Integral and Vector Calculus
Prof. Hari Shankar Mahato
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 12
Improper Integral (Contd.)

Hello students. So, up until last class we looked into comparison test and examples involving comparison tests. There is or there are a couple of more tests in the in the context of Improper Integral where we use them to talk about the convergence of those improper integral. So, I believe we have covered sufficient examples on comparison tests and you will be able to work out few more examples in your [shine/assignment] assignment. And today we will start with mu test.

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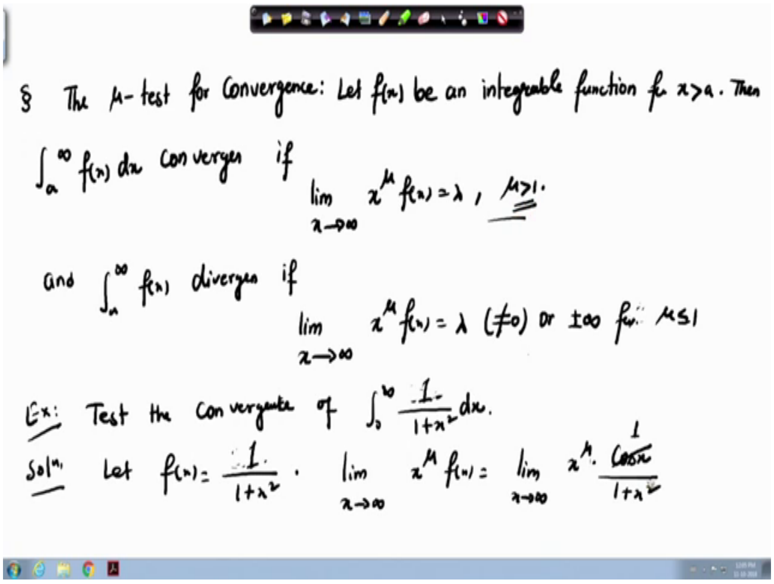

So, test of convergence using mu test. So, the mu test for convergence. So, the statement goes like this let f x be an integrable function for x greater than a. Then integral from a to infinity f x d x converges if limit x goes to infinity x to the power mu f x equals to lambda for mu greater than 1 and integral from a to infinity f x d x diverges. If limit x goes to infinity x to the power mu f x equals to lambda which is non zero or plus minus infinity if mu is all for mu is less than equal to 1.

So, for mu so, if mu is greater than 1 then we can say that for this limit then we can say that the given integral improper integral is convergent if it is not if mu is less or equal to

1, but we are still getting the finite value then in that case the given integral is divergent. So, let us see a few examples where we can talk about the convergence of the given improper integral.

So, example 1 test so, let us let us consider the similar example to test the convergence test the convergence of 0 to infinity let us say we have cos x we have cos x by 1 plus x square d x. So, then in that case we can write it so, the solution. So, let f x equals to cos x by 1 plus x square and if I take limit x goes to infinity x to the power mu f x then this will be limit x goes to infinity x to the power mu times cos x by 1 plus x square. And if I take mu as if I take mu as greater than 2 or how to say at least mu equals to 2, then in that case this whole thing will be equal to 1.

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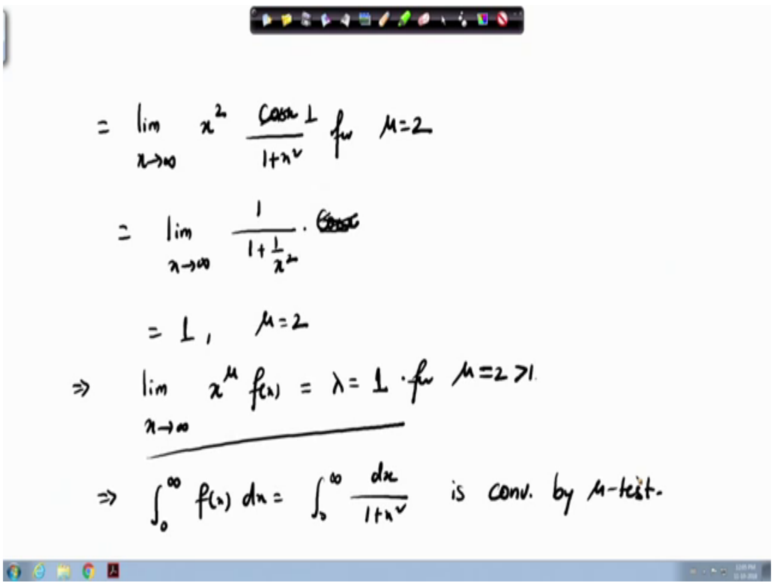

So, if I take mu greater mu equals to 2. So, let us say if I take mu equals to 2 x cos x by 1 plus x square for mu equals to 2. Then in that case I can be able to write this as 1 by. So, this is basically 1 by 1 plus x square times cos x and this whole thing x squared times f x and this whole thing would converse to or for the time being.

