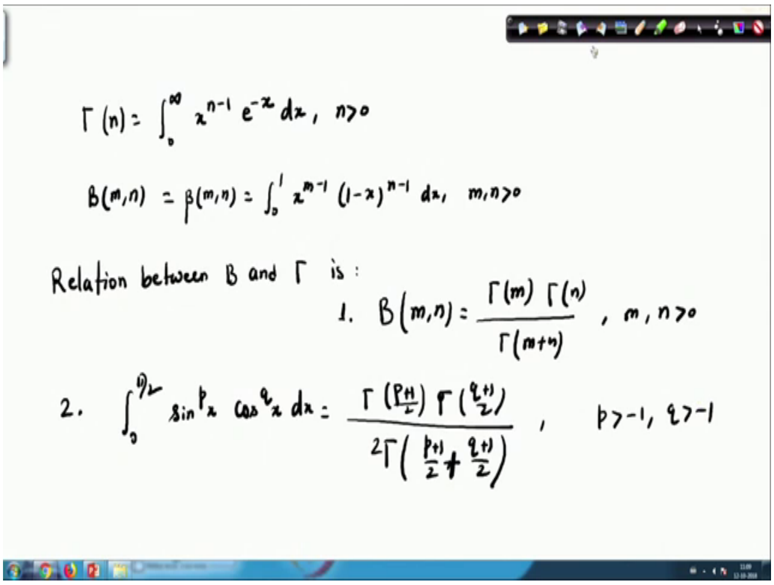

So, in the last class we saw that the gamma function is defined as gamma n equals to integral from 0 to infinity x to the power n minus 1 e to the power minus x dx where, n is a positive number. And, beta function is defined as beta m, n equals to or you can also write beta m, n; it is up to you whatever notation you prefer integral from 0 to 1 x to the power m minus 1 times 1 minus x to the power n minus 1 dx, where m and n are both positive numbers. So, both of these two integrals are of course, improper integral for certain values of m and n and we have shown that these two improper integrals are also convergent for this m, n and considered to be positive.

Now, there is a very nice relation between beta and gamma function although it does not seem like just looking at these two integrals because, one of them involves e to the power minus x. Whereas, the other one does not involve any kind of exponential function although we will see that these two integrals are actually related. However, the proof is not that simple and if we get into the proof then it will be slightly complicated. So, I will skip the proof, but I am going to write the relation because it will help us solve several problems for those of the students who are interested in the proof you can look into the books which I have recommended. It is not that difficult is that it is to lengthy and keeping the time constraint in mind it is really not worth doing in the class. However, the proof is interesting and I would suggest to look into them look into those books.

So, the relation between the relation between beta and gamma function is the relation between beta and gamma function is given by beta m, n equals to gamma m times gamma n divided by gamma m plus n, where m and n are both positive numbers. So, this is the first relation between beta and gamma function and just if let us say if we are asked to calculate the value of let us say beta half and beta half. Then in that case we can write it as gamma half and gamma half divided by gamma 1 and if we know the value of gamma half, then we can be able to calculate the value of beta half and half.

So, let us see how we can calculate the value of gamma half. So, this is our first relation. Now, the second relation is between the trigonometric function and gamma function. So, this says that integral from 0 to pi by 2 sin p to the power x times cos q to the power x dx equals to gamma p plus 1 by 2 times gamma q plus 1 by 2 divided by 2 times gamma p plus 1 by 2 comma q plus 1 by 2 sorry plus so, p plus 1 by 2 plus q plus 1 by 2. So, this is an another relation between beta between gamma function and the trigonometrical function. So, how do we prove this relation? So, the proof of this relation is fairly simple.

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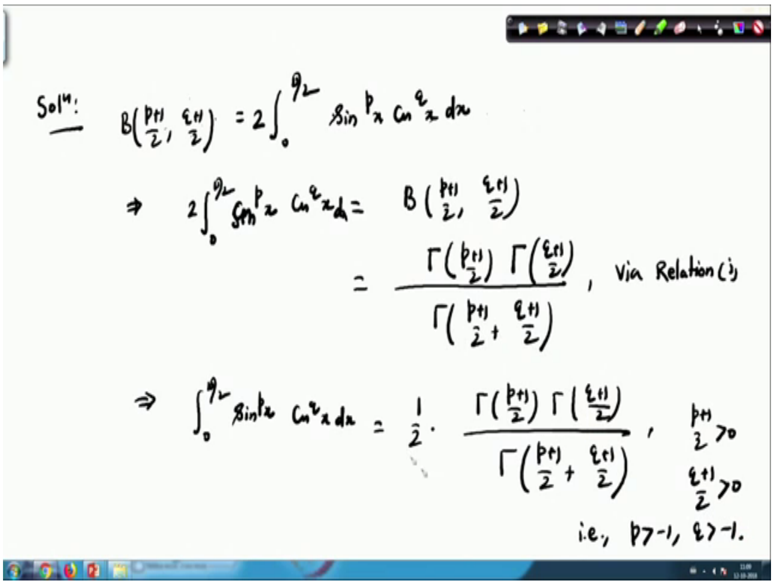

