Integral and Vector Calculus
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Lecture – 13
Introduction to Beta and Gamma Function

Hello, students. So, in the last class we looked into the concepts of Improper Integrals and how do you show that those improper integrals are convergent or not. So, we know that we call an integral as an improper integral when you have some kind of how to say problem with the lower limit or upper limit; for example, if any one of the limits are minus infinity or plus infinity or if you have some kind of infinite discontinuity in your integrand then those integrals are categorized as improper integral and.

Then, there are different-different methods with the help of which you can be able to show their convergence, then there are several tests and we also worked out few tests with the help of which we can be able to show the convergence of an improper integral.

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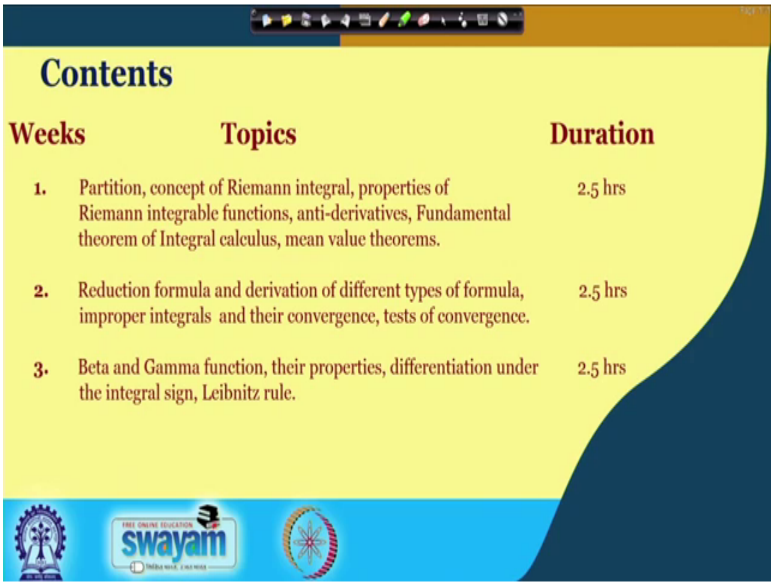

So, that was up until the last lecture. Today, we will jump into ah a new topic in our integral calculus section which is basically beta and gamma function.

So, will base will so, will basically look into this topic beta and gamma function here and.

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So, we will basically look into beta and gamma function today and we will look into their properties, the convergence and some other results related to beta and gamma function.

So, let us start with the definition of gamma function.

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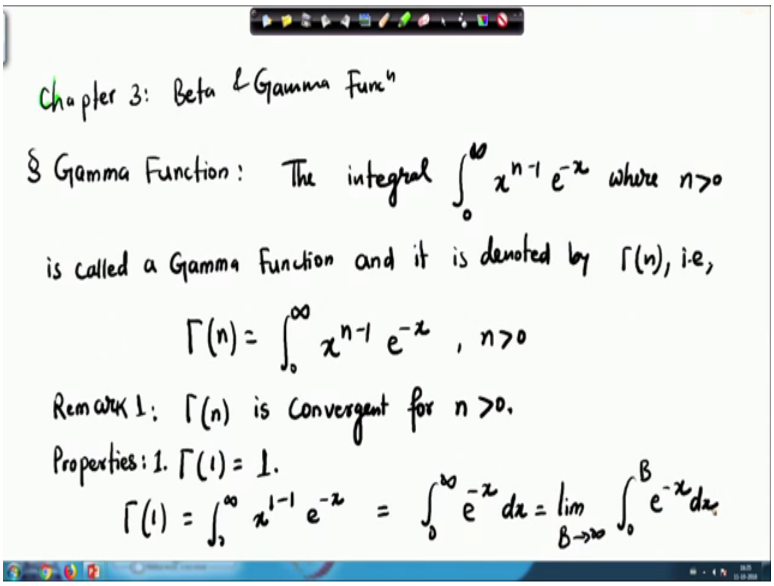

So, to start with this is our sorry chapter 3. Chapter 3 is Beta and Gamma function; beta and gamma function and we will basically start with gamma functions.

