Integral and Vector Calculus
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Lecture - 11
Improper Integral (Contd.)

Hello students. So, upon till last class we looked into different types of Improper Integrals, where you have the either the upper limit or the lower limit as infinity or you have some kind of infinity discontinuity in the function itself. This all those types of integrals are categorized as improper integral. We also worked out few examples and we also looked into some test or we defined some test which will assure the convergence of an improper integral.

Before working out examples on those test I will give you one or two more examples about improper integral where, the convergence depends on certain type of factor involving in the integral itself. So, let us see what do I mean by those examples.

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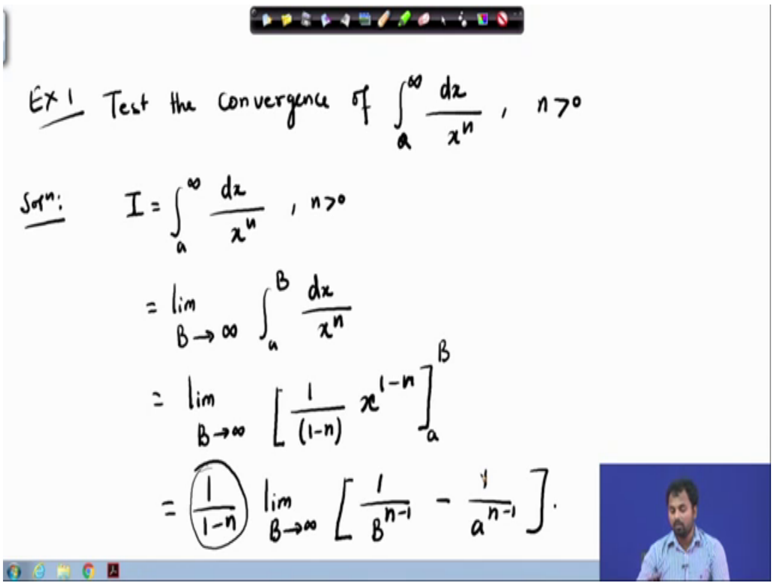

So, to start with example test the convergence; test the convergence of integral of type 0 to infinity dx by x to the power n, where n is any let us say here we have a to infinity where n is any positive number. So now, in this case what we have is one of the limits as infinity. So, I will write this integral as I integral from a to infinity dx by x to the power n where n is a positive number. So, for we do not know for what value of n this integral would converge or diverge. So, at the moment we have to just test the convergence and see for what values we can get the convergence.

So now, if I write it in our traditional improper integral forms. So, I will write as B goes to infinity integral from a to B dx by x to the power n, now evaluating this integral is fairly easy. So, we know the value of the integral would be 1 by 1 minus n times x to the power 1 minus n x to the power 1 minus n and then I will close this times. So, this the integral would run from so, arrange would run from a to B. And, now if I substitutes I will take 1 minus n outside and if I substituted the values then this will reduce to limit B goes to infinity 1 by B to the power n minus 1 minus 1 by a to the power n minus 1, where n is any positive number.

Now, we can see here that this integral so, first of all it does not matter whether B goes to infinity or not, if n is 1 then in that case this here would be undefined. So, if n is 1 then this here would be undefined. So, our very first criteria is that n cannot be less n cannot be 1; now let us check whether so, the first observation; first observation is n cannot be 1.

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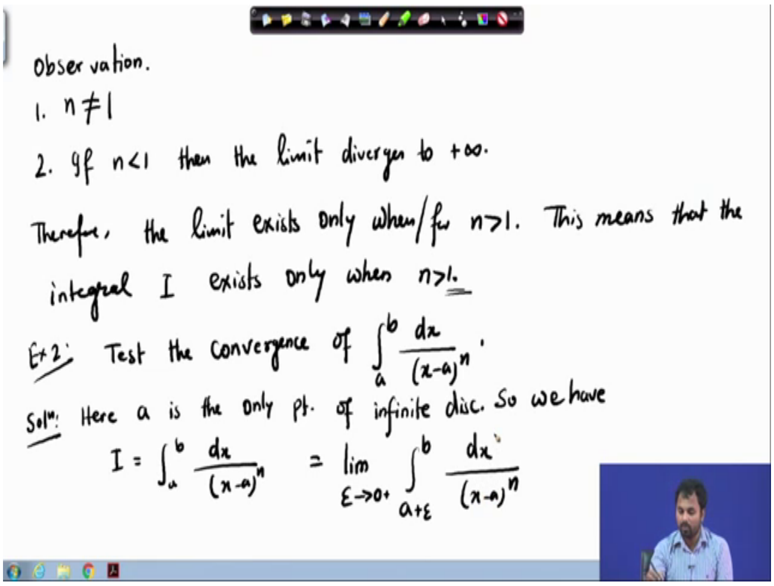