If I assume that instead of cos x if I let us say just start with just to make the example a little bit simpler because I do not want to complicate it. So, let us let us start with just 1 instead of cos x if I take just 1 then what would happen. Then if I take 1 here let us say instead of cos x I have 1 then it will be a little bit straightforward. So, instead of cos x lets say I have 1. So, then in that case this will be 1 by 1 plus x squared and then the whole limit will be 1 for mu equals to 2.

So, the message which I want to convey is limit x goes to infinity x to the power mu f x is equals to a lambda. A lambda which is basically our 1 for mu is equals to 2 which is greater than 1. So, having this limit here which is of course, finite for mu is equal to 2 which is of course, greater than 1, we can be able to apply this mu test.

So, I just wanted to show you with a simple example. So, having cos x would little complicate a little bit. So, let us not go there first of all. So, if I have 1 by 1 plus x square we have already worked out this example several times, but I would like to show the application of mu test here. So, let us say you have the integral improper integral of this time. So, we multiply by x to the power mu to the function f x and then I am evaluating this limit.

So, if I choose x equals to a mu equals to 2 which is greater than 1 then in that case the integral which is in this case. So, then in that case the integral 0 to infinity f x d x which was integral from 0 to infinity d x by 1 plus x square is convergent by mu test. So, this is one such example of mu test for this simple problem. Next we can see an another example, let us let us see that.

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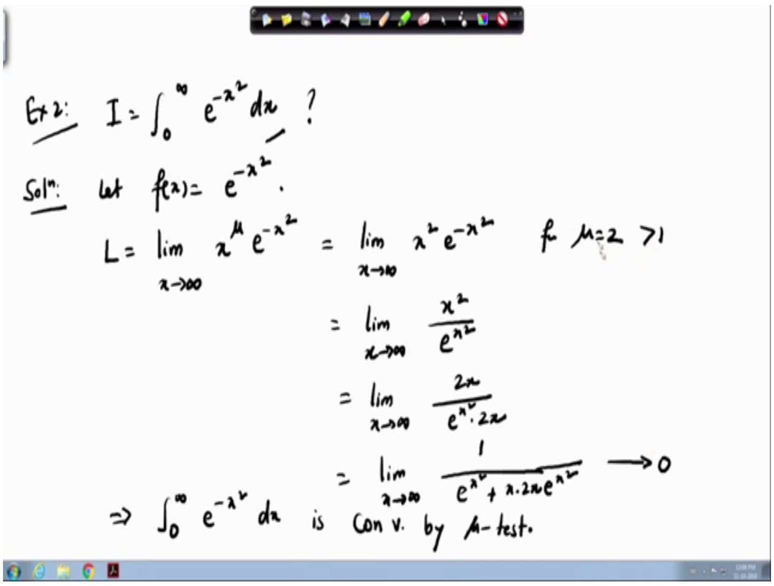

So, now we can go to a bit complicated examples.

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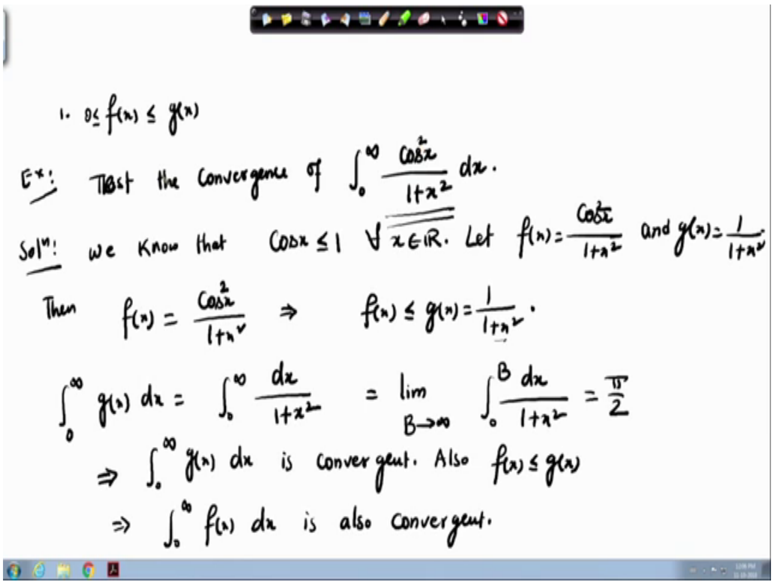