So, the solution or the proof of that relation goes like this. If you remember we have shown that beta p, q equals to integral from. So, we had this relation integral from 0 to pi by 2 2 cos 2p minus 1 times x and a sin 2q minus 1 x dx. So, we had proved this relation.

Now, if I substitute in place of p, p plus 1 and in place of, q plus 1 then this will reduce to so, this will reduce to 2 times integral from 0 to pi by 2 cos p to the power x and sin q to the power x. And, or we can we can take the other relation, we do not we do not need to take this relation. We have we can take the other relation between beta and gamma function. The other relation is we have derived this one. So, the other relation is beta p plus 1, beta p plus 1. We have this relation which we have derived in the previous class beta p plus 1 by 2 times a comma q plus 1 by 2 is actually sin p to the power x times cos q to the power x.

So, this relation we have already derived in the previous class and this can be written as 2 times sin p to the power x times cos q to the power x is equals to beta p plus 1 by 2 comma q plus 1 by 2 and now, here I will use the relation – 1 and based on relation – 1, I can write gamma p plus 1 by 2 times gamma q plus 1 by 2 divided by beta m plus n. So, beta gamma m plus n so, this can be written as gamma of p plus 1 by 2 plus q plus 1 by 2. So, here we have used via relation – 1. Let us see what is relation – 1.

So, in relation – 1, beta m, n can be written as if I replace m by p plus 1 by 2 and n by q plus 1 by 2 then I can write gamma p plus 1 by 2 gamma q plus 1 by 2 d divided by gamma p plus 1 by 2 plus q plus 1 by 2. So, this is the relation which we have used here and this can be written as. So, this can be written as integral from 0 to pi by 2 sin p to the power x times cos q to the power x dx equals to half times and gamma p plus 1 by 2 times gamma q plus 1 by 2 divided by gamma p plus 1 by 2 plus q plus 1 by 2.

So, this is the required relation and here we know that gamma is defined for all the positive numbers. So, in order to have this one as positive, so, gamma is defined for all n greater than 0. So, in order to have this one positive I can write p plus 1 by 2 greater than 0 and q plus 1 by 2 greater than 0 that is our p must be greater than minus 1 and q must be greater than minus 1. So, this relation is true. So, here I can write the condition that p is greater than minus 1 and q is greater than minus 1. So, for p and q both greater than minus 1 this relation holds. Now, let us see how we can obtain the value of gamma half.

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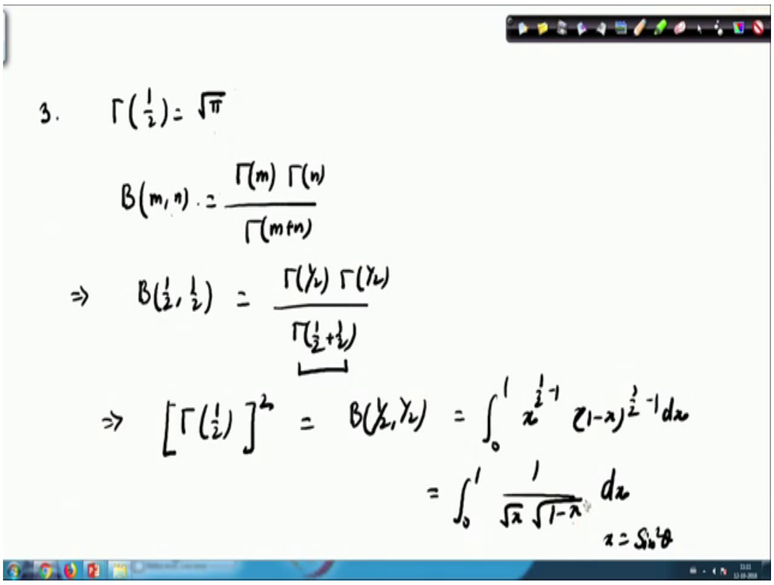

So, to obtain the value of gamma half that will be our relation – 3. So, gamma half is square root of pi by 2 or square root pi sorry, not by 2 is simply square root of pi, alright. So, from previous relation so, from previous relation we have gamma, so, beta p plus 1 by 2 q plus 1 by 2 or we can take beta m, n relation. So, from the previous relation we know that beta m, n is basically gamma m times gamma n divided by gamma m plus n. So, here if I take if I take m and n equals to half then this will become beta half half and this will be gamma half times gamma half divided by gamma half plus half.

