Integral and Vector Calculus
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Lecture – 01
Partition, Riemann intergrability and One example

Hello students. Today, we are going to begin our very first lecture on this Integral and Vector Calculus. And as I have told you that we will basically start with the integral calculus section at first, and then we will move to the vector calculus part. So, to begin with just to give you an overview of the course, I have already done that in the introductory video.

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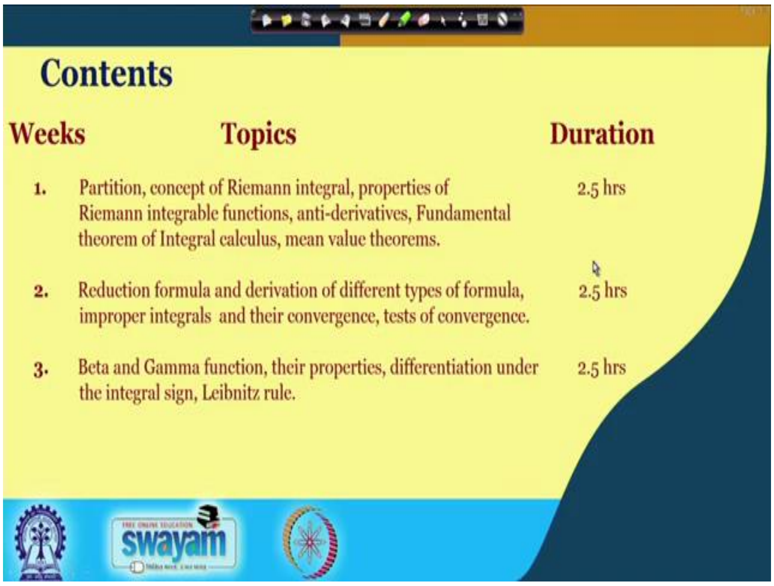

So, we will first start with the partition, and concepts of Riemann integral, and properties of Riemann integrable functions and so on. So, today we will start with the partition. And we will try to explain the concepts of Riemann integrable functions.

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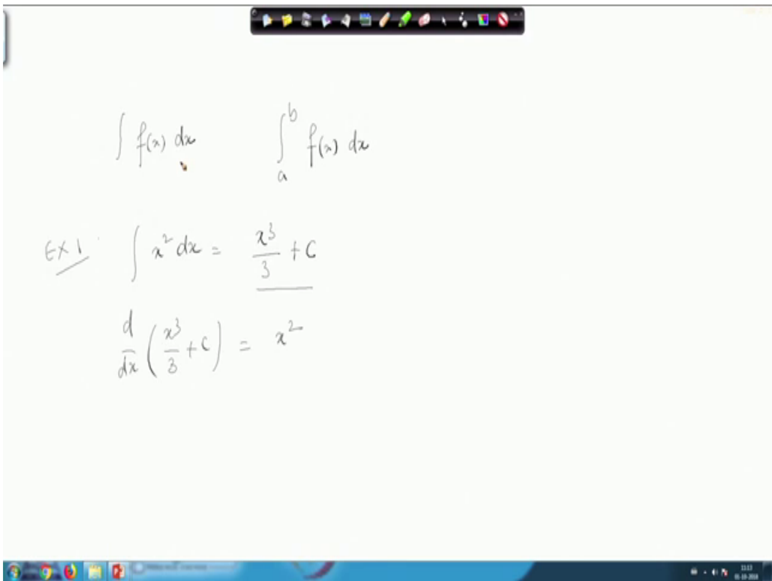

So, let us so first of all an integral calculus up until your plus 2 level, you may be familiar with the integral of the type $\int f(x)dx$. So, this is basically an indefinite integral. And you also have done something like integral from $\int_a^b f(x)dx$.. So, this is basically definite integral. And we know that if we integrate this function, so this function $f(x)$ they most of the time they are really nicely behaved functions. So, you can do integration or you can do differentiation of these functions, because they are very well behaved in a way.

And one such example could be that let us say example 1. So, you have integral of x square dx. And if you integrate, then it is basically $\frac{x^3}{3}+c$, where C is any arbitrary constant. So, you can see that the answer of this integral on the left hand side is basically a very nicely behaved function. So, x^3 is a continuous as well as differentiable function, and C is basically a constant function.

So, it is always something goes in the direction of finding the derivative of this right hand side. So, if we differentiate the right hand side $\frac{x^3}{3}+c$, then this will be basically x^2, because the derivative of the constant is 0, and derivative of the first term is basically x^2.

