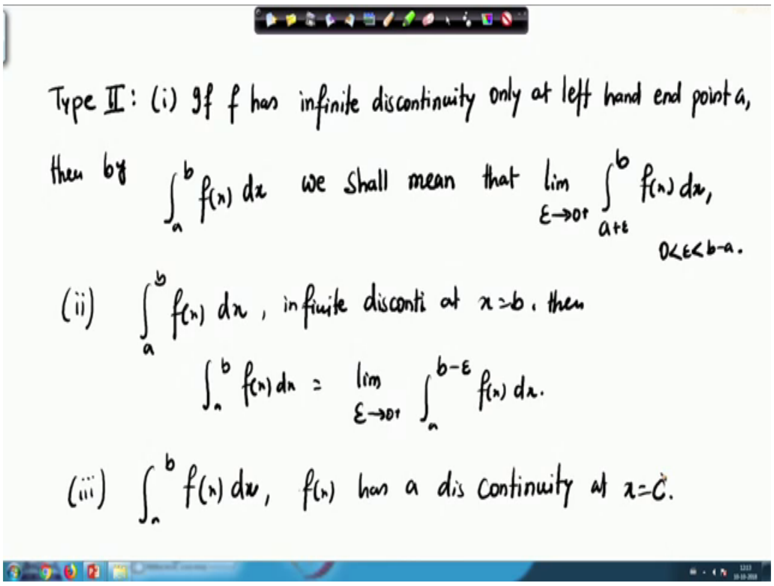
Integral and Vector Calculus
Prof. Hari Shankar Mahato
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 10
Improper Integral (Contd.)

Hello students. So, we will start our next section of Improper Integral. And in this section we will basically look into the improper integral of type II. And the type II type of improper integral are nothing but where you have the discontinuity in the function f.

(Refer Slide Time: 00:45)



So, to define that let us write type II type of improper integral, so if the function f has infinite discontinuity only at left hand end point a. Then by a to b f x dx; we shall mean that limit epsilon goes to 0 positive integral from a plus epsilon to b f x dx, where 0 less than epsilon less than b minus a. So that means, if we have an infinite discontinuity let us say at the left endpoint. At the left endpoint then in that case we add a little bit epsilon where epsilon is a arbitrary small positive real number to the left endpoint and then this will give us the value of the integral.

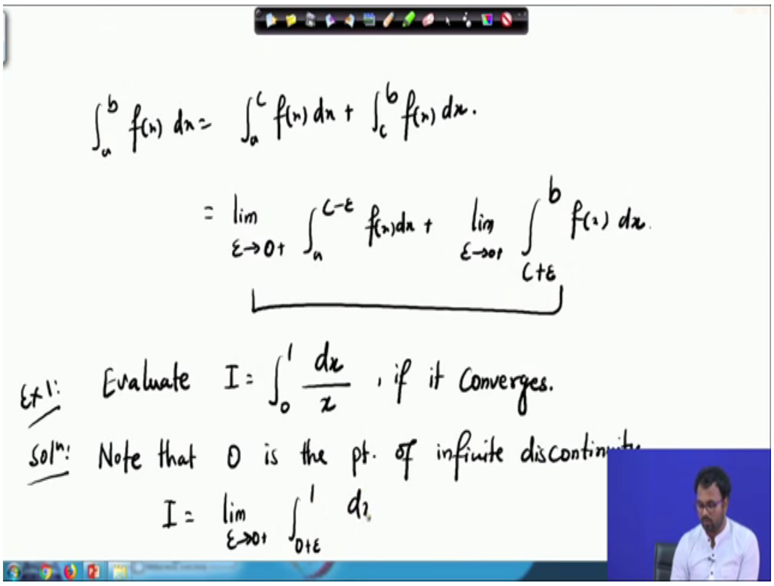
So, we just get rid of that point where the discontinuity happens and then we try to evaluate the integral and afterwards we make epsilon go to 0, and that limit will actually be the limit of our improper integral.

Similarly let us say if we have a discontinuity. So, I am avoiding the whole statement. So, what I am trying to say is that if we have infinite discontinuity, so infinite discontinuity at the point x equals to b or the right hand endpoint then in that case the value of the integral a to b f x dx would be equal to limit epsilon goes to 0 plus a to b minus epsilon f x dx. So that means, we again we cannot go beyond the interval b. So, we have to stay inside the interval otherwise the range of integral would be beyond the interval a to b.