So, the integral, it is defined as the integral 0 to 1 sorry 0 to infinity x to the power n minus 1 e to the power minus x, where n is greater than 0 is called a gamma function and it is denoted by gamma. And, so, this is a Greek letter capital gamma and that is we will write gamma n equals to integral from 0 to infinity x to the power n minus 1, e to the power minus x, where n is a positive number real number actually.

And, in the class in the previous class we saw that this integral the gamma, gamma integral or gamma function is convergent for all n greater than 0. So, we can make a remark 1: gamma n is convergent for n greater than 0, we just saw in the previous class. So, this is first remark and there are several properties that we can associate with this gamma integral and some of which are.

So, let me write it as properties. So, the first property is gamma 1 equals to 1. This one is pretty straightforward to see. So, to prove this, so, let us say this is our property – 1. So, to prove this I can write gamma 1 equals to integral from 0 to infinity x to the power 1 minus 1 e to the power minus x. Now, this can be written as integral from 0 to infinity x to the power 0 which is 1 and therefore, we will end up with e to the power minus x.

Now, this is a classic example of improper integral. So, we can write it as integral limit B goes to infinity integral from 0 to B e to the power minus x dx.

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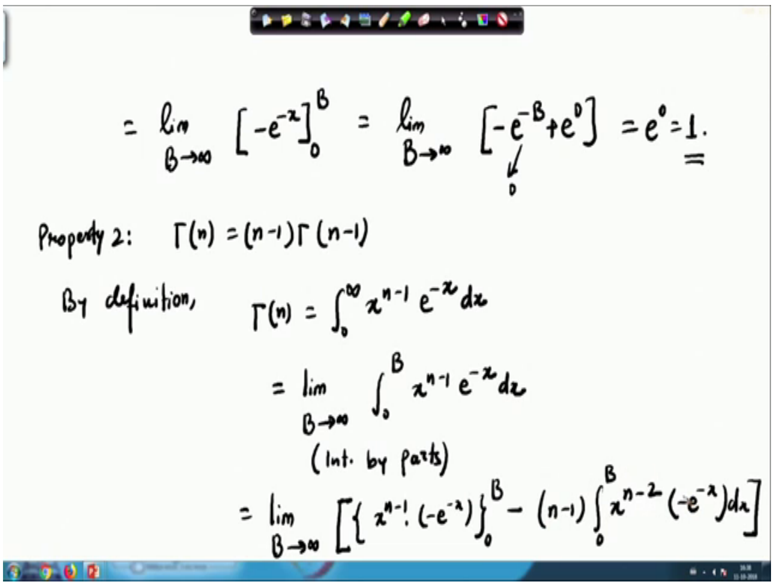

Now, after integration we will obtain limit B goes to infinity e to the power minus x would be minus e to the power minus x and we will have integral from 0 to B then this will be limit B goes to infinity we will have e to the power minus B minus minus plus e to the power 0. So, when B goes to infinity e to the power minus infinity will basically equal to 0. So, this term will go to 0 and since this term is a constant, it will remain unaffected by the limit and therefore, we will have e to the power 0, which is basically one and this is the first property that the value of gamma 1 will be 1. So, this property is verified.

Next, we will prove that second property is property 2 is gamma n equals to n minus 1 times gamma n minus 1. So, let us see how we can prove this. So, first of all by definition so, by definition by definition we have gamma n equals to integral from 0 to infinity, x to the power n minus 1 e to the power minus x dx.

So, this can be written as limit B goes to infinity integral from 0 to be x to the power n minus 1 e to the power minus x dx and now, we will integrate this by parts. So, if I integrate this by parts let me write in a small bracket integration by parts integration by parts. So, if I integrate this by parts then what will happen we will have limit B goes to infinity x to the power n minus 1 times minus e to the power minus x limit from 0 to B minus n minus 1 times x to the power n minus 2 and then minus e to the power minus x dx.