So, the denominator will become gamma half and we know that the value of gamma half is 1. So, I can write this one as gamma half whole square divided by 1. So, I am not writing that 1 and beta half a half can be written as integral from 0 to 1 x to the power half minus 1 times 1 minus x to the power 1 minus x to the power half minus 1 dx. So, this will be basically integral from 0 to 1 root x by root x times root x minus 1 minus root x 1 minus x, sorry. So, this will be root of 1 minus x dx.

Now, this is a classical example of a method of substitution. So, I substitute x equals to sin squared theta and then you do some trigonometrical simplification which I am pretty sure you can be able to do that.

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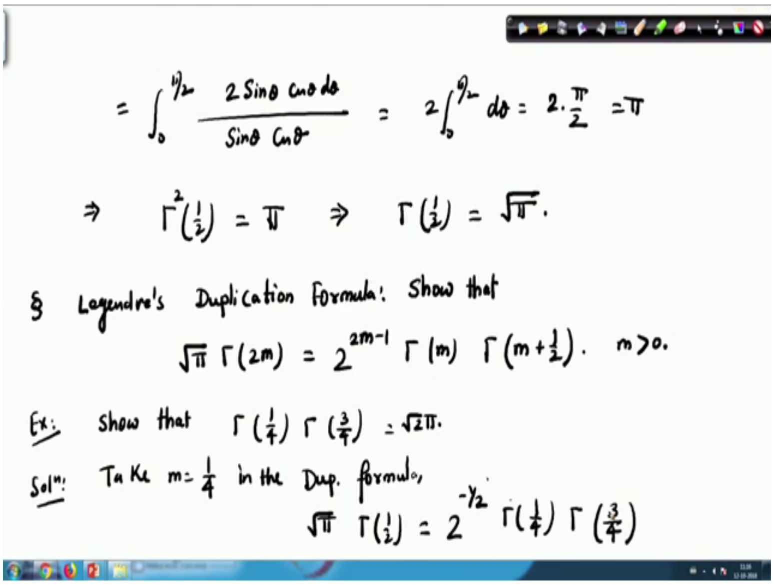

So, here ultimately you will end up with integral from 0 to pi by 2 to sin theta times cos theta d theta divided by square root of sin theta is sin square theta is sin theta and a square root of 1 minus sin square theta is cos square theta so, ultimately cos theta. So, this will be 2 times integral from 0 to pi by 2 d theta. So, ultimately we will obtain 0 times pi by 2. So, this is pi; that means, we have gamma half square gamma half square equals 2 pi and from here it implies that gamma half is nothing, but square root of pi and this is what we needed to proof. So, the value of gamma half is square root of pi.

So, this is also a very important relation in gamma function and we often how to say require the value of gamma half while calculating any type of any type of in how to say integral or a formula. Next, our next formula is we call it as Legendre's duplication formula. Of course, it is proof is a little bit extensive or a little bit lengthy. So, I will avoid the proof. However, I am I am just going to write the formula. So, the formula says that show that a square root of pi times gamma to m equals to 2 to the power to m minus 1 times gamma m times gamma m plus half and this is our required duplication from where m is any positive number.

Now, this formula here I mean in order to prove this we start with the definition of gamma function and then we do some trigonometrical substitution; it is not very complicated, it is just a little bit lengthy. So, I leave this proof up to the students and also you can look into the book of Santi Narayan and other authors where you can find this trivial proof there. What I am interested is to show the application of these formulas that how you can calculate different values of gamma; for example, we know gamma 1 is 1, but what is gamma half? And, you see with the application of this result here we can be able to calculate the value of gamma half.