So, what we are doing is we just get how to say get rid of that point in a way that we subtract a small epsilon from b, and that will give us the value of the integral b minus epsilon doing this b minus epsilon here. And that will, evaluating this limit will give us the value of the integral a to b.

And another type would be let us say we have an interval a to b f x dx. So, this interval will have a point. So, f x has a discontinuity, so within this interval f x has discontinuity at x equals to c. So that means, in between that interval a to b there exist a point c where the function is not continuous.

(Refer Slide Time: 04:35)



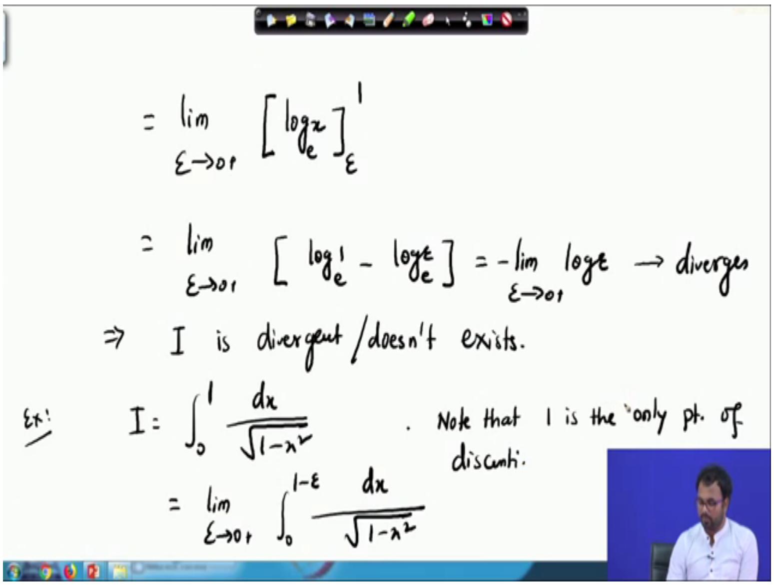
So, in there exist a point c where the function is not continuous. And then what we will do? We will write the integral a to b f x dx as a to c f x dx plus c to b f x dx. Like we did for the limits where both the limits were infinite. So, here also the point of discontinuity we can break the integral into the points of discontinuity.

And now here I can write limit epsilon goes to 0 plus a to c minus epsilon f x dx, and here I can write limit epsilon goes to 0 plus or delta goes to 0 plus c plus epsilon to b f x dx. So, and if both of these 2 limits exist then in that case that will give us the value of the integral.

And suppose if you have more than one point of discontinuity let us say if there is a point called d then we basically write integral from a to c plus c to d plus d to b so that means, if we have finite number of infinite discontinuity we can break the integral into finite number of sub integrals. And at each one of these sub integrals we put the limit of this type. Limit epsilon goes to 0 positive and then we evaluate those finite sub integrals and that will give us the value of the integral.

So, now let us work out few examples based on the above definitions. So, our first example would be evaluate I equal to 0 to 1 dx by x if it converges. So, here we can see that 0 is the point of infinite discontinuity. So, note that 0 is the point of infinite discontinuity or only point or only point of infinite discontinuity we can use the term only is the only point of infinite discontinuity. And so I can write my I as limit epsilon goes to 0 positive 0 plus epsilon to 1 dx by x.

(Refer Slide Time: 07:24)



So, then this will be limit epsilon goes to 0 plus and integral of 1 by x is log of x to the base E epsilon to 1 and then this can be written as limit epsilon goes to 0 positive log of 1 to the base e minus log of epsilon to the base e.