So, this is our first observation let us say, I can write it here as observation. So, we have got at least one value of n where this integral would not be defined or that or that that limit would not exists. Now, 2nd observation is, let us go back to the previous slide. Now, 2nd observation is if B goes to infinity and if n is less than 1, here if n is less than 1 then in that case this will become B to the power something negative and that will go in the numerator.

And, then it will become B to the power something positive and since it B goes to infinity this whole thing will go to infinity if n is less than 1. So, in that case this n cannot be less than 1 because, if n is less than 1 then this whole thing will go to infinity. So, the 2nd observation is if n is less than 1, then the limit diverges or limit the limit diverges to plus infinity. So; that means, if n is less or equal to 1 then in that case this limit; this limit here or this limit here would not exists.

And therefore, if we want to test the convergence therefore, the limit exists only when or only for when or for n greater than 1; let us check. So, if n is greater than 1 then this will be 1 by B to the power some positive and if B goes to infinity then this term will go to 0, a is any for the moment let us assume that a is also any positive number. So, a is any positive number and if n is positive then this whole thing would also be finite and ultimately the limit would exists. So; that means, here this integral I this implies that so, or this means that; this means that the integral I; the integral I exists only when n is greater than 1. So; that means, it is convergent exists or we can also write it is convergent for n greater than 1.

You see in this particular example, not only we had to be careful about the improperness of the integral, we also have to be careful about the ranges for this n for which we can see whether it is convergent or divergent . So, this was a very nice example where, we could see that for n greater than 1 this integral I, this improper integral was convergent and for n less or equal to 1 this integral I is also divergent. So, that is we have verified here.

And another example in this context could be you may come across these type of examples in your study. So, test the convergence of integral a to b dx by x minus a whole to the power n. So, again here so, here none of the limits are infinity. So, the limits are finite, but the function is creating a problem. So, at the left end point this function has infinite discontinuity. So, we can write that here a is the only point of discontinuity; is the only point of infinite discontinuity and another thing is we have to be careful about this n. So, for what values of n this integral would be convergent or would be divergent.

So, a is the only point of discontinuity. So, we have we know that what we do in this situations, we first of all write I and then the value this integral x minus a to the power n would be equal to value of these of the limit. So, we add a small epsilon just to get away from the discontinuity. So, we add a little bit epsilon in that left end point and then we write this as dx by x minus a whole to the power n and now we can integrate.

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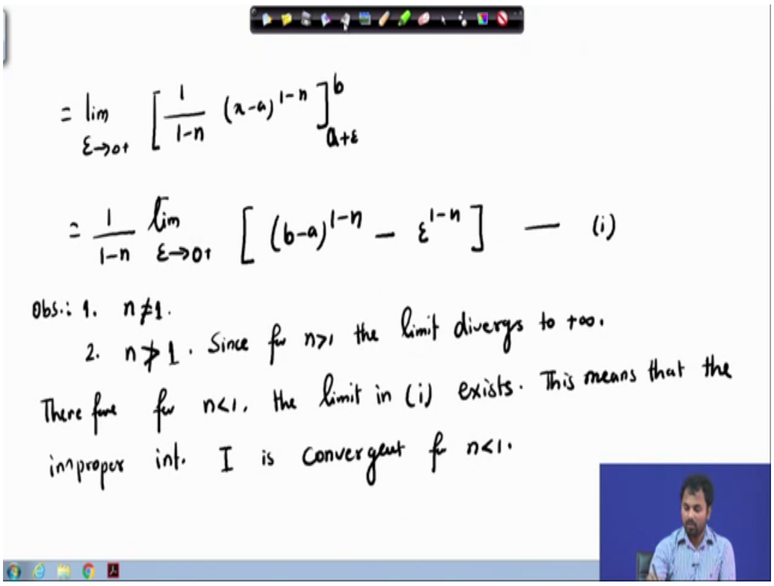