So, let us consider example and also I would like to point out that in previous class, we considered an example of this type cos x by 1 plus x square. It is better to put a square here. So, if I put a cos square x here cos square x here. So, we can test the convergence of this problem so, not the cos x, but cos square x so, that it is rather more how to say then in contrast with the comparison test of inequality type. So, the original problem should be stated as 0 to infinity cos square x by 1 plus x squared d x. Then in that case it will be in contrast with the with the comparison test of inequality type.

Now, that was just one remark or comment on my previous lecture. Next example is we test the convergence so, I am not writing the whole statement. So, we test the convergence of 0 to infinity e to the power minus x square d x. So, we want to test the convergence of this integral here. So, let us define our function f x as e to the power minus x square and next we want to evaluate let us say the limit L is equals to limit x goes to infinity.

So, here in this case we have a problem at x equals to infinity because, the upper limit is infinity and I write x to the power mu times e to the power minus x square. Now, for the time being if I take x square instead of mu if I take 2 then this is x square e to the power minus x square for mu equals to 2. Then this can be written as limit x goes to infinity x square by e to the power x square and next this one is basically infinity by infinity form if you remember from L'Hopital rule. So, this is basically infinity by infinity form.

So, I can differentiate both numerator and denominator and this will be written as 2 x divided by e to the power x square, then differentiation of x square would be 2 x. And then I can differentiate the whole thing again and this will be limit x goes to infinity 1 by e to the power x square plus x times 2 x times e to the power x square.

So, when x goes to infinity this whole thing will be 1 by infinity. So, this whole thing will go to 0 so; that means, the limit or the lambda of this limit is it is a finite number and it is going to 0 for mu equals to 2 which is; obviously, greater than 1. And if it is greater than 1 then from the come mu test for the convergence we can say that the integral is convergent. So, from here we can say that integral from 0 to infinity e to the power minus x square d x is convergent by mu test. So, you see it is always about guessing the correct value for mu.

So, for what value of mu you can be able to show that the limit is finite and if it is finite then we have to see for what values of mu is that happening. So, if it is happening for mu greater than 1, then only the integral is convergent like in this case and if it is happening for me less or equal to 1 then in that case the integral is divergent. So, choosing the correct value of mu plays an important role here.\

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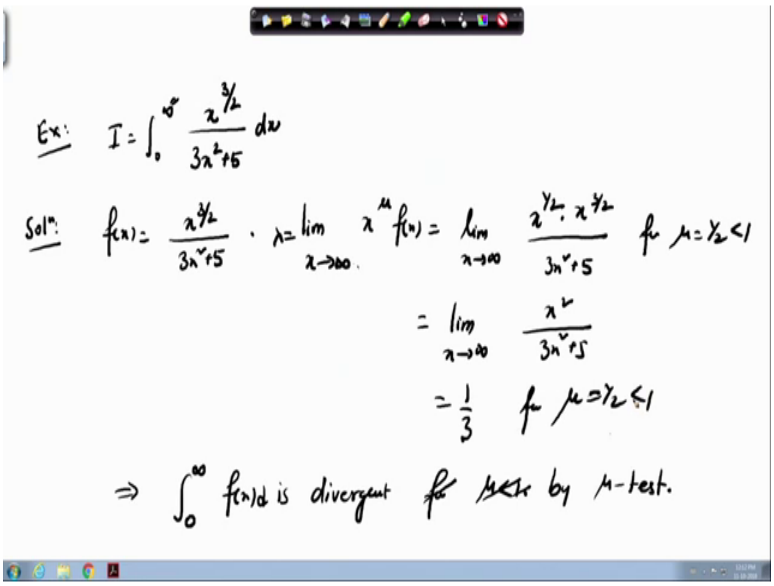

We can so, there is another example where we have the divergence part. So, let us consider the integral from 0 to infinity x to the power 3 by 2 divided by 3 x square plus 5 d x. So, obviously, here we choose f x as our x to the power 3 by 2 3 x square plus 5 and since one of the limit is infinity that is why it is an improper integral. So, let us find the limit or lambda from our mu convergence test. So, lambda is basically x to the power a limit x goes to infinity, there should not be any plus here and x to the power mu f x d x sorry f x.