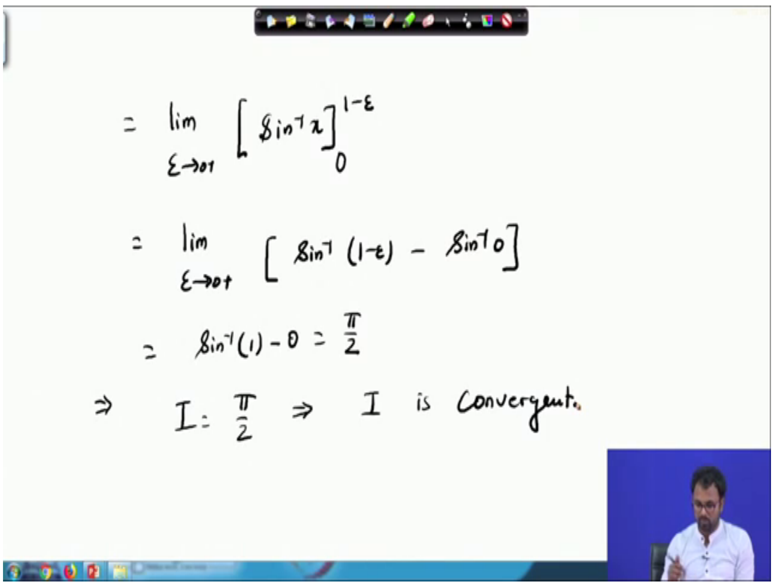
Now, log of 1 is 0 and log of epsilon; now log of 1 is 0, but this here; so minus log of epsilon. So, when epsilon goes to 0 when epsilon goes to 0 this log of epsilon will go to minus infinity. So that means, the whole term will go to a plus infinity. So, this log of epsilon will go to minus infinity as epsilon goes to 0 positive. So, this whole thing will go to a plus infinity that means, this whole thing is divergent. So, it diverges and from here we can say that our integral I is divergent or does not exist or does not exists, all right.

Next, let us consider an example where we can at least show the convergence. So, let us consider I equals to integral from 0 to 1 dx by 1 minus x square, square root of 1 minus x square. So, we have to always identify the type of improper integral. So, whether you have the discontinuity in the function or whether you have improperness in the limit.

So, once you identify that, so here in this case the limits are all right because the limits are finite 0 to 1 however, we can see that the function is discontinuous at x equals to 1. As I was saying there is a strong possibility you may have a function which is discontinuous at 2 points or 3 points and so on and then in that case we have to break this integral here into that many number of sub integrals.

So, here what would happen is, we will first get rid of this discontinuous points 0 to 1 minus epsilon dx by square root of 1 minus x square.

(Refer Slide Time: 10:03)

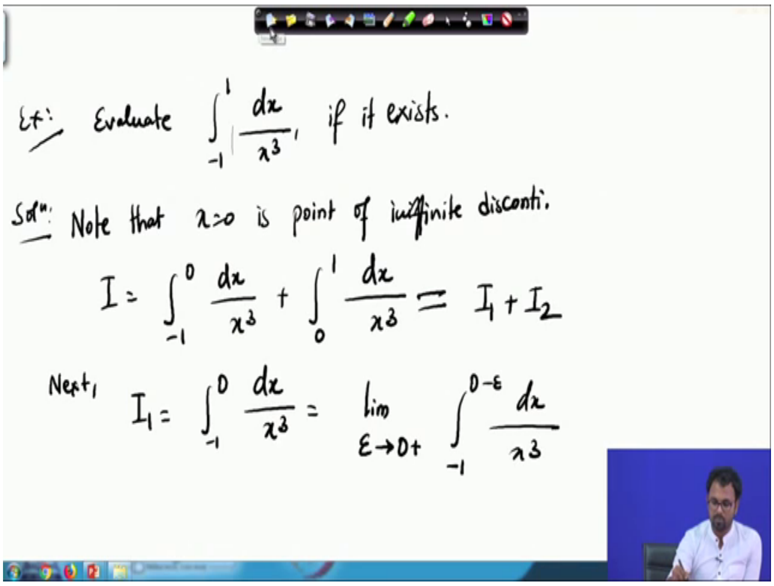


Now, we can write this as limit epsilon goes to 0 positive. Here we can put a small comment that, note that 1 is the only point of discontinuity, all right.

So, next we can write it as the value of the integral. So, here we have a square root of 1 minus x square. So, this will be basically sin inverse x. So, sin inverse x integral from 0 to 1 minus epsilon. So, integral from 0 to 1 minus epsilon and then we substitute the limits. So, this will be sin inverse 1 minus epsilon minus sin inverse 0. So, sin inverse 1 minus epsilon when epsilon goes to 0 this will be basically sin inverse 1 minus sin inverse 0 is 0 and sin inverse 1 is pi by 2. So, therefore, the value of the integral I is actually pi by 2 and from here we can say that our integral I is convergent, all right.