So, it is always goes in a way that how can we find the derivative of this function x^2, here. So, whatever function you have in the integral. So, these functions these $f(x)$ here, they are called as integrand. So, whatever integrand, you have in the integral they can be a differentiation of some other function. And these functions $f(x)$, they are sometimes called as anti-derivative. So, in a way you have to find out a function, which is a derivative of this function $f(x)$, and that will be your answer to your integral. So, this is what we did up until our plus 2 level.

And now, we will basically look into the Riemann integral. So, those integrals were quite simple, where you had to find out the derivative of the function $f(x)$ in the integral. In Riemann integral, we will look at integrals in a little bit different way. And we will start first of all with partition.

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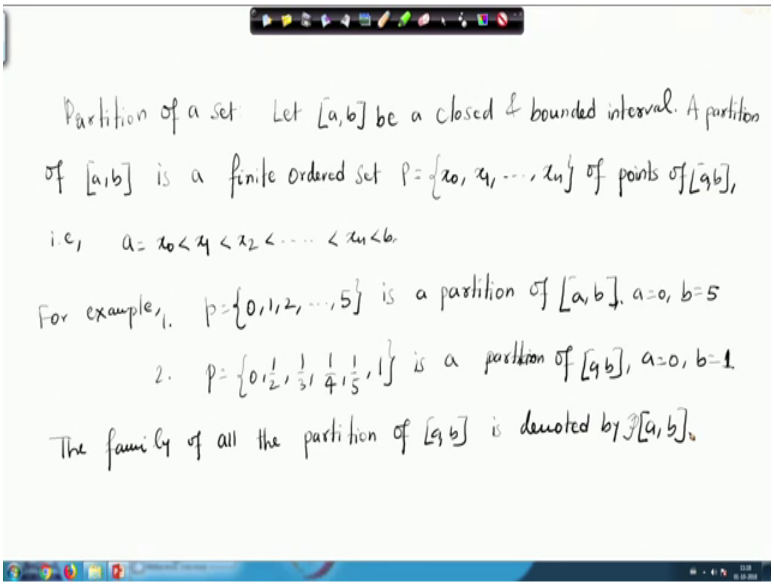

So, just to start with partition of a set so, by partition of a set, what do we mean by that? I said to start with, we will assume let $[a, b]$ be a closed and bounded interval. So, of course it is a closed interval and a and b are finite.

Then a partition of a, b is basically a finite ordered set P, which is defined as let us say x_0, x_1, x_2 up to x_n of points of a, b, which means that is a is equals to x_0, x_0 less than x_1, x_1 less than x_2, and so on. So, it is basically finite, and it is also ordered. So, you have x_0 less than x_1 less than x_2 and so on. So, such kind of finite ordered set P is basically called as the partition of a, b of this closed interval a, b.

For example, we can consider for example $P = 0,1,2$ up to 5 is a partition of a to b this is 1^{st} example. Second example could be $P = 0, \frac{1}{2}, \frac{1}{3}$ so here a and b so for sorry here a is 0, and b is 5. Similarly, for example 2 if we consider $P = 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots 1$, then this is a partition of a to b, where a is 0, and b is 1. So, you can consider any type of partition for this interval a to b.

And the family of all the partitions of a, b is denoted by p excuse me let me erase this. So, it is denoted by p of a, b. So, this is basically the family of all partitions of the close interval a, b. Now, if we let me go to a new page.

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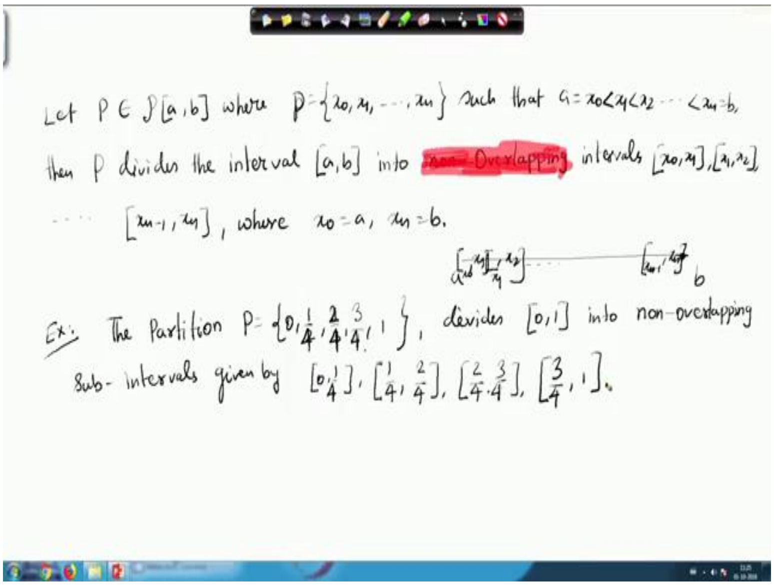