So, these formulas although they are proof is a little bit lengthy, they are very handy in order to in order to obtain different values of gamma half or beta half beta half and 1 by 4 or 1 by 2, things like that. So, these formulas are quite handy and we are basically interested in their application. So, let us see with the help of this formula what else we can we can do. So, our next result is or example is show that let us say show that gamma 1 by 4 times gamma 3 by 4 is equal to square root 2 pi. So, let us see how we can prove it. So, the solution here we take m equals to 1 by 4. So, take 1 by 4 equals to m m equals to 1 by 4 in the duplication formula.

So, if I substitute m equals to 1 by 4 what would happen. So, we will obtain square root of pi times gamma half and this one will be 2 to the power 1 by 4 minus 1. So, I am going to write minus half. So, just to save some time I can write it as to the power minus half; gamma 1 by 4 times gamma this one is 1 by 4 plus half so, basically 3 by 4.

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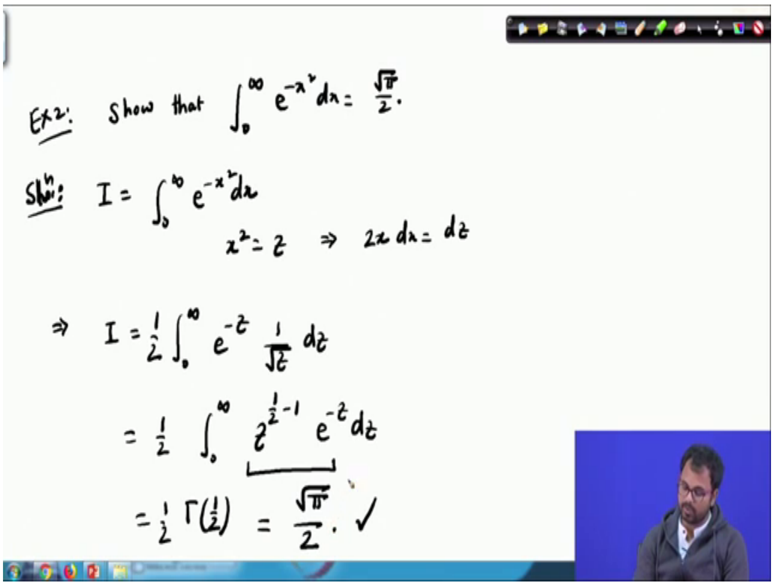

So, this will result into gamma 1 by 4 times gamma 3 by 4 equals 2 2 times I have a square root of pi and then I have gamma half. So, gamma half is again square root of so, square root of pi and this one is square root of 2. So, this is square root of 2 times square root of pi times the square root of pi is equal to square root of 2 times pi. So, you see although we needed to prove gamma 1 by 4 times gamma 3 by 4 we did not use at all the integral definition of gamma we just used this formula. So, in order to getting into the integral and then trying to prove something which could have been a little bit how to say complicated or extensive in a way we just use this formula and with the help of which the whole proof was like three lines long. So, as you can see these formulas are proving to be very handy in order to solve any type of gamma formula here.

So, next example could be our next example is the example so, show that integral from 0 to infinity e to the power minus x squared dx equals to square root of pi by 2 a solution. So, here when we see on the right hand side if it is square root of pi; that means, it must result in to gamma half in some way. So, there instinct the basic instinct would say that it has to be gamma half and; that means, here it would somehow lead to gamma half times some factor; that means, half. So, it has to be in some way gamma half and let us see whether we can obtain gamma half or not.

So, let us say we have I equals to our in given integral or the left hand side. So, this is our left hand side I can also write left hand side or I that that is not a problem here. Now, let us substitute x square equals to z. So, then in that case this will be 2x dx equals to dz and when x is 0, z is 0, when x is infinity z is infinity. So, from here we will have I equals to integral from 0 to infinity this will be half e to the power minus z dx will turn into dz and now, it is divided by 1 by x and x is 1 by. So, x is square root of z. So, this will be square root 1 by square root of z.

So, I can write this as half integral from 0 to infinity z to the power half minus 1 times e to the power minus z dz, is it not? Because half minus 1 is minus half and then that z to the power minus half will come into denominator and then it will turn into a positive power. So, I can be able to write this thing in this way now this is the definition of gamma half if I take m equal m equals to half then this is nothing, but gamma half actually from the definition of gamma function.

So, this is our gamma half is it not? And we know that gamma half is square root of pi by 2 and this is what we needed to prove here. So, you see using just some simple formulas, we are not getting into any complicated calculation, we can be able to prove that this integral is actually square root of pi by 2. So, this is one another application of gamma integral there is. There is one more example that we can that we can prove.