So, this is a one such example where we where we have subtracted the epsilon from the upper limit and we just have to calculate the integral and then pass on that epsilon goes to 0 and that will give us the value of the integral I.

(Refer Slide Time: 11:56)



So, we have covered the examples of both of the types. Next, another example could be let us say of this type. So, evaluate integral from minus 1 to 1 dx by x cube, dx by x cube if it exists. So, first of all let us see the type of this integral.

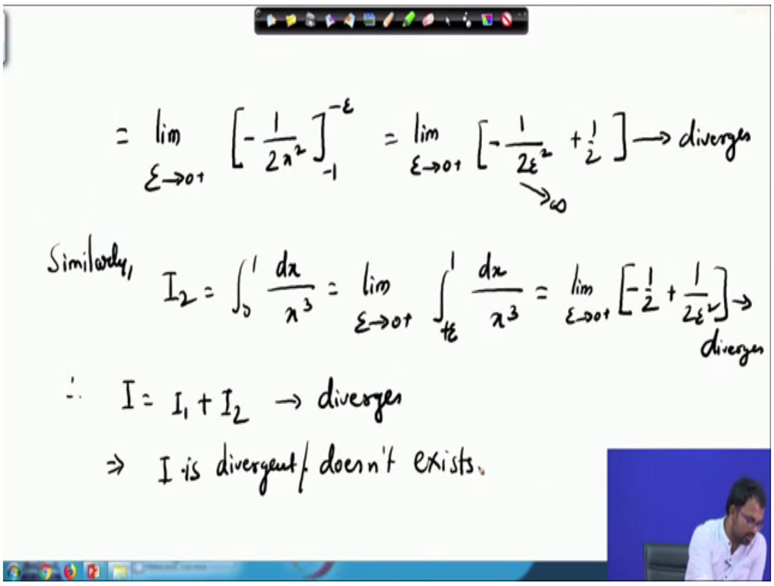
So, here the range of integration is minus 1 to 1. So, the range is pretty much fine pretty much because they are finite. Now, at x equals to minus 1 and x equals to 1 this function is also not unbounded. So, we can say that the function is not unbounded between minus

1 to 1, however that is not true because at x equals to 0 which is a point between minus 1 to 1 this function is unbounded. So, now, it falls into the category of this type. So, it falls into the category of this type, this here. And then we have to split the whole interval integral into 2 sub integrals. So, let us go back to that slide.

So, here we can write note that x equals to 0 is the point of infinite discontinuity, all right. And then I can write my integral I as minus 1 to 0 dx by x cube plus integral from 0 to 1 dx by x cube and now I can write these 2 integrals as I 1 plus I 2, where I 1 is this one and I 2 is this one.

So, next we will look into our integral I 1. So, I 1 is minus 1 to 0 dx by x cube. So, this can be written as limit epsilon goes to 0 plus and since the discontinuity has the upper limit I will subtract a little bit epsilon from there dx by x cube.

(Refer Slide Time: 14:37)



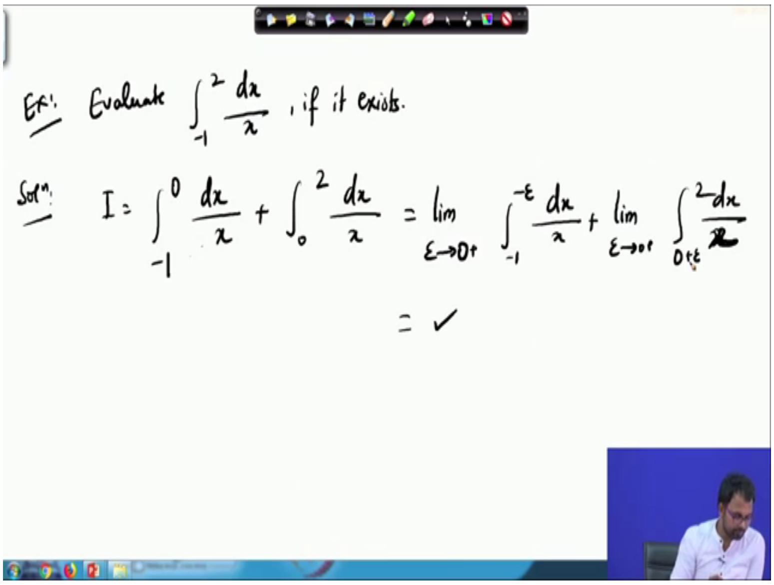
So, then the value of the integral, the value of the integral would be minus 1 by 2 x square integral from minus 1 to minus epsilon and if I substitute the limit then this will be epsilon goes to 0 positive minus 1 by half epsilon square minus minus plus half. Now, when epsilon goes to 0 positive this 1 by 2 epsilon square this will not exist. So, this will diverge to infinity. So, this whole thing, so this will diverge to infinity, so this whole thing diverges, this whole thing diverges.