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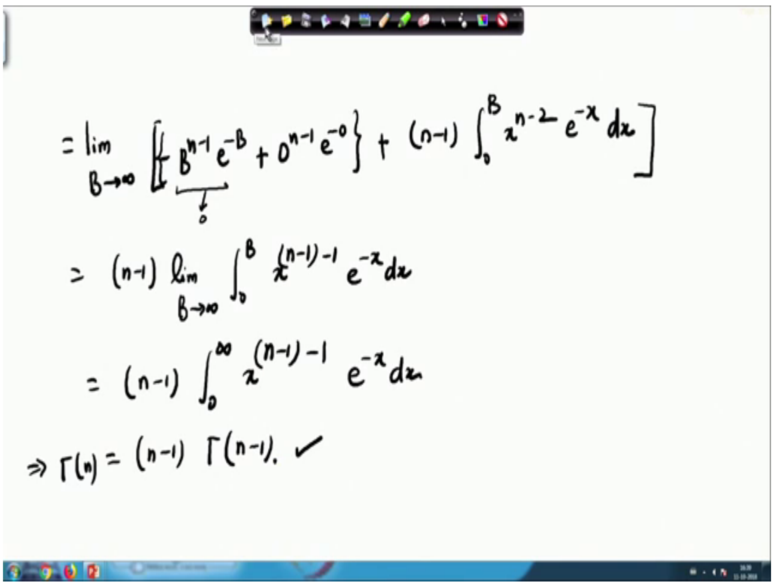

So, here we will have limit B goes to infinity we will obtain B to the power minus B to the power n minus 1 e to the power minus B plus 0 to the power n minus 1, e to the power minus 0 and then this one is minus n minus 1 integral from 0 to B x to the power n minus 2 and then this minus will come here. So, this will be e to the power, so, plus e to the power minus x. So, this will be plus e to the power minus x dx.

So, now, here so, so, we have bracket here and this one can be put in a curly bracket. So, now, when B goes to infinity this whole term will go to 0, because this will go to infinity and this one will go to 0. So, ultimately the product will go to 0 and this is anyway 0. So, we are left with n minus 1 times limit B goes to infinity integral from 0 to B, x to the power n minus 1 minus 1 e to the power minus x dx. So, in short we can be able to write n minus 1 times integral from 0 to infinity x to the power n minus 1 minus 1 e to the power minus x dx.

Now, this is nothing, but this definition here. So, if I replace n by n minus 1, then this will become the definition for gamma n minus 1, right. So, let us replace gamma n by n minus 1, then in that case this one will be the definition of gamma n minus 1 which is basically our gamma n and therefore, this is what we needed to prove, alright. So, the second property is established.

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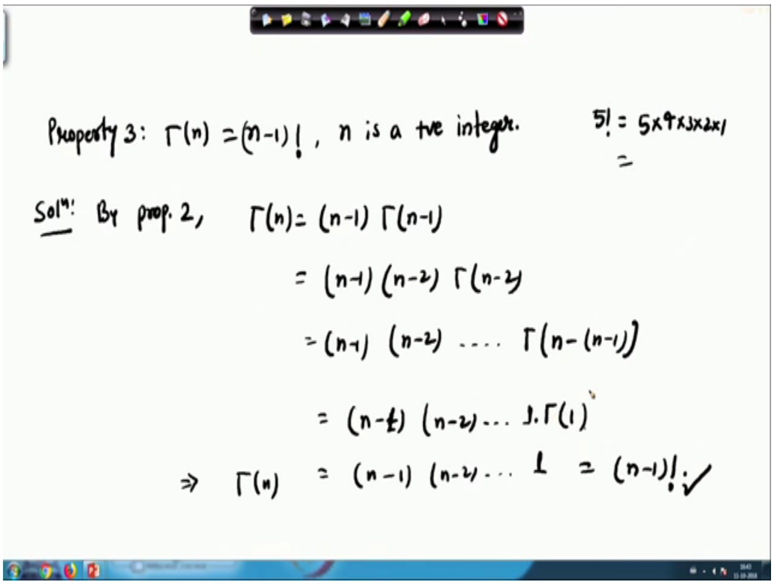

Now, we have the third property. So, property 3 which is basically gamma n equals to n factorial n minus say n minus 1 factorial, where n is positive integer. So, you know what a factorial means for example, factorial of 5 is 5 times 4 times 3 times 2 times 1. So, that is what we mean by factorial. So, you multiply the number and then it is previous number one by one and that will give you the factorial of that particular number.