So, if we integrate we can write this as limit epsilon goes to 0 positive; once we integrate this will reduce to 1 by 1 minus n x minus a whole to the power 1 minus n integral from a to b a plus epsilon to b. And, then we substitute the limit we take n out first of all why sorry 1 by 1 minus n and then we write limit epsilon goes to 0 positive b minus a whole to the power 1 minus n minus, it will be epsilon to the power 1 minus n alright.

So, now what we can see again like the previous example, if n is 1 so, we can write as small observation. So, 1st observation is if n is 1 then this whole thing will be undefined or this whole thing would not exists. So, n cannot be 1 that is the first observation. Now, what is the 2nd observation here? If n is; if n is greater than 1 if n is greater than 1 then this whole thing will become negative. So, epsilon to the power negative and if it is epsilon to the power negative then we can write it as 1 by epsilon to the power that number that real number. Now, if epsilon goes to 0 then the whole 1 by epsilon will go to infinity and then this whole limit will become undefined. So that means, n cannot be greater than 1 because, then in that case this limit will diverse to infinity.

So, the 2nd observation is n cannot be greater than 1 like we wrote down here in the previous example. So, n cannot be greater than 1 because, since for n greater than 1 the limit diverges to plus infinity; now what happens for n less than 1. So, if n is less than 1 then in that case of course, this is a positive quantity if n is less than 1 then this one will also be will also be defined. And, if n is less than 1 then this will be epsilon to the power some positive real number and when epsilon goes to 0 epsilon to the power that positive real number will go to 0. So, then in that case this limit will be defined.

So, we can write therefore, for n less than 1 the limit let us call this as equation 1, the limit in equation 1 exists. This means that; this means that the integral I, the integral or the improper integral improper integral I is convergent for n less than 1. And of course, it will be divergent for all n greater or equal to 1. So, this one was also very nice and interesting example, where we had a discontinuity in the function at one of the end points. And, that integral I mean the convergence of that integral involved this n as a factor.

So, this n plays a very important role when we talked about the convergence of this improper integral and for n less than 1, this improper integral is convergent where as if n is greater or equal to 1 we just saw that it will be divergent. So, this is an another type of examples where, an another type of example where how to say the convergence of the integral depended on depend is dependent on some other factors involved in the integrand.

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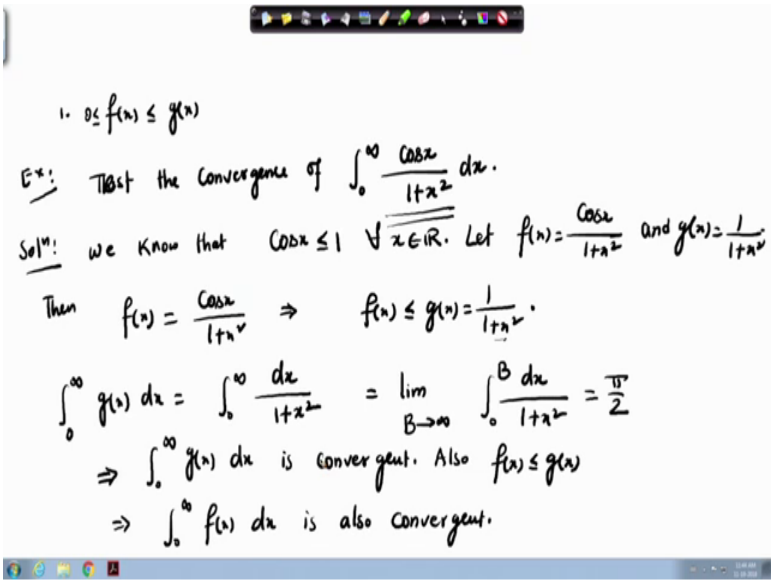

So, this is something I wanted to show you all and now we are ready to look into the examples of based on comparison test. So, we defined the comparison test as of this type. So, the first comparison test says that if you have two functions f x and g x and if they satisfy this type of inequality. And, integral of this g x is convergent, then in that case we can talk about the convergence of the integral involving f x and then there was a limit test where we evaluate the limit.