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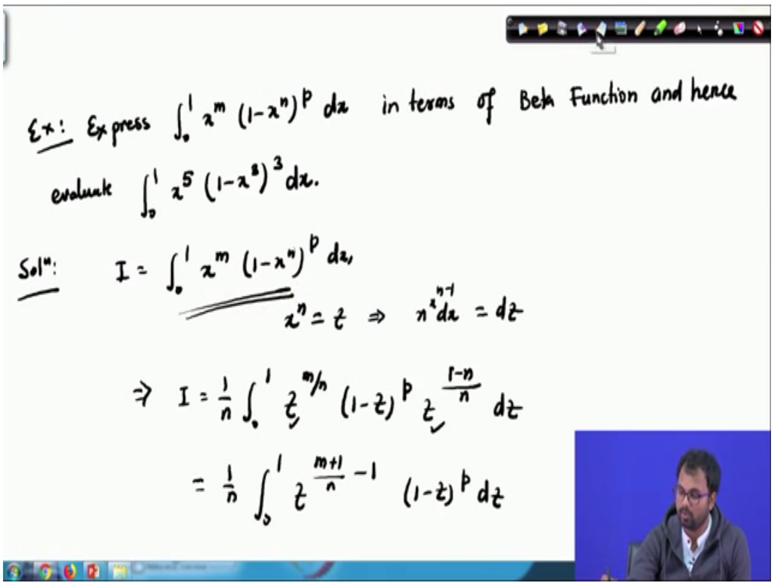

So, it says express integral from 0 to 1 x to the power m times 1 minus x to the power n whole to the power p dx in terms of beta function in terms of beta function and hence evaluate integral from 0 to 1 x to the power 5 times 1 minus x to the power 3 whole to the power 3 dx.

So, solution so, first of all I will give an integral let me write it as I equals to 0 to 1 x to the power m 1 minus x to the power n whole to the power p dx. So, if I substitute. So, first of all in order to have this in terms of beta function, this has to be x to the power m minus something and this has to be 1 minus x to the power p minus something or n minus something; so, we have to get rid of this x to the power and if we want to express it as a beta function.

So, I substitute x to the power n equals to z, then this will be n times dx x to the power n minus 1 equals to dz and when x is 0, z is 0 when x is 1, z is 1. So, this will remain as it is. So, the limits will remain unchanged; this will be 1 by n, z to the power m by n 1 minus z to the power p and this will be z to the power 1 minus n by n dz because, we have to find out the value of x to the power. So, this x to the power n minus 1 will come in the, and denominator and then in that case x would be z to the power 1 by n. So, it will be n minus 1 by n and then the whole thing will go in the numerator and then it will turn into something like this.

Now, this can be written as 1 by n integral from 0 to 1 I will put this and this together. So, this will be m plus 1 m plus 1 by n minus 1 and this will be 1 minus z to the power p dz, is it not? Now, this is this is our x to the power m minus 1 in the beta function formula and this is 1 minus x to the power n.

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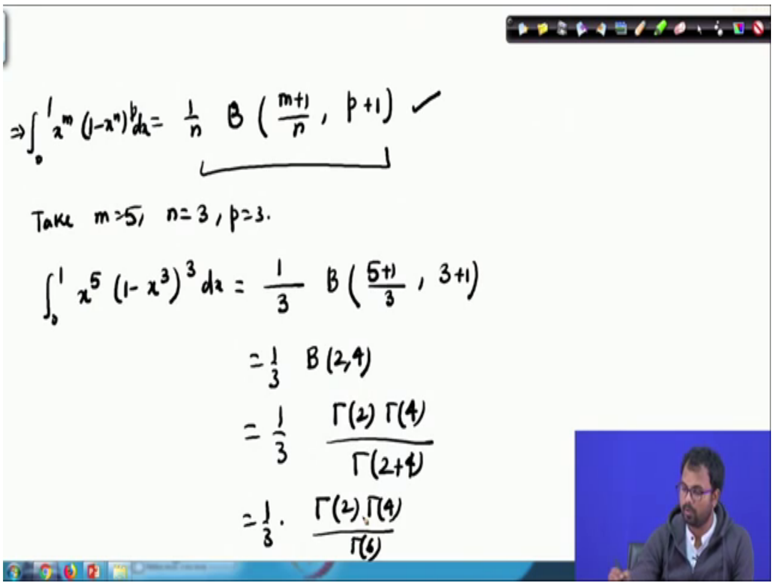

So, I can write this whole thing as 1 by n beta of or capital B of m plus 1 by n comma p plus 1; because in beta function this was supposed to be p minus 1, but since it is p; that means, I can be able to write it as p plus 1 minus 1 and hence you have here p plus 1, alright. So, that means, I was able to or we were able to explain express this integral here this integral here in terms of a beta function, alright. So, let me write it as integral from 0 to 1 x to the power m 1 minus x to the power n whole to the power p dx equals to this.