So, then this is basically limit x goes to infinity. If I choose mu equals to half, then this will become x to the power half three times x to the power 3 by 2 divided by 3 x square plus 5 for mu equals to half which is; obviously, less than 1. Then this will reduce to limit x goes to infinity x square by 3 x square plus 5 and this can be written as 1 by 1 by 3 plus 5 by x square.

So, ultimately this value will be 1 by 3 for mu equals to half which is less than 1. Now, this falls into this category that if mu is less or equal to 1 and if the value is non 0 then in that case the integral is divergent. So, from here we can write that ; so, here we can write that integral from o to infinity f x d x is divergent f x d x is divergent for mu less than 1 or by mutest for mu less than 1 or we can write by mu test.

So, that is the way we show the divergence as I was saying having this value of mu is a very vital in order to ensure the convergence or divergence of the given improper integral. We will work out one last example of an integral which is very vital in integral calculus.

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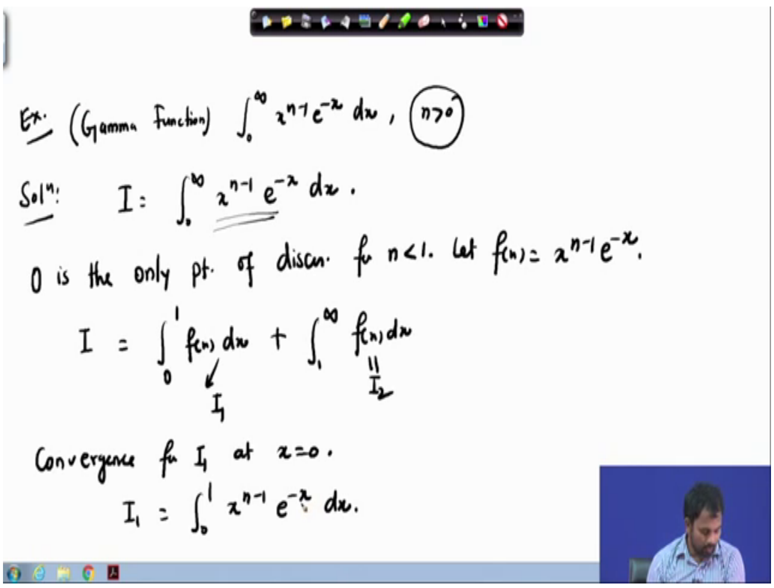

So, the example goes like this we will of course, look into a gamma function and beta function in the next class. But we see that the integral 0 to infinity x to the power n minus one e to the power minus x d x which is defined as gamma function is also an improper integral. And here I am just writing n is positive I am not writing the range for which n is this improper integral is convergent or not we will test its convergence now.

So, for the time being let us take n is greater than 0 and then we will say for what values of n this improper integral is convergent. So, let us test its convergence. Now, let me write I equals to integral from 0 to infinity x to the power n minus 1 e to the power minus x d x. So, first of all if n is greater than 1 if n is greater than 1, then this integral this integrand is defined.

However, this is still an improper integral because you have one of the limits we have one of the limits or the upper limit as infinity. Now, when n is less than 1 then in that case this whole thing will start creating problem because then in that case 0 will be the point of infinite discontinuity. Since it will become x to the power minus something and then that will come at the denominator and then when x goes to or at x equals to 0 that denominator at that particular part will be undefined.

So, at x equals to 0 this function or this integrand has an infinite discontinuity. So, we can write that 0 is the only point of discontinuity for and less than 1 right next let us take f x equals to x to the power n minus 1 times e to the power minus x. So, then I can write my integral I as the sum of two integrals 0 to 1 x to the power n minus 1 e to the power minus x which is the given function f x then 1 to infinity f x d x.

So, this one is our integral I 2 this one is our integral I 1. So, for integral I 1 we have infinite discontinuity at x equals to 0 for n is less than 1. So, let us proceed so, convergence for I 1 at x equals to 0. So, I 1 is integral from 0 to 1 x to the power n minus 1 e to the power minus x d x.