So, first of all; so, first of all we will; so, first of all we will look into we will look into the example of this type example of example on comparison test. So, test the convergence test the convergence of integral from 0 to infinity cos x by 1 plus x square dx. So, this one was our comparison test of inequality type and the second one is comparison test of limit type. So, let us see what kind of comparison test we can use here. So, looking at cos x we know that; we know that cos x is less or equal to 1 for all x in R.

So, since we know this it is pretty much intuitive to use comparison test of inequality type because, sometimes looking at the integral you can able to make out that what kind of comparison test I need to use. So, just look at this integral and this integrand and just looking at the integrand you may get an idea that since, cos x is less or equal to 1 I may have to use comparison test of inequality type.

And in order to do so, let f x is let us define f x is cos x by 1 plus x square and g x as 1 by 1 plus x square. So, then our f x our f x is cos x by 1 plus x square. And, from here f x is less or equal to g x which is equal to 1 by 1 plus x square right because, cos x is less or equal to 1 ah. So, we have this inequality. So, we found a function g x; we found a function g x which is which is what to say dominating this function f x. So, which is acting like an upper bound for this function f x; so, the first criteria satisfied. And, the second criteria is to check whether this integral g x is convergent or not. So, g x dx equals to integral from 0 to infinity dx by 1 plus x square right. And, this can be written as limit B goes to infinity 0 to B dx by 1 plus x square.

We have just worked out an example like this in our previous class on improper integral. So, this is basically tan inverse x and when B tends to infinity this whole thing will go to pi by 2. So that means, this integral from 0 to infinity g x dx is convergent and it is converging to pi by 2 and it is also dominating the function f x. So, from here we can say that so, I we can put all this into any statement. So, the statement is integral from g x to dx is convergent or exists and also f x is less or equal to g x. So, from here we can say that integral from 0 to infinity f x dx is also convergent.

So that means, just looking at the function g x and its properties we can talk about the convergence of this function f x. So, all we have do is to find an inequality like this. And, if we can able to find or locate an inequality like this we can talk about its convergence and that will assure the convergence of the given improper integral.

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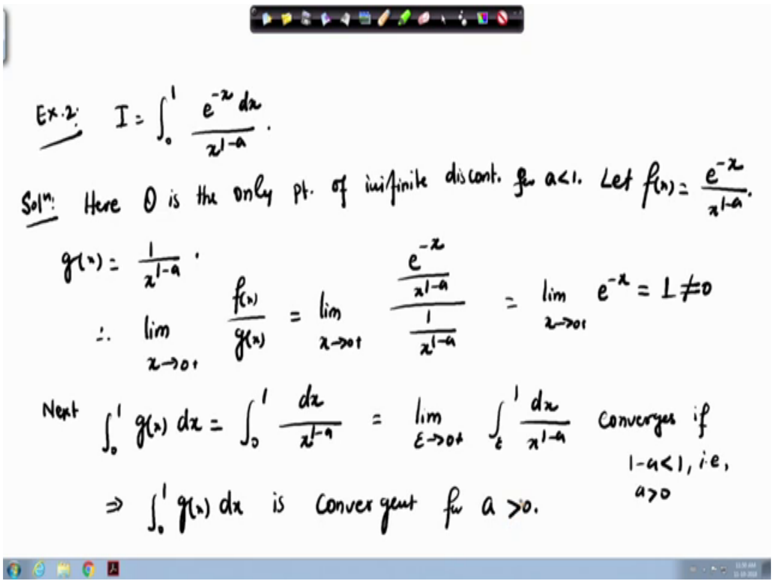

Now, let us look into an another example which is based on the limit type which is basically based on the limit type. So, the limit type example is of this type. So, example 2 test the convergence, I am avoiding this statement and I am just writing the integral. So, we have to test the convergence of the integral 0 to 1 e to the power minus x dx by x to the power 1 minus a. So, we have to test the convergence of this integral. Now, this integral is only undefined or has infinite discontinuity for a less than 1 at the point x equals to 0.