Now, if we assume that P is one such partition of the closed interval $[a, b$, where our P is basically consists of x_0, x_1, x_2 up to x_n. And such that such that a is x_0 less than x_1 less than x_2 up to x_n. Then this P, divides the interval $[a, b]$ into non-overlapping intervals non-over lapping intervals x_0 to x_1, x_1 to x_2,… up to x_{n-1} to x_n, where our x_0 is a and x_n is b.

So, if we consider a partition P of this interval $[a, b]$, which is basically our given closed interval. And if this partition P is defined by this set x_0, x_1, x_2 up to x_n, then this P will divide this closed interval into non-overlapping sub intervals. So, if you take the union of these of these sub intervals, then in that case you will basically get your closed interval $[a\ b]$.

Now, this term here non-overlapping, it means that these a sub intervals x_0 to x_1, x_1 to x_2, I mean they do not intersect, I mean they do not have any points common except for the endpoints, so that means x_0 to x_1, so where this interval x_0 to x_1, ends you have interval x_1 to x_2, beginning from there. But, they do not have any other common point, so that means these intervals they are non-overlapping.

And in other words it means that if you have your interval a to b let us say, then in that case your x_0 to x_1, will be here, then x_1 to x_2, will be beginning from here and so on. Then you have x_{n-1}, to x_n, which is basically your point b, so that is how these sub intervals are defined.

We can look into one example. So, one such example could be the partition P the partition $P = 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots 1$. Let us say divides 0 , 1 into non-overlapping subintervals, sub intervals given by so those sub intervals are $0, \frac{1}{2}$, sorry $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ and so is $0, \frac{1}{4}, \frac{1}{3}$ or we can have this as $0 , \frac{1}{4} , \frac{2}{4},$ and then $\frac{3}{4}$ that this one will be much more interesting. ah

So, let us say you have $0 , \frac{1}{4} , \frac{2}{4}, \frac{3}{4},$ and then I am taking the end point as 1; just to make the concepts, so little bit clear, instead of complicating it so 1. Then in that case the subintervals can be given by $0 ,\frac{1}{4}$, $\frac{1}{4}$ to $\frac{2}{4}$, then $\frac{2}{4}$ to $\frac{3}{4}$ and finally $\frac{3}{4}$ to 1. So, this partition P which has points as 0 , $\frac{1}{4}$, $\frac{2}{4}, \frac{3}{4}$ and 1, we will divide the interval [0,1] into non-overlapping sub intervals given by these 4 subintervals. So, this is an example of a partition for the interval [0,1] .

Now, that we have cleared out the concept of partition, we will basically look into the Riemann integrability, because Riemann integrability involves these concepts of partition and sub intervals. But, before we do the Riemann integrability, we will look into what is the upper sum and the lower sum sometimes they are also called as well Riemann upper sum or Riemann lower sum, so to do that we will start with a new page actually.

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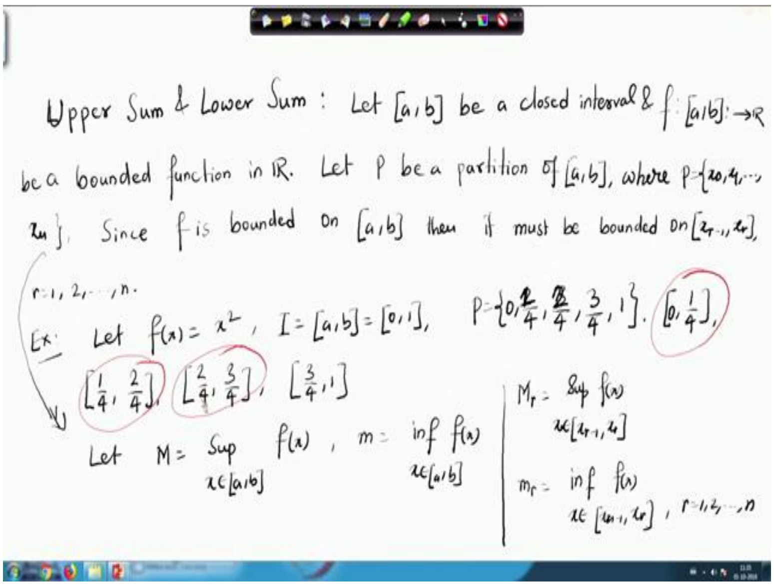

So, again so our point of discussion is upper sum and lower sum. So, upper sum and lower sum again, we will assume that let $[a, b]$ be a closed interval. So, just to give you an idea when we say a closed interval that means, both the endpoints of that interval are included. And when we say an open interval, then the endpoints are not included, and then you write it open interval something like this so this is the way to write an open interval all right.