So, here in this case we say that gamma n equals to n minus 1 factorial. So, this is very straightforward to see. So, by property – 2, by property 2 we have gamma n equals to n minus 1 times gamma n minus 1. Now, I use this ah how to say iterative formula to write gamma n minus one as n minus 2 times gamma n minus 2, right.

So, this formula for the formula for gamma n minus one can be replaced with a formula of this type. So, that is that is what I am doing I can write n as n minus 1 this n minus 1 as n minus 2 times gamma n minus 2. Similarly, I will proceed and I can be able to write gamma n gamma n minus n minus 1, right and this one will become gamma 1 and the value of gamma 1 will be finally,. So, the value of gamma 1 would be 1 times gamma 1 and value of gamma 1. So, here is n minus 1; so, n minus 1 n minus 2 dot dot and so on, value of gamma 1 is again 1. So, I can write it as 1.

So, you see we have basically n minus 1 times n minus 2 times n minus 3 up to 1 and therefore, this can be written as n minus 1, factorial and this is what we needed to prove that gamma n equals 2 n minus 1 factorial, right. So, we can see that this gamma n follows this kind of formula in minus 1 factorial.

So, after gamma function after gamma function we will look into the definition of beta function and we will also try to analyze its properties that how do we define also whether we can prove it is convergence or not things like that.

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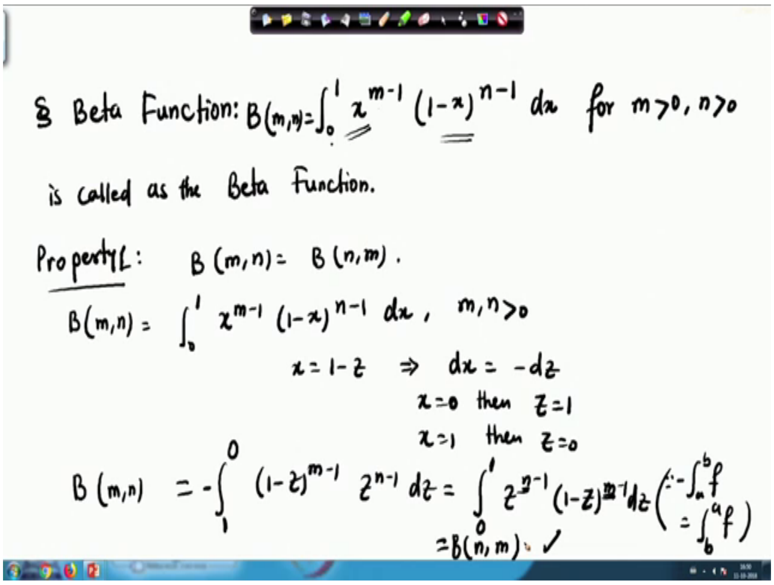

So, let us start our next topic beta function beta function.

So, an integral of type 0 to 1 x to the power m minus 1 times 1 minus x to the power n minus 1 dx for m positive and n positive is called as the beta function is called as the beta function and it is denoted by B of m commaa n equal equals to 0 to 1 x to the power m minus 1 times 1 minus x to the power n minus 1. Some people also write it as beta instead of writing B, they use the symbol beta because it is called as the beta function, but we will write B for this for this integral.

And, here we can see that this is an improper integral because for when n is less than 1; so, when n is less than 1, then in that case this one will become 1 minus x to the power something negative and if one if we have 1 minus x to the power something negative then that will come in the denominator so, that it will become 1 minus x to the power something positive and at x equals to 1, we will have a point of discontinuity.