Because, if a is greater than 1 then this whole thing will go to numerator and then this integrand is defined, I mean if does not have any kind of infinite discontinuity. So, it is only creating problem when a is less than 1 so, let us write it. So, the solution here 0 is the only point of infinite discontinuity for a less than 1. Because, for a greater equal to 1 this integral is always defined this integrand sorry this integrand is always defined.

So, now we will only check the convergence for a less than 1. So, let f x equals to e to the power minus x x to the power 1 minus a. And, we take g x as 1 by x to the power 1 minus a then we can evaluate the limit the point, where we have the discontinuity which is 0. So, let us write x goes to 0 positive f x by g x, then this is basically limit x goes to 0 positive f x is our e to the power minus x divided by x to the power 1 minus a divided by 1 by x to the power 1 minus a. So, this whole thing will reduce to limit x goes to 0 positive e to the power minus x which is 0.

So, in the comparison test of limit type the limit first of all has to be non-zero. So, this is a non-zero limit and this is basically all lambda from the definition of the comparison test of limit type. Now, first thing is assured. Now, second thing is next we have to check whether integral from 0 to 1 g x dx is convergent or not, because if they do then in that case both f x and g x would converge together or diverse together. So, this one will be 0 to 1 g dx by g x is 1 by x to the power 1 minus a. So, this integral; this integral it can be seen since a is less than 1; a is less than 1, then in that case a is less than 1, then in that case we have limit epsilon goes to 0 positive epsilon to 1 dx by x to the power 1 minus a.

Then we do the integration like we did before and then we substitute epsilon in the integral in the value of the integral and then we make epsilon goes to 0. So, this is very straight forward to do here and we can be able to see that this integral here converges, it is very straight forward to see that this integral converges if 1 minus a is less than 1. So that means, a is greater than 0 am I right. So, a goes on that side so, a is greater than 1. So, which means that; which means that integral from 0 to 1 g x dx is convergent; is convergent for a greater than 0; so, for all a greater than 0.

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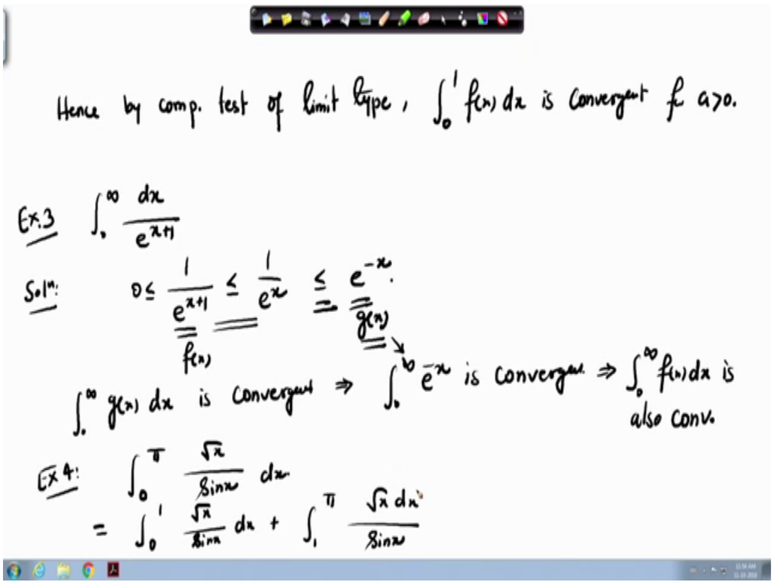

And hence by comparison test by comparison; so, I am just writing the short form of it. So, by comparison test of limit type; of limit type integral from 0 to 1 f x dx. So, f x was e to the power minus x divided by x to the power 1 minus a is convergent; is convergent for a greater than 0. So, although the function had let us go back; so, although the function has 0 has the point of infinite discontinuity for a less than 1, we just showed that the integral is always convergent why I am comparison test of limit type for all a greater than 0. So, this is one such case where we use comparison test of limit type, we also saw a comparison test of inequality type, we can work out one or two more examples.