Now, we are supposed to calculate the value of this. So, in order to calculate the value of the given integral take m equals to 5 and n equals to 3 and p equals to 3. So, then we have integral from 0 to 1 x to the power 5 1 minus x to the power 3 whole to the power 3 dx equals to. So, our n is 3 and beta m is 5 plus 1 by 3 and p is 3. So, this is this. So, I can be able to write 1 by 3 beta 2 comma 4.

Now, I will use the beta and gamma relation here. So, then this will reduce to gamma 2 times gamma 4. So, I believe this was relation 1 and then we have gamma 2 plus 4. So, this will reduce to 1 by 3 times gamma 2 times gamma 4 divided by gamma 6.

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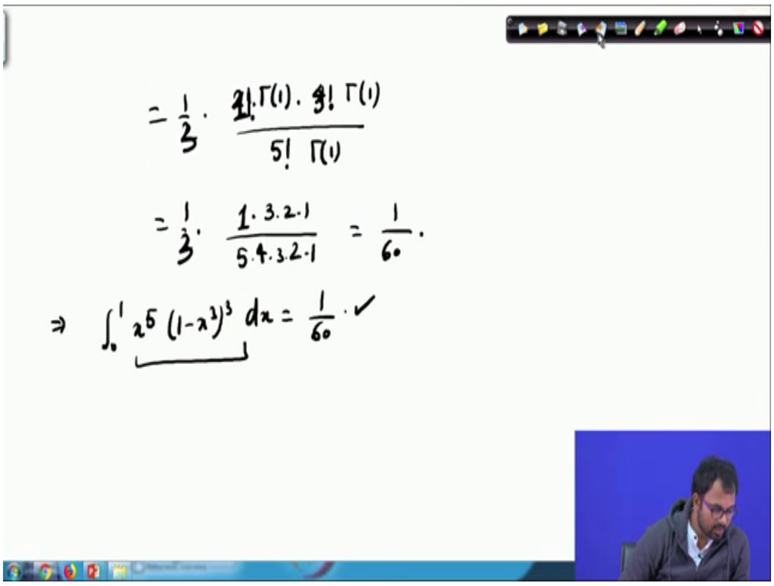

And, now I can be able to write this whole thing uses using that gamma factorial relation; so, that means, gamma 2 is 2 times gamma 1 and gamma 4 is 4 factorial 4. So, let us write it in terms of factorial. So, factorial 2 times 2 comma 1, factorial times factorial 4 times gamma 1 divided by factorial n minus 1. So, factorial 3 times gamma 1 and then I have here gamma 6 it will be factorial 5. So, it will be factorial 5 times gamma 1.

So, all the gamma 1's will get how to say will be 1. So, this will be 1 by 2, 2 times 3 times 2 times 1 and then this will be 5 times 4 times 3 times 2 times 1. So, this will get cancelled; 2, 2 cancels. So, the answer will be and then we also have here I believe 1 by 3, sorry. So, we also have 1 by 3 here. So, we will have 1 factorial times gamma 1 and this one will be 1 again so, 3 and then this, ok; so, 1 by 60. So, this is the value of that integral. So, integral from 0 to 1 x to the power 5 1 minus x to the power 3 whole to the power 3 3 dx equals to 1 by sixty and this is what we needed to show.

So, here we saw that if we are asked to evaluate a certain integral of this type we just have to use this formula here the 1 which we have derived, put the values of m and n and p to obtain the given problem and then we just have to calculate these gamma functions which does not even involve calculating an integral, we just have to remember some formulas. So, like gamma n is n minus 1 factorial. So, gamma 2 is 1 factorial gamma 4 is 3 factorial and gamma 6 is 5 5 factorial and with the help of which we can be able to calculate these gamma function this integral here.

So, similarly there can be some other problems as well and I believe you can be able to solve them I will also include those problems in our in our assignment sheet. So, that you can be able to practice more and probably I will solve one more example from gamma function in our next lecture and then we will close this topic and start with differentiation under the integral sign.

So, thank you for your time and I will see you in the next class.