So, at x equals to 1 when n is less than 1 we will have a point of infinite discontinuity similarly at x equals to 0 when x is when m is less than 1 then we have a point of infinite discontinuity. So, for both m and n less than 1 at the point x equals to 0 and x equals to 1 we have the point of infinite discontinuity. So, beta m, n is an improper integral.

Now, in order to show it is convergence we will follow the steps what we did in case of gamma function and we can be able to show that this beta function is convergent. So, we split the integral 0 to 1, f into 0 to half and then half to 1. So, that we have only one point of discontinuity in each one of these sub-integrals and then we show the convergence at 0 and convergence at 1. And, of and we can be able to see that both of these two sub-integrals are convergent and therefore, the whole beta function will be convergent. So, I leave that task up to the students and you can look into any book for the proof itself it is follows the similar same steps like we did for the gamma function ah.

Today, we are going to analyze some properties of beta function. So, let us do that. So, first property. So, property – 1 which is beta m, n equals to beta n, m. So, that means, you can switch the indices and it will still remain the same, let us see. So, the proof or the solution would go like this. So, beta m, n equals to integral from 0 to 1, x to the power m minus 1 1 minus x to the power n minus 1 dx where our m and n are both positive numbers.

Now, if I substitute x equals to let us say 1 minus z then in that case we will have dx equals to minus of dz and when x is 0 when x is 0, then z is 1 and when x is 1 then z is 0. Therefore, our beta m, n would reduce to integral from 1 to 0 our x would become 1 minus z to the power m minus 1 and 1 minus x would become z to the power n minus 1 and for dx I have a dz.

Now, since we know that since we know a small property of integral calculus that if you have minus of integral from a to b f x dx then this will be equals to integral from b to a f x dx so, that means, this minus will get the absorbed basically when you are changing the changing the range of integration in a way. So, here we had a to b if we switch the limit points from a to b to b to a and in that case this minus will get reabsorbed.

So, will do the same thing here and then this will turn into integral from 0 to 1. z to the power n minus 1 1 minus z to the power m minus 1 dz and if you look at the definition of beta function then instead of z we have instead of x we have z and instead of m we have n, instead of n we have m. So, z is basically a variable. So, it does not matter whether we are using x or z it does not matter. However, you can see that instead of m we have n, and instead of n we have m. So, that means, this is nothing, but the definition of beta n, m because the indices are switched.

So, here what we have is beta n m actually and this shows that your beta m n is equals to beta n, m for all m and n positive. So, the first property is taken care of.

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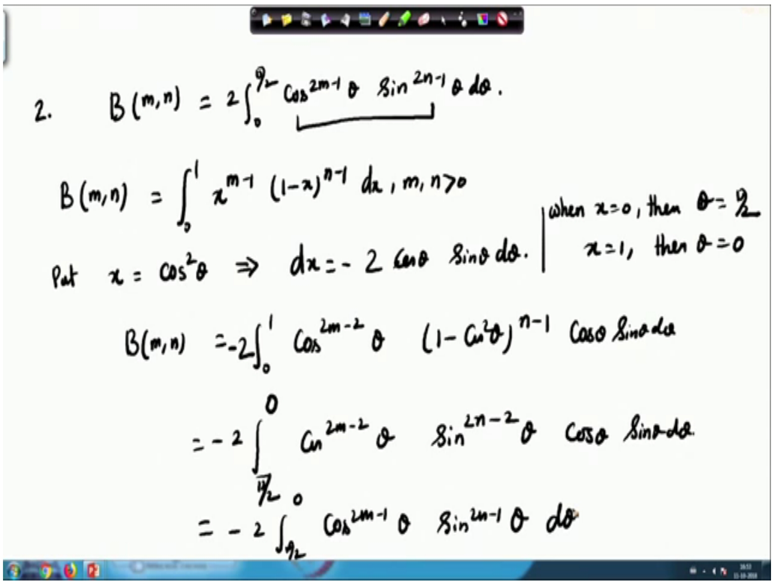

Now, we will look into the property – 2. So, the second property says that beta m, n equals to 2 times integral from 0 to pi by 2 cos of 2m minus 1 theta times sin of 2n minus 1 theta d theta. So, that means, instead of having an algebraic expression we now, have a trigonometrical expression in the integrand and from looking at this trigonometrical expression it is very straightforward that we have to do some kind of substitution for x in our in our formula.