And if since we are writing the 2 sub integrals, if any one of them if both of them are convergent then the whole thing is convergent, but if any one of them is divergent and the other one is convergent then the whole thing will again be divergent and if both of them are divergent then the whole thing will again be divergent.

So, here our I 1 is divergent similarly we can see that, similarly the integral I 2, I 2 which is basically integral from 0 to 1 dx by x cube which is equals to limit epsilon goes to 0 positive 0 minus epsilon so that means, minus epsilon 0 plus epsilon so that means, plus epsilon to 1 dx by x cube. And if we evaluate this integral then here we will obtain basically limit epsilon goes to 0 positive. This one will be minus half plus 1 by 2 epsilon square and then this one is also divergent. So, this is also diverges.

So that means, therefore, our I equals to I 1 plus I 2 they are both divergent and this implies that our integral I is altogether is divergent is altogether divergent or it does not exist or does not exist, all right. So, just splitting the interval into 2 sub intervals we evaluate the each one of the sub intervals and if any one of them is divergent or if both of them are divergent then in that case that the whole interval I would be divergent and if both of them are convergent then the whole into integral I would be convergent. So, this is one way to check whether the given integral is convergent or not.

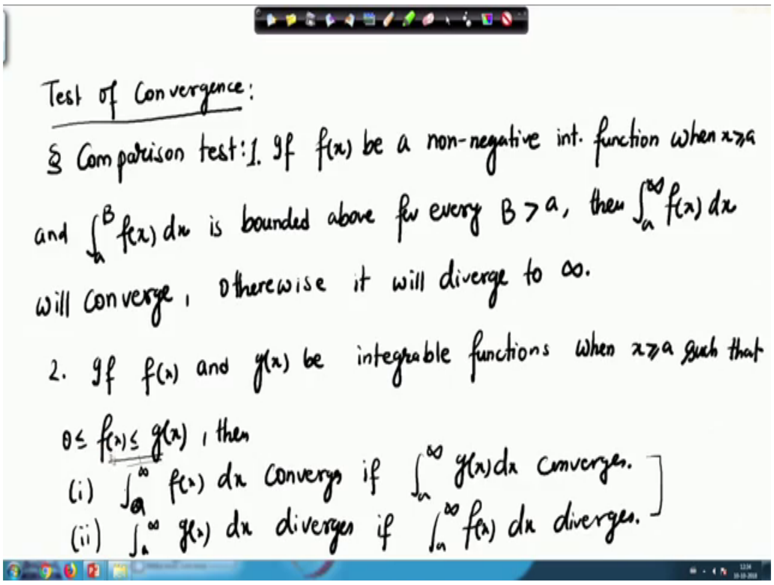
(Refer Slide Time: 17:40)



Next, similarly another example could be evaluate minus 1 to 2 dx by x if it exists. So, I will just give some hints for this example. So, I can write I as limit epsilon goes to or first let us write the integral into 2 sub integrals. So, here again 0 is the point of discontinuity we can see that clearly. So, I can write it as minus 1 to 0 dx by x plus integral from 0 to 2 dx by x, and then we can write it as limit epsilon goes to 0 positive minus 1 to minus epsilon dx by x plus limit epsilon goes to 0 positive 0 plus epsilon 2 to dx by 2, and then dx by x sorry.

And then we just evaluate each one of these individual integrals and then we can be able to see whether one of them or both of them are divergent or convergent, based on that we can draw our conclusion. So, this was a pretty this is pretty much straight forward example.