Now, in this case we will always start, we will always take a closed interval unless we are told otherwise. Now, here we take $[a, b]$ as a closed interval. And f mapping from $[a, b]$ to R be a bounded function in R. The when we say bounded function, it means that it is bounded from above, it is bounded from below, otherwise we will say that f is a bounded function from above that means, it may not be bounded from below and vice versa basically.

Now, for example when we say that a bounded function $f(x)$ equals to let us say x^2 between the interval [1,2] is a bounded function. So, in that case the upper bound would be 4, and the lower bound would be 1, so that is what we mean by the bounded function. And we say let P be a partition of f of sorry not f, but it is the partition for the closed interval $[a, b]$ I beg your pardon. So, it is the partition of the closed interval [a, b], where our partition P has the points x_0, x_1 and so on up to x_n . And again it follows that $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

So, since now f is bounded on $a, b,$ then it must be bounded on x_{r-1} to x_r , where r can be $1, 2, 3$ up to n. Here the idea is I mean if you have a function f, which is bounded on the whole interval a, b, then certainly it is bounded on every subintervals. And so here x_{r-1} , x_r can for every value of our r can be of any one of those sub intervals. And if the function f is bounded on the entire interval, then obviously it will be bounded on that one small sub interval.

For example, let us say we have $f\ (x)$ equals to x^2 and our closed interval $[a, b]$ is equals to [0 ,1], so where a is 0, and b is 1. Then the partition P, we can consider as 0, we divide it in a four interval, so $\frac{2}{4}$, then $\frac{3}{4}$ and sorry 0, $\frac{1}{4}$, then $\frac{2}{4}$, then $\frac{3}{4}$, and then 1 then. This partition P divides this interval 0,1 into non-overlapping sub intervals.

And you can see that if we pick any one of these sub intervals, so here the possible sub intervals are $0, \frac{1}{4}$, then we have $\frac{1}{4}$ to $\frac{2}{4}$, then we have $\frac{2}{4}$ to $\frac{3}{4}$, and we finally have $\frac{3}{4}$ to 1. So, these are the four sub intervals for this closed interval $[a, b]$.

And if you consider any one of these sub intervals, you can see right away that this function f is bounded, it is mainly because of the fact that the function f is bounded on the whole interval $[a, b]$. So, it is it will be always bounded on one of these sub intervals. So, similar condition holds here as well. And from here we can also see that if the function f is bounded on the entire interval a, b, then in that case you have an upper bound and a lower bound for this interval $[a, b]$.

However, if you consider these sub intervals, then on each of these sub intervals you have an upper bound and then you have a lower bound. So, you can see that on this interval 0,4, the lower bound is 0 and the upper bound is $\frac{1}{16}$. However, in the interval $\frac{1}{4}$,$\frac{2}{4}$ your lower bound is $\frac{1}{16}$, and upper bound is $\frac{1}{4}$. So, similarly on every one of these sub intervals, you will get an upper bound and then you will get a lower bound.

So, now we will define basically these upper bounds and lower bounds for the whole interval a, b and also for these small sub intervals. So, for this function f, now we are back to this definition,. We write let M equals to supremum of all the x in a, b such that super over all the x supreme of the function $f(x)$ for all x in the closed interval $[a, b]$. And small m is equals to infimum of the function $f(x)$ for all x in the closed interval $[a, b]$. So, this supremum and infimum are so all the upper bound and the lower bound for this function $f(x)$. on the interval a , b.

Similarly, now for these subintervals x_{r-1} to x_r; we define capital M_r r as the supremum of the function $f(x)$ on the interval x_{r-1} to x_r.And m_r is equals to infimum of all the x infimum of the function $f(x)$ for all the x in x_{r-1} to x_r, where r is running from $1, 2, 3$ up to n all right ok. This page is full, so we go to the next page.