So, let us try to prove that. So, we know that beta m, n equals to integral from 0 to 1 x to the power m minus 1 times 1 minus x to the power n minus 1, right. x to the power m minus 1 times 1 minus x to the power n minus 1 where m and n are both positive. So, now substitute or we put x equals to cos square theta then from here dx would be minus 2 cos theta sin theta d theta. And, therefore, our beta m n will become integral from 0 to 1 cos of 2m minus 2 theta times 1 minus cos square theta whole to the power n minus 1 and then dx is minus of 2 cos theta sin theta d theta.

So, this will be minus 2 integral and sorry. So, we all have to take care of these limits. So, let us let us see what is what would be the limit. So, when x is 0; so, here so, when x is 0 then theta is pi by 2 and when x is 1 then theta is 0. So, this can be written as integral from pi by 2 to 0 cos of 2m minus 2 theta and then this one will be sin square theta. So, this will term into turn into sin to the power 2n minus 2 theta cos theta sin theta d theta.

Now, we can bring one cos theta here and then 2m minus 2 plus 1 will become 2m minus 1. So, this one will become minus of 2 integral from pi by 2 to 0 cos 2m minus 1 theta times sin 2n minus 1 theta d theta and now, we will use the formula or the property which we just mentioned here that if you have minus of a to b f x dx, then it can be written as b to a f x f x dx.

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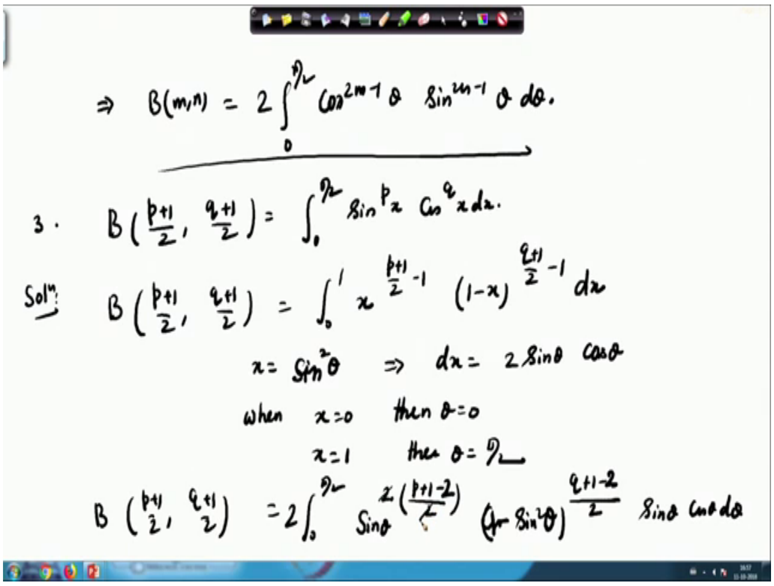

So, I can use that property and write the integral beta m n as 2 times integral from 0 to pi by 2 cos of 2 m minus 1 theta times sin of 2n minus 1 theta d theta and this is what we needed to prove, right. So, these formula say our algebraic expression can be reduced into a trigonometrical expression for this beta function.

Now, we can also be able to write we can also be able to write next property. So, property – 3: beta of p plus 1 by 2 times q plus 1 by 2 equals 2 integral from 0 to pi by 2 sin to the power p x and cos to the power q x dx. We will need these properties to prove some example to solve some examples. So, let us see how we can prove it. So, we know our beta is p plus 1 by 2 times q plus 1 by 2 equals to integral from 0 to 1 x to the power m minus 1. So, m minus 1 is p plus 1 by 2 minus 1 and 1 minus x is a whole to the power n minus 1. So, this 1 is q plus 1 by 2 minus 1 d of x.