So, let me just write the examples and I am pretty sure you can be able to do it by yourself, I will just give some hint. So, we have to check the convergence of integral of this type. So, from here we can easily establish these inequalities. So, 0 less or equal to 1 by e to the power x plus 1 because, it is always positive for all x. And for all x between 0 to infinity and then this one is less or equal to e to the power x. This is also very straight forward and then this one is less than or equal to e to the power minus x.

Now, if I assume this as our function g x and this is our function f x then in that case we all have all we have to check is whether this integral dx is convergent or not from the comparison test of inequality type. Because, we have got the inequality, all we have to show is that this integral is convergent or not and it is very straight forward. So, from here it is very straight forward and easy to show that this integral here is convergent minus x is convergent. And, if this integral is convergent; if this integral is convergent then from the comparison test of inequality type we can write that dx is also convergent is also convergent.

So, handling examples of this type basically, I can also give you examples where we have limit to evaluate the limit very quickly. So, let us say we have something like integral form. So, let us say we have integral example of this type integral from 0 to pi root x by sin x so, dx. So, here we will basically divide this integral. So, here we will basically divide this integral, let us say from between two sub integral. So, 0 to 1 root x sin x dx plus 1 to pi root x dx by sin x. And, then what we do is in order to check the convergence at 0 because, 0 is the only point where this function is having infinite discontinuity. And, in the second sub integral pi is the only point where this function have infinite discontinuity. So, we can check the convergence.

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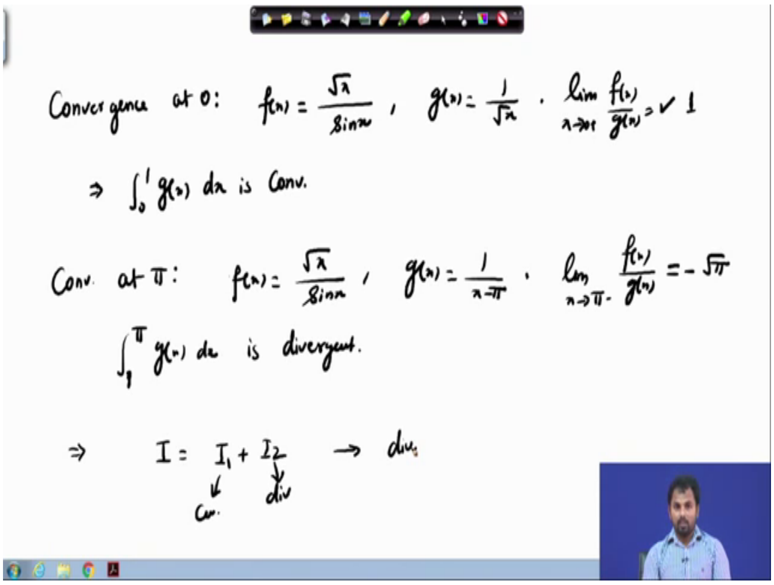

We can check the convergence first of all at 0. So, convergence at 0 and in order to do that we consider our function f x equals to root x by sin x and g x has 1 by root x. And, then we do the limit of then we do the limit of f x and g x as x goes to 0 positive. And we can be able to see that this limit is basically 1 so, this limit is basically 1 and we can also be show the we can also be able to show that integral from 0 to 1 g x dx is convergent is convergent and therefore, the first sub integral would be convergent.

Similarly, we can check the convergence at pi and in order to do that I will take my function f x as square root of x by sin x. And I will take my function g x as 1 by x minus pi, I can again evaluate the limit and it can be able it can be shown it can be shown. So, pi minus pi minus I can be able to show that this limit f x g x would be equal to minus square root of pi. And I can also be able to show that integral from 1 to pi g x dx is actually divergent. So, this I can be able to show that.

So, since this one is divergent our original sub integral this one will be divergent; however, our sub integral I 1 is convergence. So, convergence plus divergence means that the whole integral is divergent. So, from here I can write I is equals to I 1 plus I 2. And, since I 2 is divergent and I 1 is convergent the whole thing will be divergent. Therefore, by this comparison test of limit type I was being able to show that the given integral I is divergent. And, we will stop our lecture for today here. And, in the next class we will start with new test and we will work out few examples as well.

Thank you.