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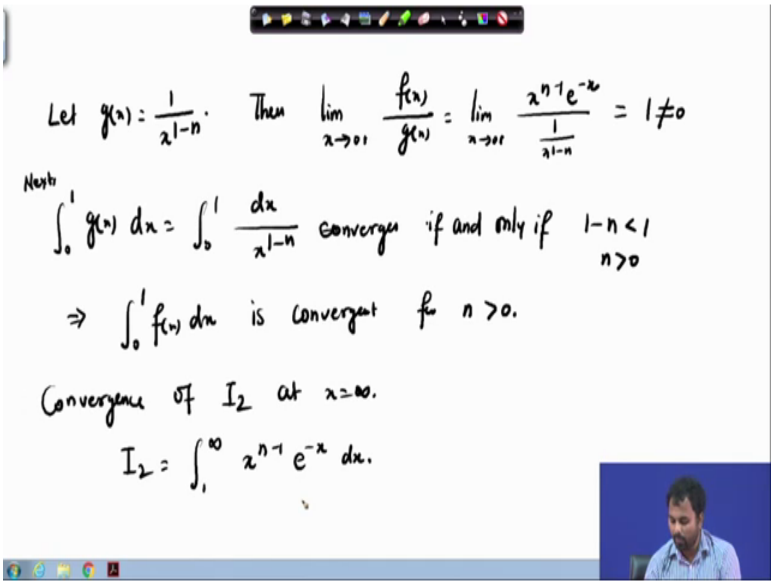

Now, in this case I choose let g x as 1 by x to the power 1 minus n. Then what will be my limit of the quotient of these two functions. So, it is 0 where we have then infinite discontinuity I can write f x by g x and this will be limit x going to 0 positive f x is x to the power n minus 1 e to the power minus x times 1 by x to the power 1 minus n. So, this will be basically 1 so, e to the power minus 0. So, this will this will lead to 1 which is; obviously, non zero and what is what is this limit here g x I can write next.

So, next this limit here integral from 0 to 1 d x by x to the power 1 minus n. So, this integral this integral is having infinite point of a this integral has an in infinite point of discontinuity has a point of infinite discontinuity at x equals to 0. So, we can write it as limit epsilon goes to 0 positive and then 0 epsilon to 1 d x by x to the power 1 minus n the things which we are doing and in the previous classes.

So, it will be fairly easy to check its convergence and I am skipping that step and I am writing it for you that this integral converges if and only if. So, if and only if 1 minus n is less than 1 so; that means, n is greater than 0 we just worked out an example like that. So, this integral converges only when n is positive so; that means, from here we can say that. So, we have a limit test we have a limit which says that the limit is non 0 and we were able to show that this integral converges only when n is positive; that means, the integral from 0 to 1 f x d x is convergent is convergent for n greater than 0.

So, even though at x equals to 0 we had a point of infinite discontinuity for n less than 1 we can be able to show that the integral is convergent the first sub integral this sub integral is convergent for all n greater than 0. So, the first part is settled now let us go to the convergence of I two at x equals to infinity. Because in case of I 2 in case of I 2 only at x equals to infinity is the is the is the improperness in the limit the function is always defined because even though if n is less than 1 this function will never have a point of infinite discontinuity in that interval one to infinity. So, our I 2 is 1 to infinity x to the power n minus 1 e to the power minus x d x.

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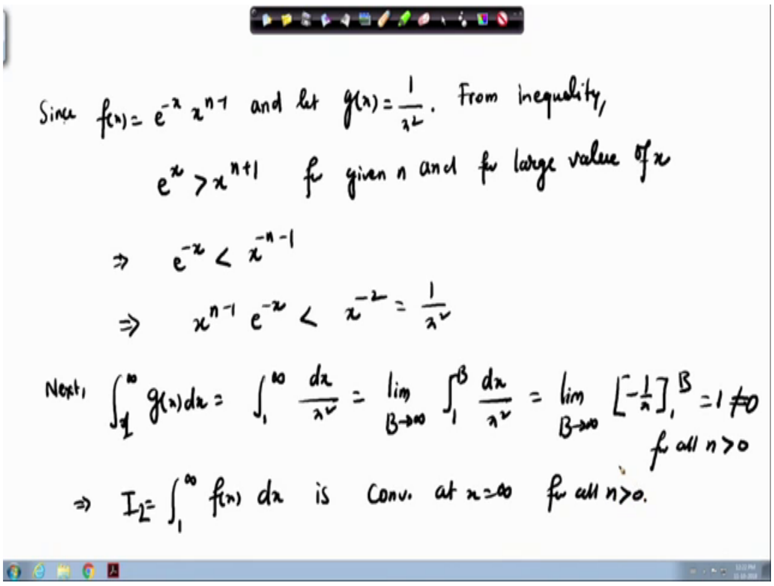

And from here we know that let us define let f x equals to e to the power minus x to the power n minus 1 and g x equals 2. So, we do not have to define f x is actually this one. So, I can modify this a little bit since f x equals to this and let g x equals to 1 by x square.