And, now, I substitute so, let us substitute x equals to. So, we will substitute x equals to cos square or x equals to. So, we have sin to the power p x. So, we substitute x equals to let say sin square theta. So, we substitute x equals to sin square theta, then this will reduce to dx equals to 2 sin theta cos theta. And, when x is 0, we will have then theta is 0 and when x is 1 then theta is pi by 2.

So, our beta p plus 1 by 2 comma q plus 1 by 2 will reduce to integral from 0 to pi by 2 we will have x to the power. So, that means, sin theta to the power sin square theta times p plus 1 minus 1 by 2 and we will have cos to the cos square. So, cos 1 minus x what I am saying is we will ultimately get up cos here. So, q plus 1 minus 2 by 2. So, here 2 by 2 and dx is 2 sin theta cos theta.

So, like previous example from here we will obtain basically cos square theta. So, one so, cos square theta so, square and 2 this will get cancelled and like the 2 and 2 here gets cancelled.

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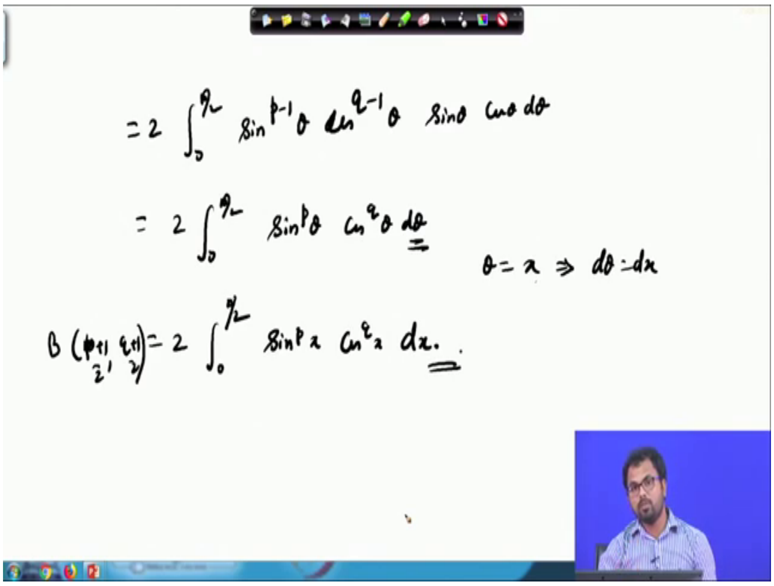

So, we will basically end up with sin to the power so, we will basically end up with 2 times integral from 0 to pi sin to the power p minus 1 theta times cos to the power q minus 1 theta then sin theta cos theta d theta. So, one sin will come here. So, it will turn into sin p theta 1 cos will come here and it will turn into cos q theta. So, let us write them. So, we will have sin p theta cos q theta d theta.

Since theta is just a variable if I substitute theta equals to z or theta equals to x it does not matter what I substitute. So, if I substitute theta equals to z or x whatever I please the value of x will be the value of theta. So, that means, when theta is 0, x is 0 when pi with when theta is pi by 2, x is pi by 2 and from here d theta equals to dx. So, everything will remain same and we can do that mainly because theta is just a variable here. So, let us change the variables and then this will reduce to 0 to 2 pi sin to the power p x cos to the power q x d theta will be dx and this is what we needed to prove. So, beta p plus 1 by 2 q plus 1 by 2 equals to 2 times integral from 0 to pi by 2 sin pi x times cos q x.

So, this was one another property of beta function and in the next class we will actually work out the examples related to beta and gamma function where we will use these properties to evaluate the value of gamma half or gamma 3 by 4 or the product of gamma 3 by 4 times gamma 5 by 8 so, examples like that. So, we can be able to evaluate the value of how do say gamma of some fraction or beta of some fraction. So, we will do such things in our next class and.

Thank you for your time.