So, from so in the previous page so here basically what we see is as I was talking in this example $f(x) = x^2$. So, here in this case, we see that on every subinterval we have a lower bound and we have an upper bound, but at the same time for this interval [0,1] the whole interval, we have a lower bound as well as we have an upper bound.

So, now there is a relation between the lower bound and upper bound on the whole interval. And the lower bound and upper bound on each one of these sub intervals, which means the lower bound on this interval on this small sub interval $0, \frac{1}{4}$, the lower bound is 0. However, the lower bound on these intervals $\frac{1}{4}$ to $\frac{2}{4}$, it is $\frac{1}{16}$. Similarly, lower bound on this interval is 1 by so $\frac{1}{4}$ and so on, so that means, the lower bound on this whole interval $[0,1]$ will be either less than or equal to the lower interval on or at the lower bound on all these sub intervals does that make sense.

So, the lower bound for this whole interval $[0,1]$ is either equal to the lower bound on these sub intervals in this case for the first interval, because the lower bound for the whole interval is 0 and the lower bound for this is smaller sub interval $0, \frac{1}{4}$ is also 0. So, it is equal to actually, but as we proceed along these sub intervals. We can see that the lower bound for this whole interval 0 comma 1 is either equal or it is less than, because on all of these sub intervals the lower bound is not 0, it is either $\frac{1}{16}$ or $\frac{1}{4}$ or something like that, so that means, there is a less or equal to relation between the lower bounds. And that relation can be written as in this fashion.

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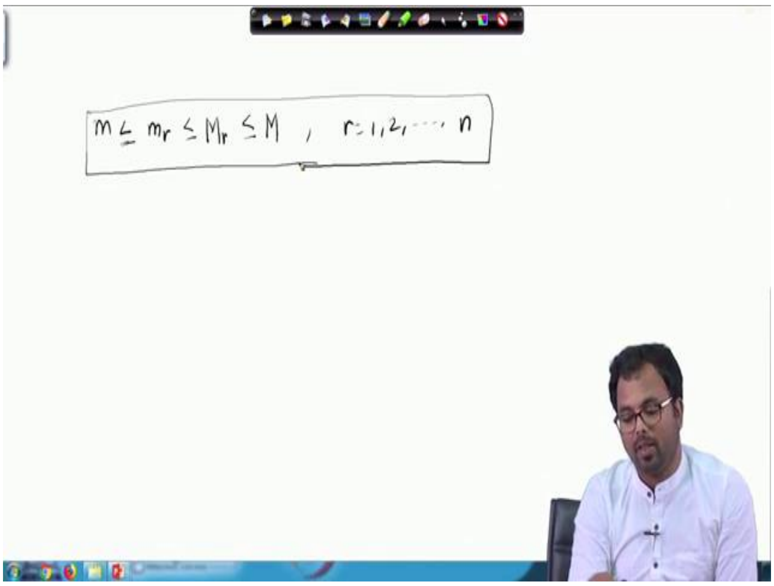

So, we can write that relation as a small m is lesser equal to small m_r. So, the lower bound of the function f on the whole interval will be lesser equal to the lower bound of the function f on all of these subintervals. And of course, the lower bound on all of these subintervals will be lesser equal to the upper bound on all of these subintervals. So, this from here to here it is obvious. Now, upper bound on all of these subintervals will certainly be less than upper bound on the whole interval because, when we talk about the upper bound, we are taking the maximum value possible for the function $f(x)$ on that interval $[0,1]$.

And since the interval 0,1 was divided into 4 sub intervals for that particular example. We are taking the maximum value of the function, and it can be attained on any one of those sub intervals. And basically, we are taking that particular value as the supremum or as the upper bound for that function f. However, in rest of the sub intervals of course, we have an upper bound of course, the function f has a maximum value at some point, but that upper bound is always lesser equal to the upper bound of the function f on the whole interval $[0,1]$.

So, that is why the upper bounds on all of these sub intervals will be lesser equal to the upper bound of the function f on the whole interval, so that upper bound is denoted by M. And here we have r running from $1, 2, 3$ up to n. So, this is a very vital relation.

And this relation says that the lower bound of the function f on the closed interval $[a, b]$ will be lesser equal to the lower bound of the function f on all the sub intervals, which is lesser equal to the upper bound of the function f on all the sub intervals, which is less than or equal to the upper bound of the function f on the whole interval $[a, b]$ for r running from 1, 2, 3 up to n. And we will conclude this lecture until here. And in the next lecture, we will begin with upper sum and lower sum, and then we will introduce the concepts of Riemann integrablility.