(Refer Slide Time: 19:39)



So, next we will look into integral of type test of convergence. So, next we will look into test of convergence. So, suppose you are given an integral and if it is not straightforward to evaluate via doing those limit procedure or via how to say the traditional method to calculate the improper integral, then we look for some tests that will help us reduce calculating those limits and integrals and just using these tests we will assure whether our given improper integral is convergent or not. So, one such test is called comparison test. So, one such test is called comparison, comparison test.

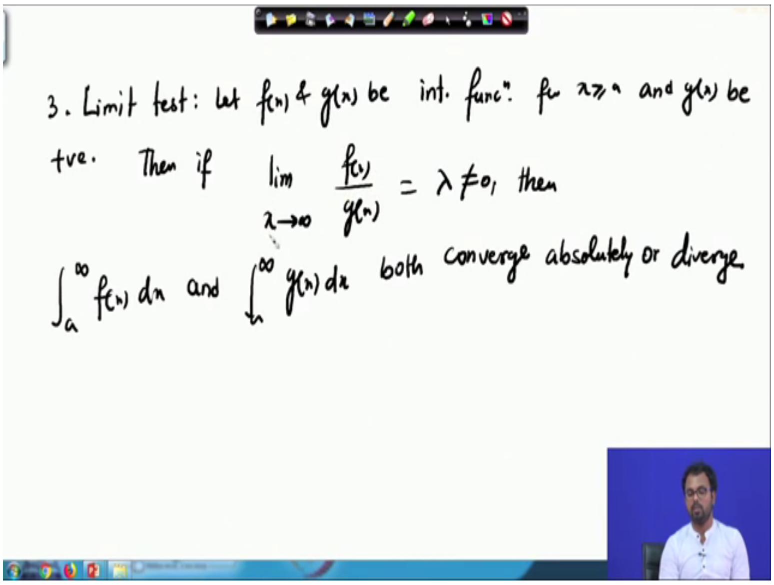
And it says that if f x be a non-negative integrable function when x is greater or equal to a and integral a to b f x dx is bounded above for every B greater than a then integral a to infinity f x dx will converge otherwise it will diverge to infinity plus infinity. So, what it says is that if we have a non-negative function f x which is integrable for x greater than or equal to a, and integral from a to B where capital B can be treated as the upper limit or upper limit of this range of integration is bounded then for every B greater than a then this integral from a to infinity will also converge, otherwise it will diverge. So, this is the comparison test of let us say, comparison test of type 1 or simply just comparison test 1, we can write it as comparison test 1. So, this is first comparison test.

And second comparison test would be, the second type would be if f x and g x be integrable functions when x is greater or equal to a, such that 0 less or equal to f x less or equal to g x then, a to integral from a to infinity f x dx converges if integral from a to infinity g x dx converges and integral from a to infinity g x dx diverges if integral from a to infinity f x dx diverges.

So, what does it mean is if we have 2 functions f x and g x and this follow some kind of comparison. So, this here this inequality is nothing, but a comparison. So, if f x is less or equal to g x then in that case if the function g x is convergent, then in that case we can say that the if the integral of the function g x is convergent, then we can say that the integral of the function f x is convergent and if the integral of the function f x is divergent then the integral of the function g x is also divergent. So, these are the 2 comparison test in a way where you compare the function f x or the given integral f x integral of f x with integral of g x by this fashion.

So, you have to make sure that whether f x is greater or equal to a less or equal to g x or not. We need to find out a function basically g x, that will sort of how to say satisfy this inequality. And based on that we can check whether the function g x is convergent or not and with the help of which we can be able to say that whether the function f x is convergent or not. So, this is one such type of comparison test.

(Refer Slide Time: 25:56)



There is a third type, third type is limit test, it is called limit test. So, it says that let f x and g x be integrable functions, be for x greater or equal to a and g x be positive, g x be positive then if limit x goes to infinity f x by g x equals to a lambda which is not equals to 0 then integral from a to infinity f x dx and integral from a to infinity g x dx both converge absolutely or diverge. So, they are both either convergent or they are both divergent.

So, this is a third type of comparison test where you basically evaluate this limit and if that limit is non-zero then in that case we can talk about the convergence or divergence of these 2 integrals. So, either they will converse converge together or they will diverse together. So, all we have to do is to find a function g x such that we can have an non-zero limit and then we can talk about the convergence.

So, based on these 3 types of comparison test we will work out few examples to make the concept clear and will do that in our next lecture. So, I will look forward to it and see you in next class.

Thank you.