Now, from inequality so, from inequality or from our basic knowledge of inequalities I can be able to write e to the power x is greater than x to the power n plus 1 for given n and for large value of x. Now this can be written as e to the power minus x is less than x to the power minus n minus 1.

Now, if I multiply both sides by x to the power n minus 1 then in that case this will be since x is running from 0 to infinity this inequality will remain unchanged. And therefore, this will be x to the power n minus 1 is times e to the power minus x is less than x to the power minus 2 which is 1 by x square. So, this is my g x this is my f x and;

obviously, this is a positive quantity so, we have the first part of our comparison test of inequality type.

Now, all we have to do is to check the value of this integral. So, next we have to check the values of all the how to say convergence of this integral. So, what is the value of this integral d x by x square. So, I can be able to write limit B goes to infinity integral from 1 to B d x by x square and then this can be written as limit B goes to infinity minus 1 by x 1 to B.

And then we substitute the limit and when B goes to infinity the first one will go to 0 therefore, the value of the integral would be 1 value of the integral would be 1 which is non zero and this is true for all n so, true for all n positive. So; that means, I equals to so; that means, I 2 equals to integral from 1 to infinity f x d x is convergent at x equals to infinity for all n greater than 0.

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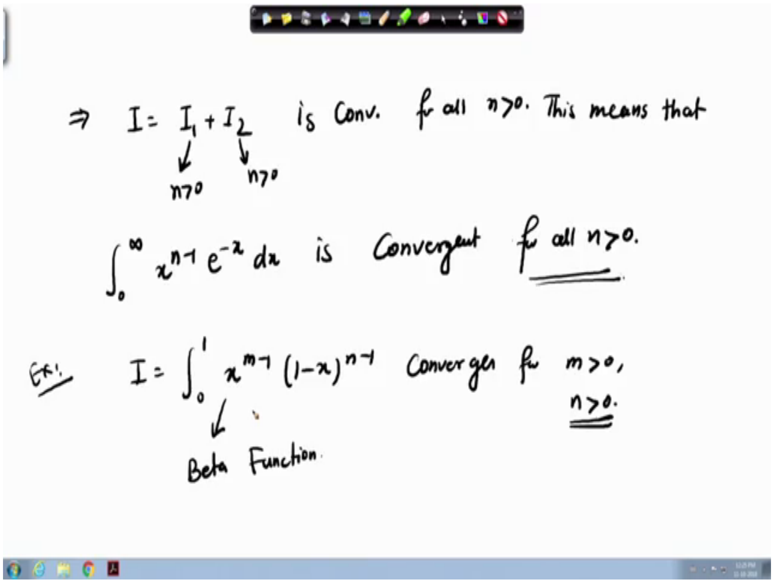

And hence from here, we can say that I equals to I 1 plus I 2 is convergent for all n greater than 0. Because I 1 is convergent for all n greater than 0 and I 2 is convergent for all n greater than 0 therefore, their sum is convergent for all n greater than 0. That means, the given gamma function given gamma function this means that the given gamma function which was 0 to infinity x to the power n minus 1 e to the power minus x d x is convergent at for sorry for all n greater than 0. And this is a very vital result in integral calculus and of course, it is also our next topic in agenda.

Similarly we have an integral of type this let us say I equals to integral from 0 to 0 to 1 x to the power m minus 1 times 1 minus x to the power n minus 1 and we have and here we can see that 0 and 1 can be the points of discontinuities for certain values of m and n. And for this integral we can also talk about such convergence and the range for m and n for which this integral is convergent.

And we will see this integral in our next chapter because, this is also in our syllabus and this integral is called as beta function beta function. So, this integral converges for m greater than 0 and n greater than 0. And this integral is also a type of improper integral and because it creates some kind of infinite discontinuity at the, for this integral at the point 0 and 1.

And we can put it in the similar fashion by dividing the integral into two sub integrals and then we look into the values of m and n for which we can talk about the convergence of this improper integral. So, will probably do that in our next class and I hope I was able to give you sufficient examples and theories on improper integral will work out for examples in your assignment sheet as well. So, thank you for today and I look forward to your next class.